Comparison of inertial manifolds and application to modulated systems*

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Abstract

We consider two dissipative systems having inertial manifolds and give estimates which allow us to compare the flows on the two inertial manifolds. As an example of a modulated system we treat the Swift–Hohenberg equation $\partial_\tau w = -(1+\partial_y^2)^2 w+\varepsilon^2 w-w^3$, $w(\tau,y) \in \mathbb{R}$, with periodic boundary conditions on the interval $(0,\ell/\varepsilon)$. Recent results in the theory of modulation equation show that the solutions of this equation can be described over long time scales by those of the associated Ginzburg–Landau equation $\partial_t v = 4\partial_x^2 v + v - 3|v|^2 v$, $v(t,x) \in \mathbb{C}$, with suitably generalized periodic boundary conditions on $(0,\ell)$. We prove that both systems have an inertial manifold of the same dimension and that the flows on these finite dimensional manifolds converge against each other for $\varepsilon \to 0$.

1 Introduction

We study the simplest system which exhibits the typical features of the theory of modulation equations, namely the Swift–Hohenberg equation (SH) with $0 < \varepsilon \ll 1$:

$$\partial_\tau w = -(1+\partial_y^2)^2 w + \varepsilon^2 w - w^3, \qquad \tau \geq 0,\ y \in \mathbb{R},\ w(\tau,y) \in \mathbb{R}. \tag{1.1}$$

Here the trivial state $w \equiv 0$ is linearly unstable with respect to perturbations of the form e^{ikx} with $|k| \approx 1$. Thus, typical solutions will show modulations of this basic periodic pattern. These phenomena are explained within the theory of modulation equations which was formally developed after Eckhaus' pioneering work [Eck65] in the years 1965 to 1970. Only recently a rigorous mathematical theory was developed in [CoE90b, KSM92]

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Here we introduce the variable $u(t,x) = \varepsilon^{-1}w(\varepsilon^{-2}t, \varepsilon^{-1}x)$ and assume periodic boundary conditions for $x \in (0,\ell)$ which means the long interval $(0,\ell/\varepsilon)$ for y in (1.1). Then, u satisfies the equation

$$\partial_t u = -\varepsilon^{-2}(1+\varepsilon^2\partial_x^2)^2 u + u - u^3, \;\; u(t,x+\ell) = u(t,x). \tag{1.2}$$

The main result of the theory of modulation equations is that solutions of this equation can be described in the form

$$u(t,x) = v(t,x)\mathrm{e}^{\mathrm{i}x/\varepsilon} + \overline{v}(t,x)\mathrm{e}^{-\mathrm{i}x/\varepsilon} + \mathcal{O}(\varepsilon) \tag{1.3}$$

where $v(t,x) \in \mathbb{C}$ satisfies the real Ginzburg–Landau equation (GL)

$$\partial_t v = 4\partial_x^2 v + v - 3|v|^2 v, \quad v(t,x+\ell) = \mathrm{e}^{\mathrm{i}\gamma}v(t,x) \in \mathbb{C}, \tag{1.4}$$

where $\gamma \in \mathbb{S}^1 = \mathbb{R}/_{2\pi\mathbb{Z}}$ has to be chosen such that $\ell/\varepsilon = \gamma \bmod 2\pi$.

Both, (SH) and (GL), are parabolic dissipative equations which are amenable to the theory of global semiflows, compact attractors and inertial manifolds, see [CFNT88, Tem88]. Here we are interested in inertial manifolds.

An inertial manifold is a finite–dimensional manifold M enjoying the following three properties:

a) M is a Lipschitz continuous manifold.

b) M is positively invariant for the semigroup $S(t)$ defined by the underlying partial differential equation, i.e $S(t)M \subset M$ for all $t \geq 0$.

c) M attracts exponentially all the orbits of the underlying partial differential equation.

As a consequence the global attractor lies completely inside any inertial manifold. For the real and complex (GL) the existence of an inertial manifold N_γ is established for instance in [DGHN88] giving a dimension proportional to ℓ^3 for our case. Results for (SH) are given in [Tab90] where the dimension of the inertial manifold M_ε can be bounded by $\varepsilon^{2/3}L^{5/3}$ for all $\varepsilon > 0$ and L being the length of the y–interval. Since we have $L = \ell/\varepsilon$ with ℓ kept fixed the dimension estimate gives $C\varepsilon^{-1}\ell^{5/3}$.

Our aim is to show that the dimension of the inertial manifold M_ε can be kept bounded for $\varepsilon \to 0$. For this purpose we use the connection between (SH) and (GL) via the modulation ansatz (1.3), which is only valid for small ε. Our result will be that the dimension of M_ε can be bounded by $C\ell^3$ independently of $\varepsilon \in (0,\varepsilon_0)$, and moreover we will show that the reduced vector field on M_ε with $\varepsilon = \ell/(2\pi q - \gamma) \to 0$, $q \in \mathbb{N}$, and $|\gamma| \leq \pi$, converges for $q \to \infty$ to the reduced vector field on N_γ.

The method relies on an abstract setting developed in Section 2 for two equations

$$\dot{u} = Ku + F(u), \quad \dot{v} = Lv + G(u).$$

(The notation is throughout such that structures associated to the second system are denoted with the letter following the letter which is used to denote the similar structure in the first system.)

We will use a splitting of $u \in X$ in three components $u_j \in X_j$ with $X = X_1 \oplus X_2 \oplus X_3$ and similarly for $v = v_1+v_2+v_3$, where X_1 and Y_1 are the spaces over which the inertial manifolds M_ε and N_γ are defined, respectively. The spaces X_k, $k = 1, 2$, are related to Y_k via isomorphisms $T_k : X_k \to Y_k$, such that $T_k K_k - L_k T_k$ and $T_k F_k(u_1+u_2) - G_k(T_1 u_1 + T_2 u_2)$ are small for $k = 1, 2$. The subspaces X_3 and Y_3 are those parts where no comparison is possible but the contributions from this part to the flow on the inertial manifold will be small.

For (GL) the space Y_1 and Y_2 are those which are spanned by the eigenfunctions $\psi_n(x) = \mathrm{e}^{\mathrm{i}(2\pi n+\gamma)x/\ell}$ for $|n| \leq n_{\mathrm{GL}}$ and $n_{\mathrm{GL}} < |n| \leq \varepsilon^{-2\delta}$, respectively, where $\dim N_\gamma \approx 4n_{\mathrm{GL}}$ and $\delta \in (0, 1)$. The spaces X_1 and X_2 are defined such that they correspond to Y_1 and Y_2 via the modulation ansatz (1.3) (without the error term $\mathcal{O}(\varepsilon)$).

2 Comparison of inertial manifolds

We consider two evolutionary systems, one for $u \in X$ and one for $v \in Y$. The spaces X and Y are Hilbert spaces. As a rule we will always use subsequent letters of the alphabet to indicate variables associated to the two different systems. For both systems we assume a splitting into three parts such that

$$\dot{u}_j = K_j u_j + F_j(u), \qquad \dot{v}_j = L_j v_j + G_j(v), \qquad j = 1, 2, 3, \tag{2.1}$$

where $u_j \in X_j$, $u = u_1 + u_2 + u_3 \in X_1 \oplus X_2 \oplus X_3 = X$, and similarly for v_j. The mappings $P_j : X \to X_j; u \mapsto u_j$ are the corresponding projections. We will also use the shorthand notation $u_{jk} \stackrel{\text{def}}{=} u_j + u_k \in X_{jk} \stackrel{\text{def}}{=} X_j \oplus X_k$.

As long as we are only interested in the existence of inertial manifolds, the distinction between X_2 and X_3 is not needed. However, it will be of importance when the comparison of the inertial manifolds $M \subset X$ and $N \subset Y$ for the u- and the v-system is studied.

The operators K_j and L_j are assumed to satisfy the estimates

$$\begin{array}{ll} \|\mathrm{e}^{tK_1}\| \leq \mathrm{e}^{-\kappa_1 t}, & \|\mathrm{e}^{tL_1}\| \leq \mathrm{e}^{-\lambda_1 t} \text{ for } t \leq 0, \\ \|\mathrm{e}^{tK_j}\| \leq \mathrm{e}^{-\kappa_j t}, & \|\mathrm{e}^{tL_j}\| \leq \mathrm{e}^{-\lambda_j t} \text{ for } j = 2, 3 \text{ and } t \geq 0. \end{array} \tag{2.2}$$

Thus, we restricted the eigenvalues of K_1, K_2 and K_3 to lie in the regions $\{\operatorname{Re}\sigma \geq -\kappa_1\}$, $\{\operatorname{Re}\sigma \leq -\kappa_2\}$ and $\{\operatorname{Re}\sigma \leq -\kappa_3\}$, respectively. The analogous statement for L_j holds.

The nonlinearities are assumed to be bounded and globally Lipschitz continuous:

$$\begin{array}{l} \|F_j(u) - F_j(\widetilde{u})\|_{X_j} \leq \sum_{k=1}^3 f_{j,k} \|u_k - \widetilde{u}_k\|_{X_k}, \\ \|G_j(v) - G_j(\widetilde{v})\|_{Y_j} \leq \sum_{k=1}^3 g_{j,k} \|v_k - \widetilde{v}_k\|_{Y_k}, \\ \|F_j(u)\|_{X_j} \leq f_{j,\infty}, \qquad \|G_j(v)\|_{Y_j} \leq g_{j,\infty}, \end{array} \tag{2.3}$$

for $j = 1, 2, 3$ and all $u, \widetilde{u} \in X$ and $v, \widetilde{v} \in Y$.

The following existence result for the inertial manifold is only stated for the u-system, but the analogous result for the v–system holds also.

Theorem 2.1
Assume that system (2.1) satisfies the estimates (2.2) and (2.3) and that there is a constant $\alpha \in (\kappa_1, \kappa_2)$ such that the contraction condition

$$\frac{1}{|\alpha-\kappa_j|}\sum_{k=1}^{3} f_{j,k} \le \frac{1}{2} \qquad \text{for } j=1,2,3, \tag{2.4}$$

holds. Then, (2.1) has an inertial manifold in the form

$$M = \mathrm{graph}(\Phi_{23}) = \{\, u \in X \;:\; u_{23} = \Phi_{23}(u_1)\,\},$$

where the functions $\Phi_{23} : X_1 \to X_{23}$ is globally Lipschitz continuous and bounded via

$$\|\Phi_j(u_1)\|_{X_j} \le f_{j,\infty}/(\kappa_j-\alpha), \quad j=2,3.$$

Proof: From now on we use the norm $\|u\| = \max\{\, \|u_j\|_{X_j} \;:\; j=1,2,3\,\}$ on X which is certainly equivalent to the original norm.

Define the space $X^\alpha = \{\, u \in \mathcal{C}^0((-\infty,0],X) \;:\; |\!|\!|u|\!|\!|_\alpha < \infty\,\}$ with the norm

$$|\!|\!|u|\!|\!|_\alpha = \sup\{\, \|u(t)\|\mathrm{e}^{\alpha t} \;:\; t \le 0\,\}.$$

Any solution of (2.1) which also satisfies $u \in X^\alpha$ solves the integral equation

$$\begin{aligned} &u = S(u_1(0),u) \text{ with} \\ &S(\xi,u)(t) = (\mathrm{e}^{tK_1}\xi - \textstyle\int_t^0 \mathrm{e}^{(t-s)K_1}F_1(u(s))\mathrm{d}s, \int_{-\infty}^t \mathrm{e}^{(t-s)K_{23}}F_{23}(u(s))\mathrm{d}s). \end{aligned}$$

From the estimates (2.2) and (2.3) we find

$$\|S_j(\xi,u)(t) - S_j(\xi,\widetilde{u})(t)\|_{X_j} \le \frac{\mathrm{e}^{-\alpha t}}{|\kappa_j-\alpha|}\sum_{k=1}^{3} f_{j,k}|\!|\!|u_k-\widetilde{u}_k|\!|\!|_\alpha$$

for $j=1,2,3$. Thus, the assumption on the matrix in (2.4) guarantees that $S(\xi,\cdot)$ is a contraction on X^α with Lipschitz constant less than or equal to $1/2$, which depends Lipschitz continuously on $\xi \in X_1$. Denote by $u = U(\xi,\cdot)$ the unique fixed point in X^α and define $\Phi_j(\xi) = U_j(\xi,0) \in X_j$ for $j=2,3$. Following the same arguments as in [FST88, DGHN88, Mi94], it is clear that the inertial manifold is as given in the theorem. QED

Using the analogous statement for the v-system we obtain an inertial manifold $N = \mathrm{graph}(\Psi_{23}) \subset Y$ and both systems can be reduced to this manifold as follows:

$$\dot{u}_1 = F_{\mathrm{red}}(u_1) \stackrel{\mathrm{def}}{=} K_1u_1 + F_1(u_1{+}\Phi_{23}(u_1)), \quad \dot{v}_1 = G_{\mathrm{red}}(v_1) \stackrel{\mathrm{def}}{=} L_1v_1 + G_1(v_1{+}\Psi_{23}(v_1)). \tag{2.5}$$

The aim is to provide conditions which allows us to compare these two reduced vector fields.

To this end, we assume that there are bijections $T_k \in \mathcal{L}(X_k,Y_k)$ for $k=1,2$ such that $T_k^{-1} \in \mathcal{L}(Y_k,X_k)$. The main philosophy here is that the u- and the v-system are comparable in the first two components whereas the third component will be small in each system such that no comparison is needed there.

We define the following quantities

$$
\begin{aligned}
&M_k(t) = T_k \mathrm{e}^{tK_k} - \mathrm{e}^{tL_k} T_k \in \mathcal{L}(X_k, Y_k),\\
&\mu_0 = \sup\{ \mathrm{e}^{\alpha t}\|M_1(t)\| \,:\, t \le 0 \},\\
&\mu_1 = \textstyle\int_{-\infty}^0 \mathrm{e}^{\alpha t}\|M_1(t)\|\mathrm{d}t, \quad \mu_2 = \int_0^\infty \mathrm{e}^{\alpha t}\|M_2(t)\|\mathrm{d}t,\\
&\rho_k = \sup\{ \|T_k F_k(u_{12}{+}u_3) - G_k(T_{12}u_{12}{+}v_3)\|_{Y_k} \,:\, u_1 \in X_1,\ \|v_3\|_{Y_3} \le \tfrac{g_{3,\infty}}{\lambda_3-\alpha},\\
&\qquad\qquad\qquad\qquad\qquad\qquad \|u_n\|_{X_n} \le \tfrac{f_{n,\infty}}{\kappa_n-\alpha} \text{ for } n=2,3 \}.
\end{aligned}
\tag{2.6}
$$

Proposition 2.2
Assume that both, the u- and the v-system, satisfy the assumptions of Theorem 2.1 with the same α. Let $M = \mathrm{graph}((\Phi_2,\Phi_3)) \subset X$ and $N = \mathrm{graph}((\Psi_2,\Psi_3))$ be the associated inertial manifolds. Moreover, let μ_j and ρ_k be defined as in (2.6). Then, for all $u_1 \in X_1$ we have the estimate

$$
\|T_2\Phi_2(u_1) - \Psi_2(T_1u_1)\|_{Y_2} \le 2\max\left\{ \mu_0\|u_1\|_{X_1} + \mu_1 f_{1,\infty} + \frac{\rho_1}{\alpha},\ \mu_2 f_{2,\infty} + \frac{\rho_2}{\alpha} \right\}.
$$

Proof: We use that $\Phi_2(\xi) = U_2(\xi,0)$ and $\Psi_2(\eta) = V_2(\eta,0)$ where U and V are the fixed points of $S(\xi,\cdot)$ on X^α and $T(\eta,\cdot)$ on Y^α. For $t \le 0$ we have the estimate

$$
\begin{aligned}
&\|T_1U_1(\xi,t) - V_1(T_1\xi,t)\|_{Y_1} =\\
&\|M_1(t)\xi - \textstyle\int_t^0 \Big(M_1(t{-}s)F_1(U(\xi,s)) + \mathrm{e}^{(t-s)L_1}[T_1F_1(U(\xi,s)) - G_1(V(T_1\xi,s))]\Big)\mathrm{d}s\|_{Y_1}\\
&\le \mathrm{e}^{-\alpha t}\Big[\mu_0\|\xi\| + \textstyle\int_t^0 \mathrm{e}^{\alpha(t-s)}\|M_1(t{-}s)\| f_{1,\infty}\mathrm{d}s\\
&\qquad + \textstyle\int_t^0 \mathrm{e}^{\lambda_1 s}\Big(\rho_1 + \sum_{k=1}^2 g_{1,k}\|T_kU_k(\xi,s) - V_k(T_1\xi,s)\|_{Y_k}\Big)\mathrm{d}s\Big].
\end{aligned}
$$

A similar estimate holds for $\|T_2U_2(\xi,t) - V_2(T_1\xi,t)\|_{Y_2}$ and with $\Delta = \max_{k=1,2} \|\!|T_1U_k(\xi,\cdot) - V_k(T_1\xi,\cdot)\|\!|_\alpha$ we obtain

$$
\Delta \le \max\left\{ \mu_0\|\xi\| + \mu_1 f_{1,\infty} + \tfrac{\rho_1}{\alpha},\ \mu_2 f_{2,\infty} + \tfrac{\rho_2}{\alpha} \right\} + \max_{k=1,2}\left(\tfrac{g_{k,1}+g_{k,2}}{|\alpha-\lambda_k|}\right)\Delta.
$$

Using (2.4) the second maximum is less than $1/2$ and the desired estimate follows. QED

In many practical cases the nonlinear error terms ρ_k can be estimated by

$$
\rho_k \le \sigma_k + \|T_k\|\frac{f_{k,3}f_{3,\infty}}{\kappa_3-\alpha} + \frac{g_{k,3}g_{3,\infty}}{\lambda_3-\alpha},
\tag{2.7}
$$

where $\sigma_k = \sup\{ \|T_kF_k(u_{12}) - G_k(T_{12}u_{12})\|_{Y_k} \,:\, u_1 \in X_1,\ \|u_2\|_{X_2} \le \frac{f_{2,\infty}}{\kappa_2-\alpha} \}$. To this end use $\|T_kF_k(u) - G_k(T_{12}u_{12}{+}v_3)\| \le \|T_k\|\|F_k(u) - F_k(u_{12})\| + \|T_kF_k(u_{12}) - G_k(T_{12}u_{12})\| + \|G_k(T_{12}u_{12}) - G_k(T_{12}u_{12}{+}v_3)\|$ together with the Lipschitz constants and the bounds on u_3 and v_3.

Theorem 2.3
Under the above assumption for all $u_1 \in X_1$ we have

$$
\begin{aligned}
\|T_1F_{\mathrm{red}}(u_1) - G_{\mathrm{red}}(T_1u_1)\|_{Y_1} \;\le\;& \Big(\|T_1K_1 - L_1T_1\| + 2\mu_0 g_{1,2}\Big)\|u_1\|_{X_1}\\
&+2\rho_1 + 2g_{1,2}\Big(\mu_1 f_{1,\infty} + \mu_2 f_{2,\infty} + \rho_2/\alpha\Big).
\end{aligned}
$$

Proof: The result is a consequence of the following estimate

$$\begin{aligned}
&\|T_1 F_{\mathrm{red}}(\xi) - G_{\mathrm{red}}(T_1\xi)\|_{Y_1}\\
&\leq \|(T_1K_1 - L_1T_1)\xi\| + \|T_1F_1(\xi+\Phi_2(\xi)+\Phi_3(\xi)) - G_1(T_1\xi+T_2\Phi_2(\xi)+\Psi_3(T_1\xi))\|\\
&\quad + \|G_1(T_1\xi+T_2\Phi_2(\xi)+\Psi_3(T_1\xi)) - G_1(T_1\xi+\Psi_2(T_1\xi)+\Psi_3(T_1\xi))\|\\
&\leq \|(T_1K_1 - L_1T_1)\xi\| + \rho_1 + g_{1,2}\|T_2\Phi_2(\xi) - \Psi_2(T_1\xi)\|.
\end{aligned}$$

together with Proposition 2.2 and $g_{1,2}/(\alpha-\lambda_1) \leq 1/2$. QED

Because of $\int_{-\infty}^0 \mathrm{e}^{t(\alpha I+K_1)}\mathrm{d}t = (K_1+\alpha I)^{-1}$ and the corresponding formula for L_1 we have $\|T_1(K_1+\alpha I)^{-1} - (L_1+\alpha I)^{-1}T_1\| \leq \mu_1/\alpha$. This implies

$$\begin{aligned}
\|T_1K_1 - L_1T_1\| &= \|(L_1+\alpha I)\Big((L_1+\alpha I)^{-1}T_1 - T_1(K_1+\alpha I)^{-1}\big)(K_1+\alpha I)\|\\
&\leq \tfrac{\mu_1}{\alpha}\|L_1+\alpha I\|\|K_1+\alpha I\|.
\end{aligned}$$

For selfadjoint and negative semidefinite operators K_1 and L_1 this can be estimated by $\alpha\mu_1$.

3 The Swift–Hohenberg equation

As already said the Swift–Hohenberg equation is given by

$$\partial_\tau w = -(1+\partial_y^2)^2 w + \varepsilon^2 w - w^3, \tag{3.1}$$

where $w(\tau, y) \in \mathbb{R}$ and $y \in \mathbb{R}$. Here we restrict our attention to the case of periodic boundary conditions where the period is given as ℓ/ε. This special scaling of the spatial variable becomes clearer in the sequel. In fact, in contrast to previous work on modulation equations we will rescale the (SH) and define $u(t,x) = \varepsilon^{-1}w(\varepsilon^{-2}t, \varepsilon^{-1}x)$, that is $t = \varepsilon^2\tau$ and $x = \varepsilon y$. The resulting equation is

$$\partial_t u = -\varepsilon^{-2}(1+\varepsilon^2\partial_x^2)^2 u + u - u^3, \qquad u(t, x+\ell) = u(t,x) \in \mathbb{R}. \tag{3.2}$$

The theory of modulation equations as presented in [CoE90b, KSM92, Sch94a] relies on the fact that the typical solutions of (3.2) can be described in the form

$$u(t,x) \approx v(t,x)\mathrm{e}^{\mathrm{i}x/\varepsilon} + \overline{v}(t,x)\mathrm{e}^{-\mathrm{i}x/\varepsilon}, \tag{3.3}$$

where v is a solution of the (real) Ginzburg–Landau equation

$$\partial_t v = 4\partial_x^2 v + v - 3|v|^2 v, \quad v(t, x+\ell) = \mathrm{e}^{\mathrm{i}\gamma}v(t,x) \in \mathbb{C}. \tag{3.4}$$

As usual this equation has to be taken as a real system for $(\operatorname{Re} v, \operatorname{Im} v)$ which is written in the complex form only for convenience.

The aim of this section is to show that (3.2) has for given $\ell > 0$ and sufficiently small ε an inertial manifold M_ε whose dimension is independent of ε and the associated reduced vector field $\widetilde{F}_1(\varepsilon, \cdot)$ converges for $\varepsilon \to 0$ to a limit which coincides with the flow on an inertial manifold N_γ for the Ginzburg–Landau equation $(3.4)_\gamma$. Notice that the periodic boundary conditions are generalized via the factor $\mathrm{e}^{\mathrm{i}\gamma}$ with $\gamma \in \mathbb{S}^1 = \mathbb{R}/_{2\pi\mathbb{Z}}$. In fact, we will only show that $F_{\mathrm{red}}(\varepsilon_k, \cdot)$ with $\varepsilon_k = \ell/(2k\pi-\gamma)$ converges for $k \to \infty$ to $G_{\mathrm{red}}(\gamma, \cdot)$ which is the flow on N_γ associated to $(3.4)_\gamma$.

3.1 The inertial manifold N_γ

In this subsection we present the existence theory for the inertial manifold for the Ginzburg–Landau equation (3.4) which is of course rather classical in the case $\gamma = 0$, see [FST88] or more specifically [DGHN88]. The case $\gamma \neq 0$ has no additional complications but we need it on the one hand to set up the notation for the comparison result and on the other hand we have to take care for the uniformity in $\gamma \in \mathbb{S}^1$.

From now on we keep $\gamma \in \mathbb{S}^1$ fixed and define $Y = \{\, v \in H^1((0,\ell)) \,:\, v(\ell) = \mathrm{e}^{\mathrm{i}\gamma} v(0) \,\}$ with the norm $\|v\|^2 = \int_0^\ell [v^2 + (\partial_x v)^2] \mathrm{d}x$. The linear operator $L : D(L) \subset Y \to Y$ is defined via

$$D(L) = \{\, v \in H^3((0,\ell)) \cap Y \,:\, \partial_x v, \partial_x^2 v \in Y \,\} \quad \text{and } Lv = 4\partial_x^2 v - v.$$

Note that $v \mapsto 4\partial_x^2 v - v$ is selfadjoint on $L^2((0,\ell))$ with the domain $\{\, v \in H^2((0,\ell) \,:\, v, \partial_x v \in Y \,\}$. The spectrum is discrete with eigenvalues $\lambda_{(n)} = -1-(\gamma+2\pi n)^2/\ell^2$, $n \in \mathbb{Z}$, and eigenfunctions $\psi_n(x) = \mathrm{e}^{\mathrm{i}(2\pi n+\gamma)x/\ell}$. Of course, considering the complex problem as a real system these eigenvalues are all double since the complex eigenfunction can be multiplied by i.

Since the generalized periodicity conditions are not standard we want to mention that the transform $\widetilde{v}(x) = \mathrm{e}^{-\mathrm{i}sx} v(x)$ with $s = \gamma/\ell$ leads to the system $\partial_t \widetilde{v} = 4(\partial_x \widetilde{v} + 2\mathrm{i}s\partial_x \widetilde{v} - s^2 \widetilde{v}) + \widetilde{v} - 3|\widetilde{v}|^2 \widetilde{v}$ with the standard periodic boundary conditions $\widetilde{v}(t, x+\ell) = \widetilde{v}(t,x)$.

Doing the standard a-priori estimates it is easy to show that there is an absorbing ball in Y, namely

$$\limsup_{t\to\infty} \|v(t)\|_Y \leq C_b \ell^{1/2}, \tag{3.5}$$

where C_b is independent of ℓ and $\gamma \in \mathbb{S}^1$. The simple dependence on ℓ follows most easily from the estimates provided for the equations on the whole real line, see e.g. [Col94, Mi97a] and in particular Lemma 3.6 in [MiS95]. Thus, we may modify $\widetilde{G} : v \mapsto 2v - 3|v|^2 v$ outside the absorbing ball to obtain a function $G : Y \to Y$ such that

$$\|G(v)\|_Y \leq C\ell^{3/2}, \qquad \|G(v) - G(\widehat{v})\|_Y \leq \mathrm{Lip}_{\mathrm{GL}} \|v - \widehat{v}\|_Y \text{ with } \mathrm{Lip}_{\mathrm{GL}} = (2 + C\ell), \tag{3.6}$$

for all $v, \widehat{v} \in Y$, where C does not depend on $\ell \geq 1$ and $\gamma \in \mathbb{S}^1$. This modification can be done as in [Mi94], Lemma 3.1, by setting $\theta(r) = \min\{r, 1\}$ and

$$G(v) = \widetilde{G}\Big(\theta(\|v\|/(C_b\sqrt{\ell}))v\Big)$$

Then using $\|v^3\| \leq C\|v\|^3$ the desired estimates (3.6) follow.

For fixed γ it is easy to find spectral gaps, however the spectral splitting cannot be taken uniformly in $\gamma \in \mathbb{S}^1$. Therefore, we now assume $\gamma \in (-\pi, \pi]$. For an n_{GL} to be chosen later we define

$$\begin{aligned} &Y_1 = \mathrm{span}\{\, \psi_k, \mathrm{i}\psi_k \,:\, k \in J_1(n_{\mathrm{GL}}, \gamma) \,\} \\ &\text{where } J_1(n_{\mathrm{GL}}, \gamma) = \mathbb{Z} \cap (-n_{\mathrm{GL}} - (\tfrac{3\pi}{2} + \gamma)/(2\pi), n_{\mathrm{GL}} + (\tfrac{3\pi}{2} - \gamma)/(2\pi)). \end{aligned}$$

Hence, the dimension of Y_1 is $4n_{\rm GL}+2$ for $|\gamma| \leq \pi/2$, and $4n_{\rm GL}+4$ otherwise. Moreover, the eigenvalues satisfy

$$\begin{aligned} \lambda_1 &= \max\{\, -\lambda_{(k)} \;:\; k \in J_1 \,\} \leq 1 + 4\tfrac{4\pi^2}{\ell^2}\big(n_{\rm GL}+c(\gamma)\big)^2, \\ \lambda_2 &= \min\{\, -\lambda_{(k)} \;:\; k \notin J_1 \,\} \geq 1 + 4\tfrac{4\pi^2}{\ell^2}\big(n_{\rm GL}+\tfrac12+c(\gamma)\big)^2, \end{aligned}$$

where $c(\gamma) = 1/4$ for $|\gamma| \leq \pi/2$ and $3/4$ else. In both cases the spectral gap $\lambda_2 - \lambda_1$ can be estimated from below by $16\pi^2 n_{\rm GL}/\ell^2$. This leads to the following result.

Theorem 3.1
There is a constant $C > 0$ such that for each $\ell \geq 1$ there exists an $n_{\rm GL} \in \mathbb{N}$ with $n_{\rm GL} \leq C\ell^3$ such that for all $\gamma \in \mathbb{S}^1$ the equation (3.4) has an inertial manifold $N_\gamma = \operatorname{graph}(\Psi_{23})$ where Ψ_{23} is a Lipschitz function from Y_1 into $Y_{23} = \operatorname{span}\{\, \psi_k, {\rm i}\psi_k \;:\; k \notin J_1(n_{\rm GL}, \gamma) \,\}$.

The result follows by writing (3.4) as the abstract system $\dot v = Lv + G(v)$, where G is the modification of $\widetilde G(v) = 2v - |v|^2 v$ such that (3.6) holds. Then, Theorem 3.1 can be applied with $\alpha = (\lambda_1+\lambda_2)/2$ and $g_{j,k} = {\rm Lip}_{\rm GL}$. At this moment the splitting of Y_{23} into $Y_2 \oplus Y_3$ plays no role. Clearly, ${\rm Lip}_{\rm GL}/|\lambda_j - \alpha|$ can be made smaller than $1/2$ by choosing $n_{\rm GL}$ sufficiently large. In particular, it is sufficient to let $n_{\rm GL} \approx C\ell^3$ for some sufficiently large C which is independent of $\ell \geq 1$ and $\gamma \in (-\pi, \pi]$.

Remark 3.2 Using the arguments in [Mi94] it is possible to reduce the dimension of the inertial manifold to $C\big(\ell^5 \log(\ell+1)\big)^{1/2}$. However, these modifications would complicate the present setting severely and hide the main emphasis in the comparison theory. In fact, using the completely different method in [Bru90] it should be possible to show that the dimension can even be chosen of order $C\ell$.

3.2 Norms and estimates for the Swift–Hohenberg equation

We return to the Swift–Hohenberg equation (3.2) and first give an a–priori estimate on the solution. We use Theorem 5.2 in [MiS95] which states for (3.1) the estimates

$$\limsup_{\tau\to\infty} \left[\sup_{z\in\mathbb{R}} \Big(\int_{y=z}^{z+1} [(w(\tau,y))^2 + (\partial_y w(\tau,y))^2]\,{\rm d}y \Big)^{1/2} \right] \leq C\varepsilon.$$

This estimate does not follow from standard energy estimates. It is derived with the help of the theory of modulation equations on the real line, not using periodicity.

Clearly this implies $\limsup_{\tau\to\infty} \|w(\tau,\cdot)\|_\infty \leq C\varepsilon$ and thus after rescaling we conclude for the solutions $u(t,x)$ of our equation (3.2) that

$$\limsup_{t\to\infty} \big(\|u(t)\|_{\rm L^2} + \varepsilon \|\partial_x u(t)\|_{\rm L^2} + \|u^3(t)\|_{\rm L^2} \big) \leq C\ell^{1/2}, \tag{3.7}$$

where C is independent of $\ell \geq 1$ and $\varepsilon > 0$ sufficiently small.

Our basic space X is chosen as ${\rm H}^1_{\rm per}((0,\ell)) = \{\, u \in {\rm H}^1((0,\ell)) \,:\, u(0) = u(\ell) \,\}$, however, we will not use the standard norm of ${\rm H}^1$, since the solutions under consideration will be of the form given in (3.3). Thus, we will use the norm defined in (3.11). Using the Fourier–series representation $u(x)=\sum_{n\in\mathbb{Z}}u_{(n)}\phi_n(x)$ with $\phi_n(x)=\mathrm{e}^{\mathrm{i}2\pi nx/\ell}$ it is clear that we expect relatively large coefficients for $|n|\approx\ell/(2\pi\varepsilon)$. Here, ϕ_n is an eigenfunction to the eigenvalue

$$\kappa_{(n)}=-1-(1-(\varepsilon 2\pi n/\ell)^2)^2/\varepsilon^2$$

of the linear operator

$$K:\left\{\begin{array}{ccc} D(K)\subset X & \to & X,\\ u & \mapsto & -\frac{1}{\varepsilon^2}(1+\varepsilon^2\partial_x^2)^2u-u,\end{array}\right.$$

where $D(K)=\mathrm{H}^5_{\mathrm{per}}((0,\ell))$. The equation (3.2) now takes the abstract form

$$\dot{u}=Ku+\widetilde{F}(u),\qquad\text{with }\widetilde{F}(u)=2u-u^3. \tag{3.8}$$

The following result shows that the Fourier modes are concentrated about $|n|\approx\ell/(2\pi\varepsilon)$.

Proposition 3.3
For each $\beta\in(0,1)$ there exists $C_\beta>0$ independent of $\varepsilon\in(0,1)$ such that all solutions $u(t)=\sum_n u_{(n)}(t)\phi_n$ of (3.2) satisfy

$$\limsup_{t\to\infty}|u(t)|_\beta\le C_\beta\sqrt{\ell},\quad\text{where }|u(t)|_\beta=\left(\sum_{n\in\mathbb{Z}}[|\kappa_{(n)}|^\beta u_{(n)}(t)]^2\right)^{1/2}.$$

Proof: The result follows easily using the variation of constants formula

$$u(t)=\mathrm{e}^{(t-r)K}u(r)+\int_{s=r}^{t}\mathrm{e}^{(t-s)K}\widetilde{F}(u(s))\mathrm{d}s,$$

the estimate $|\mathrm{e}^{tK}u|_\beta\le Ct^{-\beta}|u|_0$, where $|u|_0=\|u\|_{\mathrm{L}^2}$, and the a–priori bounds (3.7). QED

To define the splitting of X into the subspaces X_k, $k=1,2,3$ associated to $\ell\ge1$ and $\varepsilon>0$ we need the numbers $q\in\mathbb{N}$ and $\gamma\in(-\pi,\pi]$ such that $\ell=\varepsilon(2\pi q-\gamma)$. Notice that q is defined such that $\kappa_{(q)}$ is one of the four largest eigenvalue of K. Using the Fourier decomposition $u=\sum u_{(n)}\phi_n$ we define the projections

$$P_1u=\sum_{n\in I_1(\ell,\varepsilon)}u_{(n)}\phi_n,\quad P_2u=\sum_{n\in I_2(\ell,\varepsilon)}u_{(n)}\phi_n,\quad P_3=I_X-P_1-P_2.$$

where $I_1(\ell,\varepsilon)=\{\,n\in\mathbb{Z}:n-q\in J_1\text{ or }q-n\in J_1\,\}$ and where $I_2(\ell,\varepsilon)=\{\,n\in\mathbb{Z}\setminus I_1(\ell,\varepsilon):||n|-q|\le\varepsilon^{-\delta}\,\}$. Here, $J_1=J_1(n_{\mathrm{GL}}(\ell),\gamma)$ is the set defined in Subsection 3.1 and $\delta\in(0,1)$ is a parameter which will be chosen close to 1 later. We recall the notation $u_k=P_ku$ and $u_{jk}=u_j+u_k=P_{jk}u$ with $P_{jk}=P_j+P_k$. Since the function u is real we have $u_{(-n)}=\overline{u}_{(n)}$ and the symmetry of the sets I_k guarantees that u_k is again real.

With this notation it is possible to define the isomorphisms $T_k:X_k\to Y_k$, $k=1,2$, in accordance with (3.3). We define T_{12} via

$$T_{12}u_{12}=\mathrm{e}^{-\mathrm{i}x/\varepsilon}\sum_{|n-q|\le\varepsilon^{-\delta}}u_{(n)}\phi_n=\sum_{|m|\le\varepsilon^{-\delta}}u_{(q+m)}\psi_m, \tag{3.9}$$

where $\psi_n(x) = e^{ix(2\pi m+\gamma)}$ is the eigenfunction of L. Now, T_k, $k=1,2$, is obtained by restriction to the appropriate subspaces. A simple calculation shows that for $\varepsilon \in (0,\varepsilon_0)$ with $\varepsilon_0 > 0$ sufficiently small we have the identity

$$T_{12}\widetilde{F}_{12}(u_{12}) = \widetilde{G}_{12}(T_{12}u) \quad \text{for all } u_{12} \in X_{12}. \tag{3.10}$$

This identity simplifies the subsequent analysis but it is obvious that additional terms of order $\mathcal{O}(\varepsilon)$ can be allowed for.

We now define the exact norm on X which underlies our analysis. Choosing a $\delta \in (0,1)$ where $\delta \approx 1$ is the interesting case we define the weight

$$\rho_n = \begin{cases} 1+(\frac{1}{\varepsilon}-\frac{2\pi}{\ell}|n|)^2 & \text{for } ||n|-q| \leq \varepsilon^{-\delta}, \\ \varepsilon^{-2\eta}(1+\varepsilon^2 n^2) & \text{else,} \end{cases}$$

where $\eta \in \mathbb{R}$ will be chosen later. The associated norm on X is now given as

$$\|u\|_X = (\sum_{n\in\mathbb{Z}} \rho_n u_{(n)}^2)^{1/2}. \tag{3.11}$$

The choice of the norm in X was done such that the operators T_1 and T_2 preserve the norm, namely $\|T_k u_k\|_Y = \|u_k\|_X$. From Proposition 3.3 we obtain, by comparing the weights, the estimates

$$\limsup_{t\to\infty} \|u_{12}(t)\|_{X_{12}} \leq C_a\sqrt{\ell}, \qquad \limsup_{t\to\infty} \|u_3(t)\|_{X_3} \leq C_a\sqrt{\ell}\varepsilon^{2\delta-\eta}, \tag{3.12}$$

where C_a is independent of ℓ, δ, and η. The second estimate follows even easier by using $\|\widetilde{F}_3(u(t))\|_{\mathrm{L}^2} \leq C\sqrt{\ell}$ and $\|\mathrm{e}^{tK_3}\| \leq \mathrm{e}^{-\lambda_3 t}$ with $\lambda_3 \geq C\varepsilon^{-2\delta}$.

To estimate the nonlinearity $\widetilde{F}(u) = 2u - u^3$ in the norm of X involving the weight ρ we need the following result.

Proposition 3.4
For all $u = u_{12}+u_3 \in X = X_{12}\oplus X_3$ we have the estimates

$$\begin{aligned} \|P_{12}(u_{12}+u_3)^3\| &\leq \mathrm{Lip}_{\mathrm{GL}}\|u_{12}\|^3 \\ &\quad + C(\varepsilon^{\eta-\delta}\|u_{12}\|^2\|u_3\| + \varepsilon^{2\eta-\delta-1/2}\|u_{12}\|\|u_3\|^2 + \varepsilon^{3\eta-\delta-1}\|u_3\|^3), \\ \|P_3(u_{12}+u_3)^3\| &\leq C(\varepsilon^{-\eta}\|u_{12}\|^3 + \|u_{12}\|^2\|u_3\| + \varepsilon^{\eta-1/2}\|u_{12}\|\|u_3\|^2 + \varepsilon^{2\eta-1}\|u_3\|^3), \end{aligned} \tag{3.13}$$

where C does not depend on ε, δ, and η.

Proof: These results make use of the particular mode interactions in the cubic nonlinearity. We employ the weighted estimates for convolutions as derived in the appendix.

To this end set $\rho_n^{(12)} = \rho_n$ and $\rho_n^{(3)} = \infty$ for $n \in I_{12}$ and $\rho_n^{(3)} = \rho_n$ and $\rho_n^{(12)} = \infty$ else. Using the convolution of weights as defined in (A.1) we have, according to Corollary A.2, $\|P_3[u_{12}u_3^2]\| \leq K\|u_{12}\|\|u_3\|^2$ if $\rho_n \leq K^2(\rho^{(12)} * \rho^{(3)} * \rho^{(3)})_n$ for all $n \notin I_{12}$. For all the other terms similar formulae hold.

Clearly, it is sufficient to prove the result for $\eta = 0$ then the dependence on η in (3.13) follows by a simple scaling argument. Let $\chi^{(12)}(t) = 1+(2\pi t/\ell)^2$ for $|t| \leq \varepsilon^{-\delta}$ and ∞ else. Then,

$$\begin{aligned}
&(\rho^{(12)} * \rho^{(12)})_n \geq C \min\{\, \chi^{(12)}(n/2-k/\varepsilon) \;:\; k=-1,0,1 \,\}, \\
&\tau_n^{[3]} \stackrel{\text{def}}{=} (\rho^{(12)} * \rho^{(12)} * \rho^{(12)})_n \geq C \min\{\, \chi^{(12)}(n/3-k/\varepsilon) \;:\; k=-1,-1/3,1/3,1 \,\}.
\end{aligned} \tag{3.14}$$

Similarly we define $\chi^{(3)}(t) = 1+\varepsilon^2 t^2$, then with $\eta = 0$

$$(\rho^{(3)} * \rho^{(3)})_n \geq C\varepsilon^{-1}\chi^{(3)}(n), \quad \tau_n^{[0]} \stackrel{\text{def}}{=} (\rho^{(3)} * \rho^{(3)} * \rho^{(3)})_n \geq C\varepsilon^{-2}\chi^{(3)}(n). \tag{3.15}$$

These estimates are a generalized and more abstract formulation of Lemma 3.4 in [Mi94]. Additionally we have

$$\tau_n^{[2]} \stackrel{\text{def}}{=} (\rho^{(12)} * \rho^{(12)} * \rho^{(3)})_n \geq C\chi^{(3)}(n). \quad \tau_n^{[1]} \stackrel{\text{def}}{=} (\rho^{(12)} * \rho^{(3)} * \rho^{(3)})_n \geq C\varepsilon^{-1}\chi^{(3)}(n). \tag{3.16}$$

This follows easily from the convolution formula (A.1) since $\rho^{(12)}$ and $\rho^{(12)} * \rho^{(12)}$ are finite on relatively small intervals where $\rho^{(3)}$ is approximately constant. Moreover, $\sum_n 1/\rho_n^{(12)}$ and $\sum_n 1/(\rho^{(12)} * \rho^{(12)})_n$ are bounded by a constant independent of ε and δ.

Using (3.10) and the norm identity we have $\|P_{12}(u_{12})^3\|_X \leq C_1 \|u_{12}\|_X^3$ with $C_1 = \mathrm{Lip}_{\mathrm{GL}}$. With (3.14) we find $\|P_3[(u_{12})^3]\| \leq C_2 \|u_{12}\|^3$ where $C_2^2 = \sup\{\, \tau_n^{[3]}/\rho_n^{(3)} \;:\; n \in I_3 \,\} = C\varepsilon^{-2\delta}$. The remaining six estimates for the case $\eta = 0$ follow easily from $\tau_n^{[k]} \geq C\varepsilon^{k-2}(1+\varepsilon^2 n^2)$. QED

3.3 The inertial manifold M_ε

Now we can consider the nonlinearity $\widetilde{F}$ on the absorbing ball $\mathcal{B} = \{\, u_{12}+u_3 \in X \;:\; \|u_{12}\| \leq C_a,\ \|u_3\| \leq C_\beta \varepsilon^{2\delta-\eta} \,\}$ with C_a from (3.12) and C_β from Proposition 3.3. Now, define as in Section 2

$$f_{j,k} = \sup\{\, \|D_k \widetilde{F}_j(u)\|_{\mathcal{L}(X_k,X_j)} \;:\; u \in \mathcal{B} \,\}, f_{j,\infty} = \sup\{\, \|\widetilde{F}_j(u)\|_{X_j} \;:\; u \in \mathcal{B} \,\},$$

then, under the assumption $\delta \geq 1/4$ we find $f_{3,\infty} = C\varepsilon^{-\eta}$ and

$$\begin{pmatrix} f_{1,1} & f_{1,2} & f_{1,3} \\ f_{2,1} & f_{2,2} & f_{2,3} \\ f_{3,1} & f_{3,2} & f_{3,3} \end{pmatrix} = \begin{pmatrix} 2+\mathrm{Lip}_{\mathrm{GL}}+C\varepsilon^\delta & \mathrm{Lip}_{\mathrm{GL}}+C\varepsilon^\delta & C\varepsilon^{\eta-\delta} \\ \mathrm{Lip}_{\mathrm{GL}}+C\varepsilon^\delta & 2+\mathrm{Lip}_{\mathrm{GL}}+C\varepsilon^\delta & C\varepsilon^{\eta-\delta} \\ C\varepsilon^{-\eta} & C\varepsilon^{-\eta} & C \end{pmatrix}.$$

As in Lemma 3.1 of [Mi94] we find a modification $F : X \to X$ of $\widetilde{F} : X \to X$ which has the same bounds $f_{j,k}$ and $f_{j,\infty}$. Using the fact that $\lambda_3 \geq C\varepsilon^{-2\delta}$ while $\lambda_2 - \lambda_1$ is independent of ε but growing with n_{GL}, it is immediate that the contraction condition (2.4) in Theorem 2.1 is satisfied for all $\varepsilon \in (0,\varepsilon_0)$ if $2\delta > \eta > \delta > 1/2$. This gives the following result.

Theorem 3.5
For each $\ell \geq 1$ there exists an $\varepsilon_0 > 0$ such that the Swift–Hohenberg equation (3.2) has for each $\varepsilon \in (0,\varepsilon_0)$ an inertial manifold M_ε which is a Lipschitz graph over the subspace $X_1 \subset X$, whose dimension is bounded above by $C\ell^3$ where C is independent of ℓ and ε.

4 Swift–Hohenberg versus Ginzburg–Landau

We now compare the two inertial manifolds obtained in the previous section. To this end recall that keeping ℓ fixed we define for a given $\varepsilon \in (0, \varepsilon_0)$ the numbers $q \in \mathbb{N}$ and $\gamma \in (-\pi, \pi]$ via $\ell = \varepsilon(2\pi q - \gamma)$. Our aim is to compare the flow on the inertial manifold $M_{\ell/(2\pi q-\gamma)}$ with that on N_γ when q is very large.

The application of the comparison theorem is now rather easy by using the operators T_k as defined in (3.9). We first derive the estimates on the linear parts.

Lemma 4.1
For $M_k(t) = T_k \mathrm{e}^{tK_k} - \mathrm{e}^{tL_k} T_k$ we have the estimates

$$\|M_1(t)\| \leq C\varepsilon \mathrm{e}^{-t(\lambda_1 - C\varepsilon)} \text{ for } t \leq 0, \qquad \int_0^\infty \mathrm{e}^{\alpha t} \|M_2(t)\| \mathrm{d}t \leq C\varepsilon.$$

Proof: Since K_k and L_k are symmetric operators and T_k transforms the eigenfunctions of K_k into those of L_k we obtain the identity

$$\|M_k(t)\|_{\mathcal{L}(X_k, Y_k)} = \max\{\, |\mathrm{e}^{t\kappa_{(q+m)}} - \mathrm{e}^{t\lambda_{(m)}}| \;:\; m \in J_k \,\}.$$

Together with $|\mathrm{e}^\alpha - \mathrm{e}^\beta| \leq |\alpha - \beta| \mathrm{e}^{\max\{\alpha, \beta\}}$ and $|\lambda_{(m)} - \kappa_{(q+m)}| \leq C\varepsilon m(1+m^2)$ we find the given estimate on M_1 since $J_1 \subset [-n_{\mathrm{GL}}-1, n_{\mathrm{GL}}+1]$ with n_{GL} independent of ε.

For $m \in J_2$ we have $n_{\mathrm{GL}} \leq |m| \leq \varepsilon^{-\delta}$ and hence

$$\|M_2(t)\| \leq C \max\{\, \varepsilon m^3 t \mathrm{e}^{t \max\{\kappa_{(q+m)}, \lambda_{(m)}\}} \;:\; m \in J_2 \,\} \leq C \begin{cases} \varepsilon t \mathrm{e}^{-t(\lambda_2 - C\varepsilon)} & \text{for } t \geq 1 \\ \varepsilon t^{-1/2} & \text{for } t \in (0,1) \end{cases},$$

where $\max\{\kappa_{(q+m)}, \lambda_{(m)}\} \leq -Cm^2$ was used. Integration over $t \in (0, \infty)$ gives the desired result. QED

Thus for the error quantities μ_k and ρ_k defined in (2.6) we obtain

$$\mu_0, \mu_1, \mu_2 \leq C\varepsilon.$$

In order to have estimates for the nonlinearities we need to be more careful. For the construction of N_γ and M_ε we had to modify the original nonlinearities $\widetilde{F}(u) = 2u - u^3$ and $\widetilde{G}(v) = 2v - |v|^2 v$ to obtain functions $F : X \to X$ and $G : Y \to Y$ such that they coincide with the original nonlinearities inside the absorbing ball and still satisfy the same Lipschitz and supremum bounds. In order to do the comparison we now choose the modified function F according to the modified G such that the relation (3.10) remains true:

$$G(v) = \widetilde{G}(\theta(\|v_{12}\|/C_1)v_{12} + \theta(\|v_3\|/C_2)v_3), \quad F(u) = \widetilde{F}(\theta(\|u_{12}\|/C_1)u_{12} + \theta(\|u_3\|/C_3)u_3),$$

where it is important to have the same constant $C_1 = \widehat{C}\sqrt{\ell}$ in the function θ for v_{12} and u_{12}. Of course, here $\widehat{C}$ has to be taken as the maximum of C_b and C_a from (3.5) and (3.12), respectively.

Lemma 4.2
We have the estimates $\rho_1, \rho_2 \leq C\varepsilon^{\delta}$.

Proof: The above definition of G and F guarantees $T_{12}F_{12}(u_{12}) = G_{12}(T_{12}u_{12})$ which derives from (3.10) and the fact $\|T_{12}u_{12}\|_{Y_{12}} = \|u_{12}\|_{X_{12}}$. Thus, the quantity σ_k in (2.7) vanishes, and with $\|T_k\| = 1$ we are left with $\rho_k \leq \frac{f_{k,3}f_{3,\infty}}{\kappa_3-\alpha} + \frac{g_{k,3}g_{3,\infty}}{\lambda_3-\alpha}$. This gives the desired result since $\lambda_3-\alpha, \kappa_3-\alpha \geq \varepsilon^{-2\delta}$, $g_{1,\infty}, g_{k,3} \leq C$, $f_{k,3} \leq C\varepsilon^{\eta-\delta}$, and $f_{3,\infty} \leq C\varepsilon^{-\eta}$. QED

These lemmas together with $g_{1,2} \leq C$ and $\|T_1K_1-L_1T_1\| \leq C\varepsilon$ can be inserted into the estimate of Theorem 2.3 and we arrive at the following comparison result for the reduced vector field $F_{\rm red}(\varepsilon,\cdot)$ on M_ε and $G_{\rm red}(\gamma,\cdot)$ on N_γ. For clarity of the statement we formulate the result not using $\varepsilon \in (0,\varepsilon_0)$ but $\varepsilon = \ell/(2\pi q-\gamma)$ with $\gamma \in (-\pi,\pi]$ and $q > q_0$.

Theorem 4.3
Let $\ell \geq 1$ *and* $\delta \in (1/2, 1)$ *be given. Then, there exist* $n_{\rm GL} \in \mathbb{N}$ *with* $n_{\rm GL} \leq C\ell^3$ *and* $q_0 > 0$ *such that for all* $\gamma \in (-\pi,\pi]$ *the following holds:*

The Ginzburg–Landau equation $(3.4)_\gamma$ *has an inertial manifold* N_γ *with dimension less than* $4n_{\rm GL}+4$ *and a reduced vector field* $G_{\rm red}(\gamma,\cdot)$*. Moreover, for all* $q > q_0$ *the Swift–Hohenberg equation* $(3.2)_\varepsilon$ *with* $\varepsilon = \ell/(2\pi q-\gamma)$, $q \in \mathbb{N}$ *has an inertial manifold* M_ε *of the same dimension as* N_γ *and the reduced vector field* $F_{\rm red}(\varepsilon,\cdot)$ *satisfies*

$$\|F_{\rm red}(u_1) - T_1^{-1}G_{\rm red}(\gamma, T_1u_1)\| \leq C(\varepsilon\|u_1\| + \varepsilon^{\delta}). \tag{4.1}$$

Here we have chosen an approach in defining Y_{12} using the modes ϕ_n associated with $|n| \leq \varepsilon^{-\delta}$. In [Zie97] Y_{12} is defined via $|n| \leq c_0/\varepsilon$ for small but fixed $c_0 > 0$. As a result it is obtained that (4.1) is valid also for $\delta = 1$.

5 Discussion

One important result of this work is that the vector field $F_{\rm red}(\varepsilon,\cdot)$ on M_ε does not converge for $\varepsilon \to 0$ but only suitable subsequences lead to convergence against the vector field $G_{\rm red}(\gamma,\cdot)$ on N_γ. This fact follows since the flows of the vector fields $G_{\rm red}(\gamma,\cdot)$ are genuinely different. This is seen most easily by considering the steady states which are of the form

$$v(x) = r_k e^{i\omega_k x} \quad \text{where } \omega_k = (2\pi k+\gamma)/\ell \text{ and } r_k = \sqrt{(1-4\omega_k^2)/3}\,. \tag{5.1}$$

It is simple to check that for all $\ell \geq 1$ the number of such solutions changes with $\gamma \in \mathbb{S}^1$.

We note that the Swift–Hohenberg equation as well as the Ginzburg–Landau equations are gradient systems, namely with respect to the functionals

$$U(u) = \int_0^\ell \Big[\tfrac{1}{2\varepsilon^2}(u+\varepsilon^2\partial_x^2 u)^2 - \tfrac12 u^2 + \tfrac14 u^4\Big]\,dx, \quad V(v) = \int_0^\ell \Big[2|\partial_x v|^2 - \tfrac12|v|^2 + \tfrac34|v|^4\Big]\,dx.$$

Thus, the flows on the inertial manifolds M_ε and N_γ are gradient–like systems in the sense that they have a strictly decreasing Liapunov function. In particular this implies that all solutions in the global attractors have the α– and ω–limit sets in the set of equilibria.

Note however, that there is a major difference between the flow of (SH) and (GL). Both systems are translationally invariant but only (GL) has an additional phase invariance $v \mapsto e^{i\alpha}v$. In fact the set of equilibria in (GL) can be studied explicitly by reduction to a second order ODE. One finds that there the rotating waves as given in (5.1), they form a circle of equilibria since the action of space translation and phase multiplication coincides. Moreover, there are steady states for which $|v(\cdot)|$ is periodic but nonconstant. Such solutions occur always in a two-dimensional continuum of equilibria which has the topological type of a torus, see e.g. [BrR94, DoGJ95].

If we now consider generic ℓ and γ then the set of equilibria for $G_{\rm red}(\gamma,\cdot)$ consists of finitely many circles and tori as described above. Moreover, we can assume that all these manifolds of equilibria are normally hyperbolic. If we now consider the flow of $F_{\rm red}(\varepsilon,\cdot)$ with $\ell/\varepsilon = \gamma \bmod 2\pi$ then the flow is a small perturbation of $G_{\rm red}(\gamma,\cdot)$ and we conclude that all the circles and tori persist as normally hyperbolic invariant manifolds. The flow on the circles will still be trivial as the translation symmetry of (SH) forces equilibria to occur in circles. However, the flow on the tori will no longer be trivial: generically on expects that only a finite number of circles of equilibria will persist which are connected by heteroclinic orbits inside the torus. Thus, we see that the inertial manifold provides a tool to study the breakup of the normal form symmetry which is present in (GL).

A Convolutions in weighted spaces

We consider weight functions $\rho,\sigma : \mathbb{Z} \to (0,\infty]$ with $\rho_k, \sigma_k \geq \delta(1+k^2)$ for some $\delta > 0$. Associated to a weight ρ is the weighted norm $\|\cdot\|_\rho$ on via $\|u\| = \big(\sum_{n\in\mathbb{Z}} \rho_n u_n^2\big)^{1/2}$. Our aim is to estimate convolutions $w = u * v$ given via $w_n = \sum_{m\in\mathbb{Z}} u_m v_{n-m}$. To this aim we also define an operation $\rho \star \sigma$ between weight functions via

$$(\rho \star \sigma)_n = \left(\sum_{m\in\mathbb{Z}} \frac{1}{\rho_m} \frac{1}{\sigma_{n-m}} \right)^{-1}. \tag{A.1}$$

Proposition A.1
Let ρ and σ be weights as above, then $\|u * v\|_{\rho\star\sigma} \leq \|u\|_\rho \|v\|_\sigma$ for all u, v.

Proof: Let $z = u*v$, $\tau = \rho\star\sigma$, and $a_{n,m} = (\tau_n/(\rho_m\sigma_{n-m}))^{1/2}$. This implies $\sum_{m\in\mathbb{Z}} a_{n,m}^2 = 1$ for all $n \in \mathbb{Z}$. Then,

$$\begin{array}{rcl} \|z\|_\tau^2 & = & \sum_{n\in\mathbb{Z}} \tau_n z_n \big[\sum_{m\in\mathbb{Z}} u_m v_{n-m} \big] \\ & \leq & \sum_{n,m\in\mathbb{Z}} a_{n,m} \sqrt{\tau_n} |z_n| \, \sqrt{\rho_m} |u_m| \, \sqrt{\sigma_{n-m}} |v_{n-m}| \\ & \leq & (\text{Hölder}) \left(\sum_{n,m\in\mathbb{Z}} a_{n,m}^2 \tau_n z_n^2 \right)^{1/2} \left(\sum_{n,m\in\mathbb{Z}} \rho_m u_m^2 \, \sigma_{n-m} v_{n-m}^2 \right)^{1/2} \\ & \leq & \left(\sum_n \big[\sum_m a_{n,m}^2 \big] \tau_n z_n^2 \right)^{1/2} \left(\sum_{m\in\mathbb{Z}} \big[\sum_n \sigma_{n-m} v_{n-m}^2 \big] \rho_m u_m^2 \right)^{1/2} \;=\; \|z\|_\tau \|u\|_\rho \|v\|_\sigma. \end{array}$$

Dividing by $\|z\|_\tau$ gives the desired result. QED

Corollary A.2
Assume $\kappa_n \leq K^2(\rho \star \sigma \star \tau)_n$ *then* $\|u * v * w\|_\kappa \leq K\|u\|_\rho\|v\|_\sigma\|w\|_\tau$.

The proof of this result follows from the monotonicity of the norms in the weight and by applying the proposition twice.

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