Bifurcation of homoclinic orbits to a saddle–focus in reversible systems with SO(2)–symmetry*

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Abstract

Bifurcation of homoclinic orbits of reversible SO(2)–invariant vector fields in $\mathbb{R}^4$ in a vicinity of the primary homoclinic orbit H_0 is studied. Under transversality conditions on H_0 with respect to the bifurcation parameter the existence of n-homoclinic solutions is demonstrated together with the study of their parity with respect to involution. The existence of cascades of $2^k 3^m$-homoclinic solutions is established by showing their transversality and then using induction. The method relies on the construction of an SO(2)-invariant Poincaré map which, after factorization, consists of a composition of a logarithmic twist map and a smooth global map. As an application we treat the steady complex Ginzburg–Landau equation for which a primary homoclinic solution is known explicitly.

1 Introduction

We study the existence of homoclinic orbits as well as their bifurcation in reversible ordinary differential equations in $\mathbb{R}^4$ with SO(2)–symmetry. The analysis is motivated by the steady complex Ginzburg–Landau equation (cGL)

$$0 = \ddot{A} + aA + b|A|^2 A, \quad a = -\lambda^2,\ b = (1+\mathrm{i}\varepsilon)\lambda(\lambda+1), \tag{1.1}$$

which, for $\varepsilon = 0$, is known to have the explicit homoclinic solution $A(x) = (\cosh x)^{-\lambda}$. Throughout we will use the normalization $\lambda = 1+\mathrm{i}\omega$ with $\omega > 0$, such that the origin is a reversible saddle–focus. Our aim is to study the existence of homoclinic orbits for all small ε. These homoclinic orbits will then be n–homoclinic with respect to the primary homoclinic orbit H_0, which means that they wind around n times close to H_0 before they return to the origin.

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The important role of cGL for bifurcation problems on spatially unbounded domains (see [Doe93, AfM95]) led to several previous studies of its homoclinic orbits, see [Hol86, Doe96, KaMP96], where however only the case of small ω was treated. In the present work we develop a general theory for reversible SO(2)-invariant vector fields and thereby improve the known results for cGL, in particular by generalizing them to the case of large ω. This is important since the applications in $3D$ Poiseuille flow (see [AfM95, AfM98]) need ω in the range between 8 and 18. Numerical work for cGL can be found in [Lan87] where n-homoclinic solutions for several n as well as the sequences of periodic solutions close to n-homoclinic solutions were detected. Homoclinic bifurcations in generic four–dimensional reversible systems with the same linear part of saddle–focus type (hence without SO(2)–symmetry or Hamiltonian structure) are considered in [H97, Cha97]. In such systems one typically has homoclinic bifurcations only on one side of the parameter value where the primary homoclinic orbit exists.

Our methods are closely related to [MiHO92] where homoclinic orbits to a saddle–center (fixed point with eigenvalues ± 1, $\pm \mathrm{i}\omega$) in a reversible Hamiltonian system were studied. There the conserved Hamiltonian was used to reduce the dimension of Poincaré sections from 3 to 2 whereas in the present setting the SO(2)–invariance gives the desired reduction. However, the reduced Poincaré map has exactly the same features.

Our main starting point is a fixed point of saddle–focus type which is taken to be the origin. Because of the reversibility the eigenvalues of the linearization form the quadruple $\lambda, -\lambda, \overline{\lambda}, -\overline{\lambda}$ and by choosing appropriate coordinates the system can be written in the form

$$\dot{y}_1 = \lambda y_1 + F_1(\varepsilon, Y, \overline{Y}), \quad \dot{y}_2 = -\lambda y_2 + F_2(\varepsilon, Y, \overline{Y}), \tag{1.2}$$

where $Y = (y_1, y_2) \in \mathbb{C}^2$, and the equations for the complex conjugates $\overline{Y}$ are suppressed. The nonlinearities F_j are assumed to be real analytic with $F_j(\varepsilon, Y, \overline{Y}) = \mathcal{O}(|Y|^2)$. (However, vector fields with finite differentiability can be handled similarly, see [AfM98].) The SO(2)–action T_α and the reverser R (involution with $R^2 = I$) are given via

$$T_\alpha(y_1, y_2) = (\mathrm{e}^{\mathrm{i}\alpha} y_1, \mathrm{e}^{\mathrm{i}\alpha} y_2), \qquad R(y_1, y_2) = (y_2, y_1).$$

For $\varepsilon = 0$ we assume the existence of the primary homoclinic H_0. Our method relies in constructing a Poincaré map for this orbit which will allow us to find n–homoclinic orbits for small $\varepsilon \neq 0$. From the SO(2)–invariance it is clear that a homoclinic orbit exists if and only if the two–dimensional stable and unstable manifolds coincide completely and are given by rotations $T_\alpha H_\varepsilon$ of a homoclinic orbit H_ε. Moreover, from the reversibility follows that there is an $\xi \in \mathbb{R}$ such that

$$H_\varepsilon(x) = \nu_\varepsilon R\, H_\varepsilon(\xi - x) \quad \text{ for all } x \in \mathbb{R} \text{ with } \nu_\varepsilon \in \{-1, +1\}.$$

We call $\nu_\varepsilon = \mathrm{par}\,(H_\varepsilon)$ the parity of the homoclinic orbit H_ε.

The Poincaré map P is a composition of a local map $\widetilde{S}$ of Shilnikov type (see [Sh67]) and a global map $\widetilde{G}$ along the primary homoclinic H_0. It is possible to factor out the SO(2)–symmetry before these mappings are constructed as was done in [KaMP96] for the cGL. However, this leads to a three–dimensional system with a singularity at the fixed point which makes the analysis difficult. In contrast to this we first construct mappings and then factorize. We use the Poincaré section $K_{\rm in} \approx \{ (y_1, y_2) \in \mathbb{C} \,:\, |y_1| \le r, |y_2| = \delta \}$ with $\delta > 0$ and $K_{\rm out} = R\, K_{\rm in}$ which are sections at the local stable and unstable manifold of 0, respectively. Thus, the local map $\widetilde{S} : K_{\rm in} \to K_{\rm out}$, the global map $\widetilde{G} : K_{\rm out} \to K_{\rm in}$, and the Poincaré map $P = \widetilde{G} \circ \widetilde{S} : K_{\rm in} \to K_{\rm in}$ are obtained. In fact, it is more convenient to use $S = R \circ \widetilde{S}$ and $G = \widetilde{G} \circ R$, which both map $K_{\rm in}$ into itself, and take the form

$$S(\psi, w) = (\psi + \sigma(w), s(w)) \text{ and } G(\varepsilon, \psi, w) = (\psi + \gamma(\varepsilon, w), g(\varepsilon, w))$$

after choosing suitable coordinates $(\psi, w) \in \mathbb{S}^1 \times K_r$ with $(y_1, y_2) \approx (w\mathrm{e}^{\mathrm{i}\psi}, \delta \mathrm{e}^{\mathrm{i}\psi})$.

The local map S is a logarithmic twist map given exactly via $\sigma(w) = \arg w + \widehat{\Omega}(|w|)$, $s(w) = \mathrm{e}^{-2\mathrm{i}\widehat{\Omega}(|w|)}\overline{w}$ and $\widehat{\Omega}(|w|) = \omega \log(\delta/|w|)$. To show this we use the local normal form theory in [Brj71] for analytic vector fields to simplify the flow of (1.2) near $(y_1, y_2) = 0$. The case of vector fields with finite differentiability can be treated similarly with the results in [Sam82]. Note that the reverser R is involved in the definition of S and G such that $P = \widetilde{G} \circ \widetilde{S} = G \circ S$ and the reversibility amounts into the fact that G and S are involutions, i.e., $G \circ G = S \circ S = \mathrm{id}$.

The global mapping G is smooth and satisfies $G(0, \psi, 0) = (\psi + \arg \nu_0, 0)$ with $\nu_0 = \mathrm{par}\,(H_0)$. Together with $G \circ G = \mathrm{id}$ it follows that g has the expansion

$$g(\varepsilon, w) = \varepsilon a + \Gamma w + \mathcal{O}(\varepsilon^2 + |w|^2) \text{ with } \Gamma a + a = 0,\ \Gamma^2 = I,\ \det \Gamma = -1. \tag{1.3}$$

We say that the primary orbit H_0 is transversal (with respect to the perturbation in ε) if $a \neq 0$.

The set of parameters $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ decomposes into the disjoint union of E^n where E^n is the set of all ε such that (1.2) has an n–homoclinic orbit. The main results (see Section 3) can be summarized as follows:

If H_0 is transversal, then for sufficiently small $\varepsilon_0 > 0$ we have:
(1) For each $n \ge 2$ the set E^n is infinite and has 0 in its closure.
(2) Both, the set E^2 and E^3 consist of one monotone decreasing and one monotone increasing sequence converging to $\varepsilon = 0$. All the associated homoclinic orbits are again transversal.
(3) In each sequence in E^2 the parity of the associated homoclinic orbits alternates. All $\varepsilon \in E^3$ have the parity $\mathrm{par}\,(H_0)$.
(4) In $E^2 \cup E^3$ the elements of E^2 and E^3 separate each other.

An essential new feature of our work is the transversality of the 2- and 3-homoclinic orbits existing for $\varepsilon \in E^2 \cup E^3$. It is this result which allows us to use induction to produce n–homoclinic orbits which are transversal for any $n = 2^k 3^m$.

In Section 4 we present an analytical method which enables us to calculate the coefficients a and Γ in (1.3) explicitly from solving the variational equation for (1.2) around H_0. For cGL in the form (1.1) we are thus able to find a and Γ as a functions of the constant $\omega > 0$. Since $\Gamma(\omega)$ always has eigenvalues ± 1 with eigenvectors $\phi_\pm(\omega)$ with $|\phi_\pm(\omega)| = 1$ we have $a(\omega) = c(\omega)\phi_-(\omega)$ with $c : [0, \infty) \to \mathbb{R}$. The analysis in [KaMP96] shows $c(\omega) \neq 0$ for small $\omega \neq 0$. Since c is an analytic function we have transversality for all $\omega > 0$ except for an exceptional set $\{\omega_1, \omega_2, \ldots\}$ which may be finite or infinite, but it has no finite points of accumulation. Our numerical calculations give $\omega_1 \approx 8.032$ and $\omega_2 \approx 9.51$.

2 Construction of the Poincaré map

2.1 Setup of the problem

We consider homoclinic orbits in a real four-dimensional SO(2)–invariant reversible system depending on the small parameter ε. Without loss of generality we assume that the fixed point is the origin. The eigenvalue constellation of the linearization is a non-trivial quadruple $\lambda, -\lambda, \overline{\lambda}, -\overline{\lambda}$ with $\lambda = 1 + \mathrm{i}\omega$ where $\omega > 0$. After a linear coordinate transformation we may assume that the system is in the complex form

$$\dot{Y} = JY + F(\varepsilon, Y, \overline{Y}), \quad Y = (y_1, y_2) \in \mathbb{C}^2, \quad F = (F_1, F_2) \in \mathbb{C}^2 \tag{2.1}$$

with the diagonal linear part $J = \mathrm{diag}\,(\lambda_1, \lambda_2) = \mathrm{diag}\,(\lambda, -\lambda,)$ and $F_j(\varepsilon, Y, \overline{Y}) = \mathcal{O}(|Y|^2)$. Throughout we will assume that the vector field is (real) analytic in $(\varepsilon, Y, \overline{Y})$, but this is just for convenience since the case with finite differentiability can be treated similarly, see [AfM98].

The action of SO(2) and of the reverser is as follows:

$$T_\alpha(y_1, y_2) = (\mathrm{e}^{\mathrm{i}\alpha} y_1, \mathrm{e}^{\mathrm{i}\alpha} y_2), \qquad R(y_1, y_2) = (y_2, y_1).$$

Then, SO(2)–invariance of (2.1) means $F \circ T_\alpha = T_\alpha F$, and reversibility means $F \circ R = -RF$. (Of course the linear part JY has this symmetries as well.) In fact, having chosen the coordinates according to the eigenvalues as above the action is uniquely determined up to conjugacies under the additional assumption $T_\alpha R = R T_\alpha$. It is also interesting to note that there is no Hamiltonian system of the form (2.1) with the desired symmetries.

Since we want to switch between real and complex notation frequently we use the realification functor ${}_{\mathbb{R}}(\ )$ and the complexification functor ${}_{\mathbb{C}}(\ )$. For instance, ${}_{\mathbb{R}}(y_1, y_2) = (\mathrm{Re}\, y_1, \mathrm{Im}\, y_1, \mathrm{Re}\, y_2, \mathrm{Im}\, y_2)$ and ${}_{\mathbb{R}}\mathbb{C}^n = \mathbb{R}^{2n}$ denotes the realification of the complex space and ${}_{\mathbb{R}}B \in \mathbb{R}^{2n \times 2n}$ is the realification of $B \in \mathbb{C}^{n \times n}$. In the real coordinates $u = {}_{\mathbb{R}}(y_1, y_2)^T \in \mathbb{R}^4$ equation (2.1) takes the form

$$\dot{u} = f(\varepsilon, u), \quad u \in \mathbb{R}^4, \quad \varepsilon \in \mathbb{R}. \tag{2.2}$$

The induced actions on $\mathbb{R}^4$ are ${}_{\mathbb{R}}T_\alpha$ and ${}_{\mathbb{R}}R$.

Problem (2.1) is obviously reversible under the family $T_\alpha R$ of transformations. But only two of them are involutions, namely those with $\alpha = 0, \alpha = \pi$. Thus, problem (2.2) must be treated as a bireversible system with respect to R and $-R$. An important observation is that any orbit homoclinic to 0 is in fact reversible.

Lemma 2.1
Suppose that $H_\varepsilon(x)$ is a homoclinic orbit of $(2.1)_\varepsilon$ then there are unique $\nu_\varepsilon \in \{-1, 1\}$ and $\xi \in \mathbb{R}$ such that

$$H_\varepsilon(x) = \nu_\varepsilon R\, H_\varepsilon(\xi - x) \quad \text{for all } x \in \mathbb{R}.$$

The sign ν_ε is called the parity of H_ε and written as $\mathrm{par}\,(H_\varepsilon) = \nu_\varepsilon$.

Proof: Let $W^s(0)$ and $W^u(0)$ be stable and unstable manifolds of 0 and $\dim W^s(0) = \dim W^u(0) = 2$. Let $y_s(x)$ be any solution in $W^s(0)$, then by SO(2)–invariance we have $y(x) = T_\alpha y_s(x + x_s)$ for each solution y in $W^s(0)$. By reversibility, for a solution y in $W^u(0)$ we have $y(x) = T_\beta R y_s(x_u - x)$. Since the homoclinic orbit lies in $W^s(0) \cap W^u(0)$ we find

$$y_s(x) = T_{\beta-\alpha} R y_s(\xi - x) \quad \text{for all } x \in \mathbb{R}$$

where $\xi = x_s + x_u$. Evaluating at $x = \xi/2$ gives $(I - T_{\beta-\alpha}R) y_s(\xi/2) = 0$. Since $y_s(\xi/2) \neq 0$ and since $T_{\beta-\alpha}R$ has the eigenvalues $\mathrm{i}\mathrm{e}^{\pm \mathrm{i}(\beta-\alpha)}$ and their complex conjugates it follows that $2(\alpha - \beta) = 0 \in \mathbb{S}^1$. If $\alpha - \beta = 0 \in \mathbb{S}^1$ we let $\nu = 1$ and if $\alpha - \beta = \pi \in \mathbb{S}^1$ we let $\nu = -1$, then we have $T_{\beta-\alpha} = \nu I$; and thus $H(x) = \nu R\, H(\xi - x)$ as desired. QED

Simple equations of the above type with an explicit homoclinic orbit can be constructed as follows. Let $r(x)$ be a real homoclinic orbit to the two-dimensional system

$$\ddot{r} - r[G(r^2) + r^2 G'(r^2)] = 0, \quad (\text{such that } (\dot{r})^2 = r^2 G(r^2)\ )$$

where G is a smooth function with $G(0) = 1$. Then, the complex function $A(x) = r(x)^\lambda = r(x)\mathrm{e}^{\mathrm{i}\omega \log r(x)}$ solves the complex second order equation

$$\ddot{A} - A[\lambda^2 G(|A|^2) + \lambda |A|^2 G'(|A|^2)] = 0.$$

With $G(\rho) = 1 - \rho$ we obtain the important example of the time-independent complex Ginzburg-Landau equation

$$\ddot{A} - \lambda^2 A + (1 + \mathrm{i}\varepsilon)\lambda(1 + \lambda) A|A|^2 = 0, \quad \lambda = 1 + \mathrm{i}\omega,\ \omega > 0,\ A \in \mathbb{C}, \tag{2.3}$$

which has, for $\varepsilon = 0$, the explicit homoclinic solution $A_*(x) = (\cosh x)^{-\lambda}$. To our knowledge this observation appears first in [HoS72]. This example is worth keeping in mind throughout the paper.

These problems, treated as a complex system in coordinates $(A, \dot{A})$ are invariant with respect to $T_\alpha = \mathrm{diag}\,(\mathrm{e}^{\mathrm{i}\alpha}, \mathrm{e}^{\mathrm{i}\alpha})$ and reversible with the involution $R^A = \mathrm{diag}\,(1, -1)$. Transforming (2.3) to the eigenbasis of the linear part we find system (2.1) together with the specified symmetries. It is immediate that the homoclinic orbits constructed above have parity $\nu = +1$.

From now on we consider the case that (2.1) has a homoclinic orbit H_0 for $\varepsilon = 0$. Our goal is to study the existence of other homoclinic orbits for all small ε. By SO(2)-invariance and reversibility the existence of reversible orbits is a phenomenon of codimension 1, such that we expect a discrete set $E \subset (-\varepsilon_0, \varepsilon_0)$ for which homoclinic orbits exist. This point will become clearer through the following analysis.

The approach we take is that of constructing a Poincaré section and a suitable Poincaré return map which allows us to study the existence of homoclinic orbits which wind around the original homoclinic orbit n times. To this end we construct a local map which takes care of the passage of orbits near the fixed point $Y = 0$ and a global map which contains information about the flow near the homoclinic orbit. In [KaMP96] (2.3) was first reduced by SO(2)-symmetry to a real three-dimensional ODE and then Poincaré maps were constructed. To avoid the difficulties arising from the singularity at the origin, we first construct a Poincaré map and then study its symmetry properties with respect to SO(2).

2.2 The local map

Consider the analytical ODE

$$\dot{Z} = JZ + \Psi(\varepsilon, Z, \overline{Z}), \qquad \Psi = (\psi_1, \psi_2) \quad \text{with } Z = (z_1, z_2) \in \mathbb{C}^2. \tag{2.4}$$

It is said to be in normal form if

$$\psi_j(\varepsilon, Z, \overline{Z}) = z_j \sum_{\langle \Lambda, Q\rangle = 0} g_{jQ}(\varepsilon)(Z, \overline{Z})^Q, \quad \Lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (\lambda, -\lambda, \overline{\lambda}, -\overline{\lambda}) \tag{2.5}$$

where $Q = (q_1, \ldots, q_4) \in \mathbb{Z}^4$, $q_j \geq -1$, $q_k \geq 0$ for $k \neq j$, and $(Z, \overline{Z})^Q = z_1^{q_1} z_2^{q_2} \overline{z}_1^{q_3} \overline{z}_2^{q_4}$. Monomials $z_j(Z, \overline{Z})^Q$ with $\langle \Lambda, Q\rangle := \lambda_1 q_1 + \cdots + \lambda_4 q_4 = 0$ are called resonant monomials.

Lemma 2.2
There exist a neighborhood $U \subset \mathbb{C}^2$ of 0 and an analytical transformation

$$Z = Y + \Theta(\varepsilon, Y, \overline{Y}), \;\; \Theta(\varepsilon, Y, \overline{Y}) = (\theta_1(\varepsilon, Y, \overline{Y}), \theta_2(\varepsilon, Y, \overline{Y})), \quad \theta_j(\varepsilon, \cdot) : U \to \mathbb{C} \tag{2.6}$$

such that $\theta_j(\varepsilon, Y, \overline{Y})$ contains no resonant monomials and that (2.1) *is transformed into the normal form*

$$\dot{z}_1 = z_1[\lambda + \Phi(\varepsilon, |z_1 z_2|^2)], \quad \dot{z}_2 = -z_2[\lambda + \Phi(\varepsilon, |z_1 z_2|^2)], \tag{2.7}$$

with $\Phi(\varepsilon, 0) = 0$. Moreover, the transformation is SO(2)*- and R-equivariant.*

Proof: Observe that the problem (2.1) is analytical and the set of eigenvalues $\{\lambda_j\}, j = 1, \ldots, 4$ does not belong to the simple Poincaré domain, but to the more difficult Siegel domain (see [Arn83]) where analyticity of normal form transformations only holds if further strong conditions are satisfied [Brj71].

For this purpose we write (2.1) as 4 complex equations with $X = (Y, \overline{Y}) \in \mathbb{C}^4$ by adding the complex conjugate equations, i.e., $\dot{X} = \text{diag}\,(\Lambda)X + G(\varepsilon, X)$ where $G = (F_1, F_2, \overline{F}_1, \overline{F}_2)$. Of course, SO(2)-symmetry and reversibility are preserved.

We first derive the general form of the normal form under the given symmetries. Since $\overline{\lambda}/\lambda \notin \mathbb{R}$ the resonance condition $\lambda q_1 - \lambda q_2 + \overline{\lambda} q_3 - \overline{\lambda} q_4 = 0$ yields $q_1 = q_2$ and $q_3 = q_4$. From SO(2)-invariance we conclude $q_1 + q_2 - q_3 - q_4 = 0$ which implies $q_1 = q_2 = q_3 = q_4$. Since the normal form transformation (2.6) with no resonant terms preserves symmetries and reversibility (see [Brj71, Arn83, IoA92]) we arrive at

$$\dot{x}_j = x_j[\lambda_j + \Phi_j(\varepsilon, x_1 x_2 x_3 x_4)] \quad \text{for } j = 1, \ldots, 4. \tag{2.8}$$

Using $(x_3, x_4) = (\overline{x}_1, \overline{x}_2)$ and reversibility we find $(\Phi_2, \Phi_3, \Phi_4) = (-\Phi_1, \overline{\Phi}_1, -\overline{\Phi}_1)$, and thus (2.7) is established.

For the analyticity of the normal form transformation θ we use Theorem 2 of [Brj71]. There are no small denominators in this problem; Siegel's and hence Brjuno's small denominator condition is obvious, as for all $Q \in \mathbb{Z}^4$ with $\langle \Lambda, Q\rangle \neq 0$ we have

$$|\langle \Lambda, Q\rangle| = [(q_1-q_2+q_3-q_4)^2 + \omega^2(q_1-q_2-q_3+q_4)^2]^{1/2} \geq \min\{1, \omega\}.$$

It is left to check the condition A_2 from [Brj71]. It reads as follows: *There exist two power series $\zeta(\varepsilon, X), \Xi(\varepsilon, X)$ such that in the normal form* (2.5)

$$\Phi_j(\varepsilon, X) = \lambda_j \zeta(\varepsilon, X) + \overline{\lambda}_j \Xi(\varepsilon, X), \quad j = 1, \dots, 4.$$

For our special normal form (2.8) this condition is fulfilled with real $\zeta = \frac{\lambda\Phi - \overline{\lambda\Phi}}{\lambda^2 - \overline{\lambda}^2}$ and $\Xi = \frac{\overline{\lambda}\Phi - \lambda\overline{\Phi}}{-\lambda^2 + \overline{\lambda}^2}$. This completes the proof. QED

We describe the local mapping in the normal form coordinates $(z_1, z_2) \in \mathbb{C}^2$. For a homoclinic solution $H_\varepsilon(x)$ of (2.2) in Y-coordinates we know that $H_\varepsilon(x) \in U$ for $|x| \geq x_0 \gg 1$. Hence, in Z-coordinates we have $\widetilde{H}_\varepsilon(x) = H_\varepsilon(x) + \Theta(\varepsilon, H_\varepsilon(x))$ for $|x| \geq x_0$.

$$\begin{array}{ll} Z(x) = \widetilde{H}_\varepsilon(x) = (c_+ \mathrm{e}^{-\lambda x}, 0) & \text{for } x \geq x_0, \\ Z(x) = \widetilde{H}_\varepsilon(x) = (0, \mathrm{par}\,(H_\varepsilon) c_+ \mathrm{e}^{\lambda(\xi + x)}) & \text{for } x \leq -x_0, \end{array} \tag{2.9}$$

with $\xi \in \mathbb{R}$ and $c_+ \in \mathbb{C}$. To see this, we use the fact that the stable manifold of (2.7) is given by $z_1 \equiv 0$ while the unstable manifold is given by $z_2 \equiv 0$. The result on the parity is a consequence of Lemma 2.1.

We construct a local map $\widetilde{S}$ between the hypersurfaces

$$K_{\text{in}} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| \leq r, |z_2| = \delta\} \text{ and } K_{\text{out}} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| = \delta, |z_2| \leq r\},$$

where δ and r are taken small enough. Clearly, K_{out} is transversal to the unstable manifold of $Z = 0$ and K_{in} is transversal to the stable manifold. The local (Shilnikov) map induced by the flow of (2.7) is given by

$$\widetilde{S}(\varepsilon, \cdot) : \left\{ \begin{array}{ccc} K_{\text{in}} \setminus \{(0, z_2) : |z_2| = \delta\} & \to & K_{\text{out}}, \\ (z_1, \delta \mathrm{e}^{\mathrm{i}\psi_2}) & \mapsto & (\delta \frac{z_1}{|z_1|} \mathrm{e}^{\mathrm{i}\Omega(\varepsilon, |z_1|)}, \, \mathrm{e}^{\mathrm{i}(\psi_2 - \Omega(\varepsilon, |z_1|))} |z_1|), \end{array} \right.$$

where

$$\Omega(\varepsilon, |z_1|) = \frac{\omega + \mathrm{Im}\,\Phi(\varepsilon, \delta^2 |z_1|^2)}{1 + \mathrm{Re}\,\Phi(\varepsilon, \delta^2 |z_1|^2)} \log \frac{\delta}{|z_1|}. \tag{2.10}$$

This special form of $\widetilde{S}$ follows easily from (2.7) since $|z_1 z_2|^2$ is constant along solutions.

The reversibility of the problem is such that R maps K_{in} into K_{out} as well as K_{out} into K_{in}. Moreover the mapping $\widetilde{S}(\varepsilon, \cdot)$ satisfies the conditions

$$\widetilde{S}(\varepsilon, \cdot)^{-1} = R\widetilde{S}(\varepsilon, R\,\cdot) \text{ and } \widetilde{S}(\varepsilon, T_\alpha \cdot) = T_\alpha \widetilde{S}(\varepsilon, \cdot), \tag{2.11}$$

for $\alpha \in \mathbb{S}^1 = \mathbb{R}/_{2\pi\mathbb{Z}}$.

2.3 The global and the Poincaré map

The global map, which will take care of the flow along the homoclinic excursion, has to be constructed in the Y-coordinates since the validity of the Z-coordinates is necessarily restricted to a small neighborhood of $Z=0$. Thus, denote the normal-form transformation of Lemma 2.2 for $(z_1,z_2)\in\mathbb{C}^2$ by $Z=M(\varepsilon,Y,\overline{Y})$ and define the Poincaré sections

$$\widehat{K}_{\mathrm{in}}=M^{-1}(\varepsilon,K_{\mathrm{in}})\subset\mathbb{C}^2 \text{ and } \widehat{K}_{\mathrm{out}}=M^{-1}(\varepsilon,K_{\mathrm{out}})\subset\mathbb{C}^2$$

in the Y-coordinates. Since the homoclinic orbit $H_0(x)$ of (2.2) exists for $\varepsilon=0$ we conclude the existence of a smooth Poincaré mapping $\widehat{G}(\varepsilon,\cdot):\widehat{K}_{\mathrm{out}}\mapsto\widehat{K}_{\mathrm{in}}$, where as usual, $\widehat{G}$ may only be defined in a smaller set.

In order to be able to combine this map with the local map $\widetilde{S}$ we transform it back to Z-coordinates and let

$$\widetilde{G}(\varepsilon,\cdot):\left\{\begin{array}{ccc} K_{\mathrm{out}} & \to & K_{\mathrm{in}}, \\ (\delta\mathrm{e}^{\mathrm{i}\psi_1},z_2) & \mapsto & M(\varepsilon,\widehat{G}(\varepsilon,M^{-1}(\varepsilon,(\delta\mathrm{e}^{\mathrm{i}\psi_1},z_2)))). \end{array}\right.$$

This map preserves the SO(2)-symmetry and the reversibility as follows:

$$\widetilde{G}^{-1}(\varepsilon,\cdot)=R\widetilde{G}(\varepsilon,R\,\cdot),\quad \widetilde{G}(\varepsilon,T_\alpha\cdot)=T_\alpha\widetilde{G}(\varepsilon,\cdot), \tag{2.12}$$

for $\alpha\in\mathbb{S}^1$. The assumption that (2.2) has a homoclinic orbit H_0 for $\varepsilon=0$ now means that

$$\widetilde{G}(0,(\delta\mathrm{e}^{\mathrm{i}\psi_1},0))=(0,(\nu_0\delta\mathrm{e}^{\mathrm{i}\psi_1},0))\quad\text{with } \nu_0=\mathrm{par}\,(H_0). \tag{2.13}$$

This follows easily from (2.9) by shifting $x\to x-\xi/2$.

We are now in the position to define the total Poincaré map

$$P(\varepsilon,\cdot)=\widetilde{G}(\varepsilon,\cdot)\circ\widetilde{S}(\varepsilon,\cdot):K_{\mathrm{in}}\to K_{\mathrm{in}}.$$

In fact, we can avoid the use of the two sections K_{in} and K_{out} by using the fact $RK_{\mathrm{out}}=K_{\mathrm{in}}$ and defining the maps $S:K_{\mathrm{in}}\to K_{\mathrm{in}}$ and $G:K_{\mathrm{in}}\to K_{\mathrm{in}}$ via

$$S(\varepsilon,\cdot)=R\widetilde{S}(\varepsilon,\cdot) \text{ and } G(\varepsilon,\cdot)=\widetilde{G}(\varepsilon,R\,\cdot),$$

which implies $P=G\circ S$. As a consequence of the symmetries of $\widetilde{S}$ and $\widetilde{G}$ in (2.11) and (2.12), respectively, we have the following result.

Lemma 2.3
Both maps $S,G:K_{\mathrm{in}}\to K_{\mathrm{in}}$ are involutions on their domains of definition: $S\circ S=I$ and $G\circ G=I$. Moreover both maps are SO(2)-equivariant, that is $S\circ T_\alpha=T_\alpha S$ and $G\circ T_\alpha=T_\alpha G$ for $\alpha\in\mathbb{S}^1$.

Note that the mappings S and G are reversible in the sense that $S^{-1}=S$ and $G^{-1}=G$ and there are no other reversibilities left over from R.

2.4 The factorization with respect to SO(2)-symmetry

The next step is to use the SO(2)-symmetry to reduce the dimension of the problem from 3 to 2. For this purpose we introduce in K_{in} local coordinates

$$\psi = \psi_2 \in \mathbb{S}^1 \text{ and } w = z_1 \mathrm{e}^{-\mathrm{i}\psi_2} \in K_r = \{\, w \in \mathbb{C} \;:\; |w| \le r \,\},$$

such that $(z_1, z_2) = (w\mathrm{e}^{\mathrm{i}\psi}, \delta \mathrm{e}^{\mathrm{i}\psi}) \in K_{\text{in}}$. The SO(2)-action in these coordinates is $T_\alpha(\psi, w) = (\psi + \alpha, w)$ and the maps S and G take the form

$$S(\varepsilon, \psi, w) = (\psi + \sigma(\varepsilon, w), s(\varepsilon, w)) \text{ and } G(\varepsilon, \psi, w) = (\psi + \gamma(\varepsilon, w), g(\varepsilon, w)). \tag{2.14}$$

For the local map we have the following explicit expressions

$$\sigma(\varepsilon, w) = \arg w + \Omega(\varepsilon, |w|), \quad s(\varepsilon, w) = \mathrm{e}^{-2\mathrm{i}\Omega(\varepsilon,|w|)}\overline{w} \tag{2.15}$$

The complex conjugation in $s(\varepsilon, \cdot)$ arises from R in the definition $S = R\widetilde{S}$. Notice that for $w = 0$ there is the unique continuation $s(\varepsilon, 0) = 0$. In contrast $\sigma(\varepsilon, w) : \mathbb{R} \times \mathbb{C} \mapsto \mathbb{R}$ has a discontinuity in the origin and cannot be extended continuously.

The functions γ and g can be obtained from $\widetilde{G}$ via

$$(g(\varepsilon, w)\mathrm{e}^{\mathrm{i}\gamma(\varepsilon,w)}, \delta \mathrm{e}^{\mathrm{i}\gamma(\varepsilon,w)}) = \widetilde{G}(\varepsilon, (\delta, w)) = T_{-\psi}\widetilde{G}(\varepsilon, (\delta \mathrm{e}^{\mathrm{i}\psi}, \mathrm{e}^{\mathrm{i}\psi} w)). \tag{2.16}$$

The properties of the maps G and S in (ψ, w)-coordinates can be summarized as follows.

Lemma 2.4
Let $E_0 = (-\varepsilon_0, \varepsilon_0)$ for sufficiently small positive ε_0. Then, we have:

A. *The functions $\gamma : E_0 \times K_r \to \mathbb{S}^1$ and $g : E_0 \times K_r \to \mathbb{C}$ are real analytic and satisfy on their domain of definition*

$$\gamma(\varepsilon, g(\varepsilon, w)) + \gamma(\varepsilon, w) = 0 \in \mathbb{S}^1 \text{ and } g \circ g = I. \tag{2.17}$$

B. *The functions $\sigma : E_0 \times K_r \setminus \{0\} \to \mathbb{S}^1$ and $s : E_0 \times K_r \setminus \{0\} \to K_r$ are real analytic and $s(\varepsilon, \cdot)$ is a Lipschitz map on K_r. Moreover we have*

$$\sigma(\varepsilon, s(\varepsilon, w)) + \sigma(\varepsilon, w) = 0 \in \mathbb{S}^1 \text{ and } s \circ s = I. \tag{2.18}$$

C. *Let $p(\varepsilon, w) = g(\varepsilon, s(\varepsilon, w))$ then $p^{-1} = s \circ p \circ s = g \circ p \circ g$.*

The proof is straightforward from the above construction.

We note that whenever $g(\varepsilon, w) = w$ we obtain from (2.17) the relation $2\gamma(\varepsilon, w) = 0 \in \mathbb{S}^1$. Assuming that for $\varepsilon = 0$ there exists a homoclinic orbit H_0 with $\nu_0 = \mathrm{par}\,(H_0)$ we conclude with (2.13) that $\nu_0 = \mathrm{e}^{\mathrm{i}\gamma(0,0)}$. Since γ is a smooth function we conclude

$$\gamma(\varepsilon, w) = \arg \nu_0 \in \mathbb{S}^1 \quad \text{whenever } g(\varepsilon, w) = w. \tag{2.19}$$

(Note that $\arg 1 = \arg \mathrm{e}^{\mathrm{i}0} = 0 \in \mathbb{S}^1$ and $\arg(-1) = \arg \mathrm{e}^{\mathrm{i}\pi} = \pi \in \mathbb{S}^1$.) Finally the total Poincaré map $P = G \circ S$ is written in (ψ, w)-coordinates as

$$P(\varepsilon, \psi, w) = (\psi + \widetilde{\pi}(\varepsilon, w), p(\varepsilon, w))$$

where $\widetilde{\pi}(\varepsilon, w) = \gamma(\varepsilon, s(\varepsilon, w)) + \sigma(\varepsilon, w)$ and $p(\varepsilon, w) = g(\varepsilon, s(\varepsilon, w))$.

2.5 The canonical form of S and G

Notice first that the local map $s(\varepsilon, w)$ can be simplified with the use of an analytical coordinate transform to

$$\widetilde{s}(\varepsilon, v) = \mathrm{e}^{-2\mathrm{i}\widehat{\Omega}(|v|)}\overline{v} \quad \text{with } \widehat{\Omega}(|v|) = \omega \log(\delta/|v|). \tag{2.20}$$

The needed transformation is given by $v = w \exp\left(-\frac{\omega + \operatorname{Im}\Phi(\varepsilon,\delta^2|w|^2)}{\omega[1+\operatorname{Re}\Phi(\varepsilon,\delta^2|w|^2)]}\right) = w + \mathcal{O}(|w|^3)$. For brevity we rename $v \mapsto w$ and $\widetilde{s} \mapsto s$ to keep old notations for simplified transformation and use

$$S(\psi, w) = (\psi + \arg w + \widehat{\Omega}(|w|), \mathrm{e}^{-2\mathrm{i}\widehat{\Omega}(|w|)}\overline{w})$$

as the final form of our local map from $K_{\text{in}} \to K_{\text{in}}$. The main advantage is that S is now independent of ε.

In these new coordinate w the global map G still is an involution with $g(0,0) = 0$ such that we have the expansion

$$_{\mathbb{R}}g(\varepsilon, {}_{\mathbb{R}}w) = \varepsilon\, a + \Gamma\, {}_{\mathbb{R}}w + \mathcal{O}(\varepsilon^2 + |w|^2) \quad \text{with } \Gamma^2 = I \text{ and } \Gamma a + a = 0. \tag{2.21}$$

This implies that eigenvalues of the matrix Γ lie in $\{-1, 1\}$. If $\det\Gamma = 1$ then either $\Gamma = I$ or $\Gamma = -I$; however in our situation only the case $\det\Gamma = -1$ is relevant. This is seen as follows. The Poincaré map $P = G \circ S$ from K_{in} into itself is orientation preserving and hence the same is true for the factorized map $p = g \circ s$, that is $\det D_w p(w) = 1$ whenever it is defined. Hence $\det D_w s(w) = -1$ implies $\det D_w g(w) = -1$ as desired.

Having $\Gamma^2 = I$ and $\det\Gamma = -1$ there is a rotation $w \mapsto \mathrm{e}^{\mathrm{i}\alpha} w$ such that ${}_{\mathbb{R}}\mathrm{e}^{-\mathrm{i}\alpha}\Gamma\,{}_{\mathbb{R}}\mathrm{e}^{\mathrm{i}\alpha} = \begin{pmatrix} 0 & b \\ 1/b & 0 \end{pmatrix}$ with $b > 0$. For $\Gamma = \begin{pmatrix} q & u \\ v & -q \end{pmatrix}$ take $\alpha = \frac{1}{2}\operatorname{arcctg}(-\frac{u+v}{q})$. Note that such a rotation of $w \in K_r$ does change the local map s simply by $\widehat{\Omega} \mapsto \widehat{\Omega} + 2\alpha$ which amounts into the same as changing δ slightly. The canonical form of the global map in this case is

$$_{\mathbb{R}}g(\varepsilon, {}_{\mathbb{R}}w) = c\varepsilon \begin{pmatrix} -b \\ 1 \end{pmatrix} + \begin{pmatrix} 0 & b \\ 1/b & 0 \end{pmatrix} {}_{\mathbb{R}}w + \mathcal{O}(\varepsilon^2 + |w|^2) \quad \text{with } c^2 = \frac{|a|^2}{1+b^2}. \tag{2.22}$$

Despite the fact that we only need the case $\det\Gamma = -1$ in this paper the following result is kept general for possible future reference.

Lemma 2.5
Suppose that g is the real analytical involution with the expansion (2.21). Then the fixed point set $\operatorname{Fix} g(\varepsilon, \cdot) = \{\, w \in K_r \,:\, g(\varepsilon, w) = w \,\}$ has the following characterization.

A. *If $\Gamma = I$, then $a = 0$ and $\operatorname{Fix} g(\varepsilon, \cdot) = K_r$.*

B. *If $\Gamma = -I$, then $\operatorname{Fix} g(\varepsilon, \cdot) = \{\rho(\varepsilon)\} \subset K_r$ with $\rho(\varepsilon) = -\varepsilon a/2 + \mathcal{O}(\varepsilon^2)$.*

C. *If $\det\Gamma = -1$ then*

$$\operatorname{Fix} g(\varepsilon, \cdot) = \{\, (x_1, x_2) \in {}_{\mathbb{R}}K_r \,:\, \mu(\varepsilon, x_1, x_2) = 0 \,\} \tag{2.23}$$

where $\mu(\varepsilon, x_1, x_2) = b x_2 - x_1 - bc\varepsilon + \mathcal{O}(\varepsilon^2 + |x|^2)$.

Proof: The real analytical involution $Z(\varepsilon, w) = (\varepsilon, {}_{\mathbb{R}}g(\varepsilon, {}_{\mathbb{R}}w))$ on $(-\varepsilon_0, \varepsilon_0) \times K_r$ has the expansion $Z(v) = Nv + \mathcal{O}(v^2)$ with $v = (\varepsilon, {}_{\mathbb{R}}w)$ near the fixed point $v = 0$. It is analytically conjugate to its linear part around the fixed point. This is the contents of Bochner theorem (see [Bre72]) for $\mathbb{Z}_2$-actions. Thus, statements A, B, and C follow since they are obviously true in the linear case.

We give a simple and explicit proof as we want be able to conclude that the conjugation of Bochner's theorem does not change ε. In fact, the conjugating transformation is $Q(v) = v + NZ(v)$ since $NQ = (NZ + Z^2)Z = QZ$ and hence $Z = Q^{-1} \circ N \circ Q$. QED

3 n-homoclinic orbits

We return to system (2.2) for which we assumed the existence of a primary homoclinic orbit $H_0(\cdot)$ for $\varepsilon = 0$. To this orbit we fix a small tubular neighborhood $N_\delta = \{ Y \in \mathbb{C}^2 : \exists\, x \in \mathbb{R} : |Y - H_0(x)| \leq \delta \}$, i.e. $\delta > 0$ is small. For small $\varepsilon \neq 0$ there might again exist homoclinic orbits.

Definition 3.1
An orbit $H_\varepsilon(\cdot)$ of $(2.2)_\varepsilon$ which is homoclinic to 0 is called an n-homoclinic orbit with respect to the primary orbit H_0 if it winds around n times in N_δ.

From our construction of the Poincaré section we conclude that an n-homoclinic orbit has to intersect K_{in} in n different points $(\psi^j, w^j) \in K_{\text{in}}$ such that

$$(\psi^{j+1}, w^{j+1}) = P(\varepsilon, \psi^j, w^j) \text{ with } w^j \neq 0 \text{ for } j = 1, \ldots, n-1,$$

and additionally $w^1 = g(\varepsilon, 0)$ and $w^n = 0$. Here the condition $w^j \neq 0$ guarantees that we do not have a homoclinic orbit which is m-homoclinic with $m < n$.

Hence, the set of all parameter values $\varepsilon \in E_0 = (-\varepsilon_0, \varepsilon_0)$ where (2.2) has a n-homoclinic orbit is given by

$$E^n = \{ \varepsilon \in E_0 \,:\, p^n(\varepsilon, 0) = 0 \text{ and } p^j(\varepsilon, 0) \neq 0 \text{ for } j = 1, \ldots, n-1 \}.$$

3.1 Characterization of E^n

We introduce the fixed point sets for s and g, respectively,

$$\operatorname{Fix} s = \{ w \in K_r \,:\, s(w) = w \}, \quad \operatorname{Fix} g(\varepsilon, \cdot) = \{ w \in K_r \,:\, g(\varepsilon, w) = w \}.$$

From the reversibility properties of S and G stated in Lemma 2.4 we know that $2\sigma(\varepsilon, w) = 0$ on $\operatorname{Fix} s$ and $2\gamma(\varepsilon, w) = 0$ on $\operatorname{Fix} g(\varepsilon, \cdot)$. From (2.19) we already know that $\gamma(\varepsilon, w) = \arg \nu_0$ on $\operatorname{Fix} g(\varepsilon, \cdot)$, and Lemma 2.5 states that $\operatorname{Fix} g(\varepsilon, \cdot)$ is a smooth curve in K_r. The set $\operatorname{Fix} s$ decomposes into $\operatorname{Fix} s = \operatorname{Fix} s_{+1} \cup \operatorname{Fix} s_{-1} \cup \{0\}$ with

$$\operatorname{Fix} s_\nu = \{ w \in K_r \,:\, \arg w + \widehat{\Omega}(|w|) = \arg \nu \}.$$

These two sets are logarithmic spirals due to the special form of $\widehat{\Omega}$ given in (2.20).

Theorem 3.2
We have $E^{2m} = E^{2m}_{+1} \cup E^{2m}_{-1}$ *where*

$$E^{2m}_{\nu} = \{\, \varepsilon \in E_0 \ : \ p^m(\varepsilon, 0) \in \operatorname{Fix} s_\nu,\ p^j(\varepsilon, 0) \neq 0 \text{ for } j = 1, \ldots, m \,\}.$$

For $\varepsilon \in E^{2m}_\nu$ *the associated* $2m$*-homoclinic orbit* H_ε *satisfies* $\operatorname{par}(H_\varepsilon) = \nu$*. Moreover,*

$$E^{2m+1} = \{\, \varepsilon \in E_0 \ : \ p^{m+1}(\varepsilon, 0) \in \operatorname{Fix} g(\varepsilon, \cdot),\ p^j(\varepsilon, 0) \neq 0 \text{ for } j = 1, \ldots, m+1 \,\}$$

and for $\varepsilon \in E^{2m+1}$ *we have* $\operatorname{par}(H_\varepsilon) = \operatorname{par}(H_0)$*. Another useful characterization is*

$$E^{2m+1} = \{\, \varepsilon \in E_0 \ : \ p^{-m}(\varepsilon, 0) \in \operatorname{Fix} g(\varepsilon, \cdot),\ p^{-j}(\varepsilon, 0) \neq 0 \text{ for } j = 1, \ldots, m \,\}.$$

Proof: If $p^n(0) = 0$ then part C of Lemma 2.4 gives $p^{n-k}(0) = p^{-k}(0) = g \circ p^k \circ p(0) = s \circ p^k(0)$ where we used $s(0) = 0$. If $n = 2m$ this gives $p^m(0) = s \circ p^m(0)$ and thus $p^m(0) \in \operatorname{Fix} s$. If $n = 2m+1$ we take $k = m$ and find $p^{m+1}(0) = g \circ p^m \circ g(0) = g \circ p^{m+1}(0)$ since $0 = s(0)$. Thus, we find $p^{m+1}(0) \in \operatorname{Fix} g(\varepsilon, \cdot)$ as desired. Since $s(0) = 0$ and $g \circ g = I$ this is equivalent to $p^{-m}(0) \in \operatorname{Fix} g(\varepsilon, \cdot)$.

It remains to consider the parities of the associated homoclinic orbits H_ε for $\varepsilon \in E^n$. This can only be decided in the mapping without factorization. Consider the case $n = 2m+1$ first, and denote by (ψ^k, w^k), $k = 1, \ldots 2m$ the intersection points of H_ε with K_{in}. In Z-coordinates we have $(z_1^k, z_2^k) = (w^k e^{i\psi^k}, \delta e^{i\psi^k}) \in K_{\text{in}}$ and $(\widetilde{z}_1^k, \widetilde{z}_2^k) = \widetilde{G}^{-1}(\varepsilon, (z_1^k, z_2^k)) \in K_{\text{out}}$. Since $w^{m+1} \in \operatorname{Fix} g(\varepsilon, \cdot)$ we have $\psi^{m+2} = \psi^{m+1} + \gamma(\varepsilon, w^{m+1}) = \psi^{m+1} + \arg \nu_0$. With $\widetilde{G}^{-1} = RG^{-1}$ we conclude

$$\begin{aligned}(\widetilde{z}_1^{m+1}, \widetilde{z}_2^{m+1}) &= RG^{-1}(\varepsilon, (w^{m+1} e^{i\psi^{m+1}}, \delta e^{i\psi^{m+1}})) = R(w^{m+1} \nu_0 e^{i\psi^{m+1}}, \delta \nu_0 e^{i\psi^{m+1}}) \\ &= \nu_0(\delta e^{i\psi^{m+1}}, w^{m+1} e^{i\psi^{m+1}}) = \nu_0 R(z_1^{m+1}, z_2^{m+1})\end{aligned}$$

Thus, we know that there exist ξ_1 and ξ_2 such that $H_\varepsilon(\xi_1) \in K_{\text{out}}$ and $H_\varepsilon(\xi_2) \in K_{\text{in}}$ with $H_\varepsilon(\xi_1) = \nu_0 R H_\varepsilon(\xi_2)$ and Lemma 4.2 shows that $\operatorname{par}(H_\varepsilon) = \nu_0$.

The arguments for $\varepsilon \in E^{2m}_{\pm 1}$ are analogous. QED

3.2 Sequences of 2- and 3-homoclinic orbits

In order to give precise statements on the secondary homoclinic solutions we have to introduce some genericity condition on the primary homoclinic solution $H_0(x)$.

Definition 3.3
The primary homoclinic solution $H_0(x)$ *is called* **transversal** *with respect to the perturbation in* ε *if* a *in* (2.21) *satisfies* $a \neq 0$.

For the transversal homoclinic orbits it is possible to use the characterization given in Theorem 3.2 for proving the existence of n-homoclinic orbits. We starts with 2- and 3-homoclinic orbits.

Theorem 3.4
Suppose that the primary homoclinic orbit H_0 with parity ν_0 is transversal. Then for sufficiently small $\varepsilon_0 > 0$ we have

$$E^2 = \{\, \varepsilon^{(2)}_{>,k}, \varepsilon^{(2)}_{<,k} \;:\; k \in \mathbb{N}\} \quad \textit{and} \quad E^3 = \{\, \varepsilon^{(3)}_{>,k}, \varepsilon^{(3)}_{<,k} \;:\; k \in \mathbb{N}\},$$

where the sequences are ordered as follows

$$\varepsilon^{(2)}_{<,k} < \varepsilon^{(3)}_{<,k} < \varepsilon^{(2)}_{<,k+1} < 0 < \varepsilon^{(2)}_{>,k+1} < \varepsilon^{(3)}_{>,k} < \varepsilon^{(2)}_{>,k}, \tag{3.1}$$

and for the vector of quotients we have

$$\frac{1}{\varepsilon^{(2)}_{>,k}}\Big(\varepsilon^{(2)}_{>,k+1}, \varepsilon^{(3)}_{>,k}, \varepsilon^{(2)}_{<,k}, \varepsilon^{(3)}_{<,k}\Big) \longrightarrow \Big(\mathrm{e}^{-2\pi/\omega}, \mu_2, \mu_3, \mu_4\Big) \quad \textit{for } k \to \infty. \tag{3.2}$$

For $\varepsilon \in E^3$ we have $\mathrm{par}\,(H_\varepsilon) = \nu_0 = \mathrm{par}\,(H_0)$ and for $\varepsilon = \varepsilon^{(2)}_{>,<,k}$ we have $\mathrm{par}\,(H_\varepsilon) = (-1)^k$.

Proof: We are looking for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ such that $g(\varepsilon, 0) \in \mathrm{Fix}\, s$. Since the primary homoclinic orbit is transversal $g(\varepsilon, 0) = \varepsilon c(-b, 1)^T + \mathcal{O}(\varepsilon^2)$ is a smooth curve in K_r passing through the origin with nonzero speed with respect to ε. Hence, there are infinitely many intersections with the two logarithmic spirals $\mathrm{Fix}\, s$ given by

$$\frac{1}{2}\arcsin(b/\widehat{b}) + \mathcal{O}(\varepsilon) + \pi n = \widehat{\Omega}(|\varepsilon c|\widehat{b} + \mathcal{O}(\varepsilon^2)), \quad n \in \mathbb{N}, \tag{3.3}$$

where $\widehat{b} = (1+b^2)^{1/2}$. For all $n \in \mathbb{N}$ with $n \geq N_0$ there are unique solutions $\varepsilon^{(2)}_{<,n} < 0 < \varepsilon^{(2)}_{>,n}$ of (3.3). Obviously, they lie in $(-\varepsilon_0, \varepsilon_0)$ for N_0 large enough and, according to Theorem 3.2, the parities are equal to $(-1)^n$. Renumbering the sequences gives the result for E^2.

To study 3-homoclinic orbits we have to check the condition $p^{-1}(\varepsilon, 0) \in \mathrm{Fix}\, g(\varepsilon, \cdot)$ or $\mu(p^{-1}(\varepsilon, 0)) = 0$. This gives $\varepsilon\, c\, [-(1+b^2)\sin 2\widehat{\Omega}(|\varepsilon c|\widehat{b} + \mathcal{O}(\varepsilon^2)) + b] + \mathcal{O}(\varepsilon^2) = 0$. This is equivalent to solving

$$\widehat{\Omega}(|\varepsilon c|\widehat{b} + \mathcal{O}(\varepsilon^2)) = \frac{1}{2}\arcsin\frac{b}{1+b^2} + \pi n + \mathcal{O}(\varepsilon), \quad n \in \mathbb{N}, \tag{3.4}$$

which gives sequences $\varepsilon^{(3)}_{<,n} < 0 < \varepsilon^{(3)}_{>,n}$. According to Theorem 3.2 the parity of all these solutions is $\nu_0 = \mathrm{par}\,(H_0)$. Moreover, comparing (3.3) and (3.4) and using $|b|/(1+b^2) \leq |b|/\widehat{b}$ proves the result on the ordering of the sequences. The explicit formula for $\widehat{\Omega}$ gives nice expansions of the solutions such that the statement of the limits in (3.2) is easily deduced. QED

Our aim is to use induction to generate n-homoclinic orbits with $n \geq 4$ by considering H_ε as a primary homoclinic orbits to which the above theorem can be applied as well.

Theorem 3.5
The homoclinic solutions found in Theorem 3.4 are transversal for sufficiently small $\varepsilon \in E^2 \cup E^3$.

Proof: For $\varepsilon_2 \in E^2$ we consider H_{ε_2} as a primary homoclinic solution with the local map s and the global map $g_2 = g \circ s \circ g$ where $g_2(\varepsilon_2, 0) = 0$. Note that by reparametrization of ε we can always suppose $c = 1$. In the vicinity of $(\varepsilon_2, 0)$ we have the following expansion ${}_{\mathbb{R}}g_2(\varepsilon, {}_{\mathbb{R}}w) = (\varepsilon - \varepsilon_2)a_2 + \Gamma_2\, {}_{\mathbb{R}}w + \mathcal{O}((\varepsilon-\varepsilon_2)^2 + |w|^2)$ with $a_2 = \partial_\varepsilon\, {}_{\mathbb{R}}g(\varepsilon_2, s(\varepsilon_2, g(\varepsilon_2, 0))) + G_2(\varepsilon_2)S_1(\varepsilon_2)\partial_\varepsilon g(\varepsilon_2, 0)$ and $\Gamma_2 = G_2 S_1 G_1$ where $G_1(\varepsilon) = \mathrm{D}\, {}_{\mathbb{R}}g(\varepsilon, 0)$, $S_1(\varepsilon) = \mathrm{D}\, {}_{\mathbb{R}}s(\varepsilon, g(\varepsilon, 0))$, $G_2(\varepsilon) = \mathrm{D}\, {}_{\mathbb{R}}g(\varepsilon, s(\varepsilon, g(\varepsilon, 0)))$. Since g is smooth, $a_2 = a + \Gamma S_1(\varepsilon_2)a + \mathcal{O}(\varepsilon_2)$. That is why $a_2 \neq 0$ for sufficiently small ε_2 if and only if $0 \neq \Gamma(\Gamma S_1(\varepsilon_2)a + a) = S_1(\varepsilon_2)a - a$. The matrix $S_1(\varepsilon)$ can be decomposed into $S_1(\varepsilon) = \widetilde{S}_1(\varepsilon) + \widehat{S}_1 + \mathcal{O}(\varepsilon)$ with

$$\widetilde{S}_1(\varepsilon) = \begin{pmatrix} \cos 2\widehat{\Omega}(\varepsilon\widehat{b}) & -\sin 2\widehat{\Omega}(\varepsilon\widehat{b}) \\ -\sin 2\widehat{\Omega}(\varepsilon\widehat{b}) & -\cos 2\widehat{\Omega}(\varepsilon\widehat{b}) \end{pmatrix}, \quad \widehat{S}_1 = \frac{2\omega}{1+b^2}\begin{pmatrix} b & -1 \\ b^2 & -b \end{pmatrix}$$

Notice $\widetilde{S}_1(\varepsilon_2)g(\varepsilon_2, 0) = s(\varepsilon_2, g(\varepsilon_2, 0))$ and hence $\widetilde{S}_1(\varepsilon_2)a = a$. Therefore we arrive at

$$S_1(\varepsilon_2)a - a = \widehat{S}_1 a = \frac{2\omega}{1+b^2}\begin{pmatrix} b & -1 \\ b^2 & -b \end{pmatrix}\begin{pmatrix} -b \\ 1 \end{pmatrix} = -2\omega \begin{pmatrix} 1 \\ b \end{pmatrix} \neq 0$$

Thus, the transversality of the bifurcating 2-homoclinic solutions is established for ε_2 small enough.

For $\varepsilon_3 \in E^3$ the associated homoclinic orbit H_{ε_3} has the same local map s and the global map $g_3 = g \circ s \circ g \circ s \circ g$ with $g(\varepsilon_3, 0) = 0$. The expansion is ${}_{\mathbb{R}}g_3(\varepsilon, {}_{\mathbb{R}}w) = (\varepsilon-\varepsilon_3)a_3 + \Gamma_3\, {}_{\mathbb{R}}w + \mathcal{O}((\varepsilon-\varepsilon_2)^2 + |w|^2)$ where $\Gamma_3 = G_3(\varepsilon_3)S_2(\varepsilon_3)G_2(\varepsilon_3)S_1(\varepsilon_3)G_1(\varepsilon_3)$ and $a_3 = a + G_3(\varepsilon_3)S_2(\varepsilon_3)[a + G_2(\varepsilon_3)S_1(\varepsilon_3)]$, where $G_3(\varepsilon_3) = G_1^{-1}(\varepsilon_3)$, $S_2(\varepsilon_3) = S_1^{-1}(\varepsilon_3)$ and $G_2(\varepsilon_3) = G_2^{-1}(\varepsilon_3)$ since $s(\varepsilon_3, g(\varepsilon_3, 0)) \in \operatorname{Fix} g(\varepsilon, \cdot)$. Since g is smooth, $a_3 \neq 0$ for ε_3 small enough if and only if $a + \Gamma S_1^{-1}(\varepsilon_3)[a + \Gamma S_1(\varepsilon_3)a] \neq 0$ or equivalently

$$(\Gamma S_1(\varepsilon_3) - S_1(\varepsilon_3) + I)a \neq 0 \quad \text{with } S_1(\varepsilon_3) = \widetilde{S}_1(\varepsilon_3) + \widehat{S}_1 + \mathcal{O}(\varepsilon_3). \tag{3.5}$$

From $s(\varepsilon_3, g(\varepsilon_3, 0)) \in \operatorname{Fix} g(\varepsilon, \cdot)$ we conclude $\widetilde{S}_1(\varepsilon_3)a = a + \Gamma\widetilde{S}_1(\varepsilon_3)a + \mathcal{O}(\varepsilon_3)$, and hence condition (3.5) is equivalent to

$$0 \neq \widehat{S}_1(\Gamma - I)a = (1+b^2)\begin{pmatrix} -\cos 2\widehat{\Omega}(\varepsilon_3\widehat{b}) \\ b^{-1}\cos 2\widehat{\Omega}(\varepsilon_3\widehat{b}) \end{pmatrix}.$$

According to Theorem 3.4 for $\varepsilon_3 \in E^3$ we have $\sin 2\widehat{\Omega}(\varepsilon_3\widehat{b}) = \frac{b}{1+b^2} + \mathcal{O}(\varepsilon_3^2) \neq \pm 1$; and hence $a_3 \neq 0$ follows and the transversality is established. QED

3.3 n-homoclinic orbits with $n \geq 4$

An immediate corollary of Theorems 3.4 and 3.5 is the existence of cascades of transversal $2^k 3^m$-homoclinics for $k, m \in \mathbb{N}$. This follows by induction where Theorem 3.2 is applied in each step. As a consequence we find that the sets E^n with $n = 2^k 3^m$ with $k+m \geq 2$

accumulate on each point ε which corresponds to a $2^{\widehat{k}}3^{\widehat{m}}$-homoclinic orbit with $\widehat{k} \leq k$, $\widehat{m} \leq m$, and $\widehat{k}+\widehat{m} < k+m$. In fact, there is a fractal structure.

For $n \geq 5$ which are not of the above form our results are less precise. However, Lemma 3.4 in [MiHO92] is fully applicable in our situation and we conclude that between each $\varepsilon_m \in E^m$ and each $\varepsilon_{m+1} \in E^{m+1}$ there is at least one point $\varepsilon_{2m+1} \in E^{2m+1}$. By induction it is then easy to conclude that all E^n contain infinitely many points in each neighborhood of $\varepsilon = 0$, see Theorem 3.5 in [MiHO92]. The main argument there is to use the sign of the function μ defining $\operatorname{Fix} g(\varepsilon, \cdot)$.

For instance, to prove the existence of sequences of 5-homoclinic solutions we have to check the condition $\mu(p^3(\varepsilon, 0)) = 0$. Notice that $\mu(\varepsilon, 0) = -bc\varepsilon$ and $\mu(p(\varepsilon, 0)) = bc\varepsilon$. That is why if $p^3(\varepsilon_3, 0) = 0$ and $p^3(\varepsilon_2, 0) = p(\varepsilon_2, 0)$ then between ε_2 and ε_3 there is a point $\varepsilon_5 \in E^5$ such that $\mu(p^3(\varepsilon_5, 0)) = 0$. Now the existence of the sequences of 2 and 3-homoclinic values of ε described in the theorem 3.4 gives the infinite sequence of 5-homoclinic values of ε. Unfortunately, these arguments leaves the question on transversality unstudied.

We remark that all the results do not depend on the analyticity of the vector field. The transversality argument in Theorem 3.5 just needs taking one derivative. Similarly, the construction of $(2m+1)$-homoclinic orbits involves a monotonicity argument which works for continuous functions. This is contrast to the results in [MiHO92] where analyticity was needed to obtain higher order homoclinic orbits. However, analyticity can be used to show that the number of points in E^{2m+1} between any to adjacent points in E^m and E^{m+1} must be finite. Without analyticity we even may have situations where E^n contains a closed interval.

4 Calculation of Γ and a

In this section we want to show how a and Γ in the canonical form (2.21) of the global map g can be calculated. This will be achieved by studying the linearized equation (2.2) around the primary homoclinic orbit H_0. Thus, we will be able to check the transversality conditions for the complex Ginzburg-Landau equation (2.3) numerically. In particular, we obtain exact numbers for the normal form coefficients c and b in (2.22) and to check the transversality condition in Definition 3.3.

This part of the analysis will mainly use real coordinates, thus we recall $u = {\scriptstyle\mathbb{R}}(y_1, y_2)$ and set $v = {\scriptstyle\mathbb{R}}(z_1, z_2)$ for the local normal-form coordinates. By $\phi_x(\varepsilon, v(0)) = v(x)$ and $\psi_x(\varepsilon, u(0)) = u(x)$ we denote the flow maps $\phi_x, \psi_x : \mathbb{R}^4 \to \mathbb{R}^4$ of be the flows of (2.7), rewritten in real coordinates, and of (2.2), respectively. The normal-form transformation (2.6) written in real coordinates is $v = M(\varepsilon, u)$, and it preserves the symmetry and the reversibility.

We choose $\tau > 0$ such that $M(0, H_0(\tau)) \in K_{\text{in}}$ and $M(0, H_0(-\tau)) \in K_{\text{out}}$ and define the map

$$\Pi(\varepsilon, \rho, v) = (M \circ \psi_{2\tau+\rho} \circ M^{-1})(\varepsilon, Rv)$$

which maps a neighborhood of K_{in} into another neighborhood of K_{in}. Via ψ_x this map includes the global information of the flow in a tubular neighborhood along H_0 which is not contained in the normal form. By the construction Π has SO(2)-symmetry $T_\alpha \Pi = \Pi \circ T_\alpha$ for $\alpha \in \mathbb{S}^1$ and is reversible with $\Pi \circ \Pi = \mathrm{id}$. Moreover we have the expansion

$$\Pi(\varepsilon, \rho, \widetilde{H}_0(\tau) + \widetilde{u}) = \widetilde{H}_0(\tau) + \varepsilon l + L\widetilde{u} + \rho \dot{\widetilde{H}}_0(\tau) + \mathcal{O}(\varepsilon^2 + \rho^2 + |\widetilde{u}|^2), \tag{4.1}$$

where $L \in \mathbb{R}^{4\times 4}$ and $l \in \mathbb{R}^4$ satisfy $L^2 = I$ and $Ll + l = 0$. Without loss of generality we may assume $\widetilde{H}_0(\tau) = {}_{\mathbb{R}}(0, \delta)$, compare (2.9).

We first show that L and l contain all the information on Γ and a and then we give analytic expressions for them using infinite integrals over $x \in \mathbb{R}$ of Melnikov type which can be calculated numerically.

4.1 Reduction to the factorized global map g

By differentiation of $\Pi(0, 2\rho, T_\alpha \widetilde{H}_0(\tau + \rho)) = T_\alpha \widetilde{H}_0(\tau + \rho)$ with respect to α and ρ and by using $\widetilde{H}_0(\tau+\rho) = {}_{\mathbb{R}}(0, \delta\, \mathrm{e}^{-\lambda\rho})$ we find the relation $L\, {}_{\mathbb{R}}(0, \mathrm{i}\alpha+\lambda\rho) = {}_{\mathbb{R}}(0, \mathrm{i}\alpha-\lambda\rho)$ for $\alpha, \rho \in \mathbb{R}$. Writing $L = \begin{pmatrix} L_1 & L_2 \\ L_3 & L_4 \end{pmatrix}$ with $L_j \in \mathbb{R}^{2\times 2}$ gives $L_2 = 0$ and $L_4 = \begin{pmatrix} -1 & 0 \\ -2\omega & 1 \end{pmatrix}$.

The global map $G : K_{\text{in}} \to K_{\text{in}}$ is obtained from Π by restriction to K_{in} and by adjusting the travel time $2\tau + \rho$ with $\rho = \widehat{\rho}(\varepsilon, \widetilde{u})$ such that $\Pi(\varepsilon, \widehat{\rho}(\varepsilon, \widetilde{u}), H_0(\tau) + \widetilde{u}) \in K_{\text{in}}$. From (2.16) we know that $g(\varepsilon, w)$ is obtained by evaluating the first two components (Π_1, Π_2) of $\Pi \in \mathbb{R}^4$ in the form

$${}_{\mathbb{R}}g(\varepsilon, {}_{\mathbb{R}}w) = \nu_0 (\Pi_1, \Pi_2)(\varepsilon, \widehat{\rho}(\varepsilon, {}_{\mathbb{R}}(w, 0)), {}_{\mathbb{R}}(w, \delta)) + \mathcal{O}(\varepsilon^2 + |w|^2),$$

where $\nu_0 = \mathrm{e}^{-\mathrm{i}\gamma(0,0)}$ is the parity of the primary orbit.

At lowest order the travel time correction $\widehat{\rho}$ does not enter into the first two components (Π_1, Π_2). Thus, we immediately conclude by comparison with ${}_{\mathbb{R}}g(\varepsilon, {}_{\mathbb{R}}w) = \varepsilon a + \Gamma\, {}_{\mathbb{R}}w + \mathcal{O}(\varepsilon^2 + |w|^2)$ the identities

$$\Gamma = \nu_0 L_1 \quad \text{and} \quad a = \nu_0 l_1,$$

where $l = (l_1, l_2) \in \mathbb{R}^4$ with $l_1, l_2 \in \mathbb{R}^2$.

4.2 Calculation of L

This calculation can be done completely with $\varepsilon = 0$.

Define $\Psi(x) = D_u \psi_x(0, {}_{\mathbb{R}}H_0(x))$ the linearization of (2.2) around ${}_{\mathbb{R}}H_0(x)$ which satisfies

$$\dot{\Psi} = C(x)\Psi \text{ with } \quad \Psi(0) = I, \; C(x) = D_u f(0, {}_{\mathbb{R}}H_0(x)). \tag{4.2}$$

Since $\widetilde{H}_0(x-\tau) = {}_{\mathbb{R}}(\nu_0 \delta e^{\lambda x}, 0)$ for $x \leq 0$, $\widetilde{H}_0(x+\tau) = {}_{\mathbb{R}}(0, \delta e^{-\lambda x})$ for $x \geq 0$, and ${}_{\mathbb{R}}\widetilde{H}_0(x) = M(0, {}_{\mathbb{R}}H_0(x))$ the linearization of (2.7) along $\widetilde{H}_0$ gives the real system

$$\dot{\Phi} = C^* \Phi \quad \text{with} \quad C^* = {}_{\mathbb{R}}\begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}, \quad \Phi(x)\Phi(\xi)^{-1} = {}_{\mathbb{R}}\begin{pmatrix} \mathrm{e}^{\lambda(x-\xi)} & 0 \\ 0 & \mathrm{e}^{-\lambda(x-\xi)} \end{pmatrix}. \tag{4.3}$$

Note that we have avoided to use $\Phi(x)$ for $x \in (-\tau, \tau)$ since there $\widetilde{H}_0$ is not defined.

Suppose that $x, \xi \geq \tau$ such that $M(0, \psi_x({}_{\mathbb{R}}H_0(0))) = \phi_{x-\xi} M(0, \psi_\xi({}_{\mathbb{R}}H_0(0)))$. This implies by differentiation $N(x)\Psi(x) = \Phi(x)\Phi(\xi)^{-1}N(\xi)\Psi(\xi)$ with $N(x) = D_v M(0, {}_{\mathbb{R}}H_0(x))$. It follows that the real matrix

$$B(\xi) = \Phi(\xi)^{-1} N(\xi)\Psi(\xi) \tag{4.4}$$

is in fact independent of $\xi \geq \tau$. Let us denote it by $B_+ \in \mathbb{R}^{4\times 4}$. On the other hand the matrix L in (4.1) takes the form $N(\tau)\Psi(\tau)\Psi(\tau)^{-1}N(\tau)R$ which gives by reversibility and the definition of B_+ the formula

$$L = \Phi(\tau) B_+ R B_+^{-1} \Phi(\tau)^{-1}.$$

Since $\Phi(\tau)$ is explicitly known, it remains to find B_+, which can be managed as follows.

We know $\Psi(x)$ explicitly by solving the linear system (4.2). The normal-form transformation $v = M(0, u)$ can be written as

$$v = u + {}_{\mathbb{R}}\Theta_3(0, u) + \mathcal{O}(|u|^5) \tag{4.5}$$

where $\Theta_3(0, u)$ is a homogeneous part of $\Theta(0, u)$ (polynomial) of the order 3. Thus, $N(x) = [I + N_2(x) + \mathcal{O}(e^{-4x})]$ with $N_2(x) = D_u \Theta_3(0, {}_{\mathbb{R}}H_0(x))$. Finally B_+ takes the form

$$B_+ = \Phi(x)[I + N_2(x) + \mathcal{O}(\mathrm{e}^{-4x})]\Psi(x). \tag{4.6}$$

Since $\Psi(x)$ and $\Phi(x)$ only grow like e^x it is clear that omitting terms of order $\mathcal{O}(\mathrm{e}^{-4x})$ the function on the right-hand side converges to B_+ with the error bounded by $\mathcal{O}(\mathrm{e}^{-2x})$. Thus, B_+ can be obtained numerically by evaluating the right-hand side for sufficiently large x.

4.3 Calculation of l

The vector l is obtained using the first order perturbation in ε, $m(x) = \partial_\varepsilon f(0, {}_{\mathbb{R}}H_0(x))$ and the linear inhomogeneous system $\dot{y} = C(x)y + m(x)$. Since ε only appears in (2.1) through $F(\varepsilon, Y, \overline{Y}) = \mathcal{O}(|Y|^3)$ we have $m(x) = \mathcal{O}(\mathrm{e}^{-3|x|})$, and there is a unique solution which decays at least like e^x for $x \to -\infty$

$$q(x) = \Psi(x) \int_{-\infty}^{x} \Psi(\xi)^{-1} m(\xi)\, \mathrm{d}\xi. \tag{4.7}$$

Thus, the perturbed stable manifold in v-coordinates for $x \geq \tau$ is

$$V_\varepsilon(x) = M(\varepsilon, {}_{\mathbb{R}}H_0(x) + \varepsilon q(x) + \mathcal{O}(\varepsilon^2)) = {}_{\mathbb{R}}\widetilde{H}_0(x) + \varepsilon(N(x)q(x) + \partial_\varepsilon M(0, {}_{\mathbb{R}}H_0(x))) + \mathcal{O}(\varepsilon^2).$$

Recall that $\phi_x(\varepsilon, v)$ is the flow of (2.7) and $V_\varepsilon(\tau) = \phi_{\tau - x}(\varepsilon, V_\varepsilon(x))$.

Expanding in powers of ε and comparing with (4.1) gives the formula

$$l = \partial_\varepsilon V_{\varepsilon=0}(\tau) = \Phi(\tau)\Phi(x)^{-1}[N(x)q(x) + \partial_\varepsilon M(0, {}_{\mathbb{R}}H_0(x))]. \tag{4.8}$$

Since the perturbation in ε acts only on terms of the order ≥ 3 it follows $\partial_\varepsilon M(0, {}_{\mathbb{R}}H_0(x)) = \mathcal{O}(\mathrm{e}^{-3x})$. Thus, with the use of (4.4) and (4.8) the vector $l \in \mathbb{R}^4$ in (4.1) takes the form

$l = \Phi(\tau)[B(x)\int_{-\infty}^{x}\Psi(\xi)^{-1}m(\xi)\,\mathrm{d}\xi + \mathcal{O}(\mathrm{e}^{-2x})]$. With $x \to \infty$ we find convergence like e^{-2x} to the limit

$$l = \Phi(\tau)B_+\widehat{l} \quad \text{with} \quad \widehat{l} = \int_{-\infty}^{\infty}\Psi(\xi)^{-1}m(\xi)\,\mathrm{d}\xi.$$

By reversibility we have $\widehat{l} = \int_{\mathbb{R}}\Psi(-\xi)^{-1}m(-\xi)\,\mathrm{d}\xi = \int_{\mathbb{R}}R\Psi(\xi)^{-1}R(-Rm(\xi))\,\mathrm{d}\xi = -R\widehat{l}$. That is why it suffices to calculate $\widehat{l}_+ = \int_0^\infty \Psi(\xi)^{-1}m(\xi)d\,\xi$ and find $\widehat{l} = \widehat{l}_+ - R\widehat{l}_+$. This implies $Ll + l = 0$ as desired.

4.4 Application to the complex Ginzburg-Landau equation

Finally we apply the above arguments to the complex Ginzburg-Landau equation as given in (1.1) and (2.3). Clearly we can define $(y_1, y_2) = (\dot{A} + \lambda A, \dot{A} - \lambda A)$ and immediately obtain the desired form of (2.1). To calculate Γ and a in (2.21) through (4.1) we have to do the normal form transformation shown in (4.5). Then the calculation of B_+ and $\widehat{l}_+$ as indicated is straight forward and was done with a standard Runge–Kutta solver. The convergence of B_+ as given in (4.6) as well as the identity $Ll + l = 0$ served as good checks of the correctness of the result.

The numerical results are done as a parameter study in the regime $\omega \in [0, 10]$. The the coefficients $c(\omega)$ and $b(\omega)$ in the canonical form (2.22) of $g(\varepsilon, w)$ were plotted. We find $b(0) = 1$ and then an exponential decay to 0 with $b(8) \approx 10^{-12}$. For $c(\omega)$ we obtain $c(\omega) = \gamma\omega + \mathcal{O}(\omega^3)$ for small ω with $\gamma \approx 180$. Then c remains positive until it reaches its first zero $\omega_1 \approx 8.032$ and the second zero appears at $\omega_2 \approx 9.51$. Since $c : [0, \infty) \to \mathbb{R}$ is an analytic function which does not vanish identically, it follows that it can only have a discrete set $\{\omega_1, \omega_2, \ldots\}$ of zeros. This set which might be finite or countable, where in the latter case $\omega_j \to \infty$ for $j \to \infty$.

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