Global existence and uniqueness for an optical fiber laser model

Alexander Mielke†, Philip Holmes‡ and J. Nathan Kutz§

† Institut für Angewandte Mathematik, Universität Hannover, 30167 Hannover, Germany

‡ Program in Applied and Computational Mathematics and Department of Mechanical and Aerospace Engineering, Princeton University, Princeton, NJ 08544, USA

§ Program in Applied and Computational Mathematics, Princeton University, Princeton, NJ 08544 and Bell Laboratories, Lucent Technologies and AT&T Research, Murray Hill, NJ 07974, USA

Abstract. We prove global existence and uniqueness of solutions for a modified nonlinear Schrödinger equation modeling pulse propagation in an optical fiber laser. Pulse shaping is achieved by a saturable Bragg reflector: a device modeled here by non-local, nonlinear terms. Our results rely only on general model properties and are not limited to the specific case discussed. We discuss stability and bifurcation from the trivial solution and chirped soliton solutions in a special case. Numerical illustrations are provided.

AMS classification scheme numbers: 35Q55, 35B32, 35B20, 78A60, 41A60

1. Introduction

In this paper we prove global existence and uniqueness of solutions for a modified nonlinear Schrödinger equation (NLS) that models the pulse dynamics of an optical fiber laser with a saturable Bragg reflector (SBR). The SBR rather than the fiber properties is responsible for the pulse shaping and modelocking; indeed, modelocked pulses are observed in both normal (defocusing) and anomalous (focusing) dispersion fibers. While our results are motivated by a specific model, they apply to a wider class of modified NLS equations with non-local nonlinear terms.

The basic model was derived in Kutz *et al* [1], where its pulse type ("chirped soliton") solutions were investigated analytically and numerically and shown to compare well with experimental measurements (e.g. Tsuda *et al* [2, 3]; Collings *et al* [4]). The reader should refer to these papers for details; here we merely sketch the final "averaged" model equation of Kutz *et al* [Equation (13) in Ref. [1]]. The model is based on the well-known NLS equation governing pulse propagation in a suitably doped optical fiber;

$$iu_t + \frac{D}{2} u_{xx} + \alpha |u|^2 u + i\Gamma u - iG_u(t)\left(1 + \tau\partial_x^2\right) u = 0. \tag{1}$$

Here $u(x,t) \in \mathbb{C}$ is the normalized electric field envelope, D the dispersion, α the cubic (Kerr) nonlinearity, and Γ represents the intrinsic loss in the fiber [5, 6]. The parameter τ is the fiber's gain bandwidth, and

$$G_u(t) = \frac{2G_{\text{sat}}}{1 + \|u\|^2/E_{\text{sat}}} \tag{2}$$

models the saturated gain in the fiber. The L^2 energy norm $\|u\|^2$ is also written

$$E(t) = \|u\|^2 = \int_{\mathbb{R}} |u(x,t)|^2 dx \tag{3}$$

and E_{sat} is the (desired) saturation energy. The parameters $E_{\text{sat}}, G_{\text{sat}}, \alpha, \Gamma$ and τ are positive constants. The dispersion D may take either sign: $D < 0$ corresponds to normal dispersion (defocusing) and $D > 0$ to anomalous dispersion (focusing).

We have written the independent variables of Eq. (1) in standard mathematical notation, in which $x \in \Omega \subseteq \mathbb{R}$ denotes the spatial domain and t time, and solutions are defined for $t \geq 0$. In the convention of nonlinear optics, t denotes a spatial variable and x a wave frame coordinate, so that a t-independent (mathematically stationary) solution represents a wave of constant shape travelling at a uniform speed (see Refs. [5] and [6]).

The SBR relies on quantum well effects which are currently only theoretically accessible via large scale computations; consequently we model its dynamics phenomenologically. It is essentially a nonlinear mirror which reflects high amplitude incident signals with smaller loss than low amplitude ones, thus promoting pulse sharpening. It has both a fast component, which is understood to act instantaneously, and a slow component which effectively integrates over the pulse history arriving at the mirror. In the model developed by Kutz *et al* [1], these effects are collected in a nonlinear boundary condition which may be written

$$u^+ = f(|u^-|)u^-, \tag{4}$$

where $u^\pm$ are the instantaneous amplitudes of the reflected and incident waves and

$$\begin{aligned} f(|u(x,t)^-|) = & \left\{1 - \sigma_l - \sigma_f\left(1 - \frac{|u(x,t)^-|^2}{\|u^-\|_\infty^2}\right)\right. \\ & \left. -\sigma_s\left(1 - \frac{\int_{-\infty}^x |u(\xi,t)^-|^2 d\xi}{\|u^-\|^2} \exp\left[-H(x-x_m)\frac{x-x_m}{X_d}\right]\right)\right\} \\ \stackrel{\text{def}}{=} & \left\{1 - \sigma_l - \sigma_f B_f(u^-) - \sigma_s B_s(u^-)\right\}. \end{aligned} \tag{5}$$

Here $\|\cdot\|_\infty$ denotes the L^∞ (supremum) norm, $H(\cdot)$ is the Heaviside function and, in case u is continuous in x and has a unique (local) maximum, x_m denotes the location of that maximum. Intrinsic losses in the SBR are represented by σ_l and the expressions premultiplied by σ_f and σ_s respectively model the fast and slow responses of the device, with X_d characterizing the spatial "memory". Observe that the present definition of $B_f(u), B_s(u)$, which leads to shorter formulae below, differ from those of Kutz *et al* [1]. It should be noted that the σ's are typically very small positive constants, i.e., $O(10^{-2})$. As noted by Kutz *et al* [1], this model makes sense only for waves having well separated maxima, such as the chirped soliton pulse trains studied there. Moreover, as we shall see, the model as it stands does not satisfy the hypotheses of our main result and the functions $B_f(u)$ and $B_s(u)$ require minor modifications which are discussed below.

Finally, in the averaged model the nonlinear boundary (reflection) condition of Eqs. (4) and (5) is replaced by globally defined functions based on the analogy between the effect of Eq. (5) on a wave $u(x,t)$ and the effect of the linear decay over one (normalized) round-trip time

$$u_t = \beta u \qquad \Longrightarrow \qquad u(x,1) = \exp(\beta) u(x,0)$$

so that $\exp(\beta) = f(|u|)$ or, for suitably small $\sigma_l, \sigma_f, \sigma_s$

$$\beta = \ln(f(|u|)) \approx -\sigma_l - \sigma_f B_f(u) - \sigma_s B_s(u)\,.$$

The addition of this loss factor to Eq. (1) yields the final model equation:

$$u_t = \left(\tau G_u(t) + i\frac{D}{2}\right) u_{xx} + (G_u(t){-}\rho)u + i\alpha|u|^2 u - [\sigma_f B_f(u){+}\sigma_s B_s(u)]u\,, \tag{6}$$

where $\rho = \Gamma + (1{-}R) + \sigma_l$ and $1{-}R$ (≈ 0.01) has been included to account for the reflectivity R of the conventional mirror via which the laser is coupled to external devices.

Appropriate boundary conditions for Eq. (6) include periodic conditions on $-L \le x \le L$, as employed in numerical simulations, with L much greater than the spatial extent of typical structures. Alternatively, solutions may be considered on $-\infty < x < \infty$ with suitable decay at $\pm\infty$ so that L^2 and H^1 norms are finite. In either case boundary terms appearing in integrations by parts vanish identically and our results below hold. In this paper we focus on the problem defined on the whole real line with suitable decay at infinity. Modifications necessary for the periodic case should be clear.

The remainder of this paper is organized as follows. In section 2 we obtain a priori L^2 and H^1 estimates on solutions of Eq. (6) using only the following minimal assumptions on $B_f(u)$ and $B_s(u)$:

$$0 \le B_f(u)(x,t) \le 1, \quad 0 \le B_s(u)(x,t) \le 1, \tag{7}$$

which hold for the specific cases of Eq. (5). Existence and uniqueness are proven in Section 3, under the additional assumption that $B_f(u)$ and $B_s(u)$ are suitably bounded in H^1 and generate Lipschitz maps from H^1 to L^2. The latter does not appear to hold for the functions defined in Eq. (5), but minor modifications rectify this. We discuss these and other modelling issues in Section 4, and in Sections 5 and 6 we briefly discuss stability issues, existence and bifurcation of a special class of steady solutions, and we illustrate our results with some numerical simulations.

2. A priori estimates

Let $E(t) = \|u\|^2 = \int_{\mathbb{R}} u\bar{u}dx$ denote the L^2 energy ($\bar{\cdot}$ denotes the complex conjugate). We easily compute from Eq. (6)

$$\frac{1}{2}\frac{dE}{dt} = \mathrm{Re}\int_{\mathbb{R}} \bar{u}u_t\,dx = -\tau G_u(t)\|u_x\|^2 + (G_u(t)-\rho)\|u\|^2 - \int_{\mathbb{R}}(\sigma_f B_f(u) + \sigma_s B_s(u))|u|^2\,dx.$$

Using Eq. (7) this yields

$$\frac{1}{2}\frac{dE}{dt} \leq -\tau G_u(t)\|u_x\|^2 + (G_u(t)-\rho)E\,, \tag{8}$$

Recalling the definition of $G_u(t)$ in Eq. (2), inequality (8) implies that E is strictly decreasing as long as

$$E \geq E_* \stackrel{\mathrm{def}}{=} \left(\frac{2G_{\mathrm{sat}}}{\rho} - 1\right)^+ E_{\mathrm{sat}}\,, \tag{9}$$

where $a^+ = a$ if $a \geq 0$ and $= 0$ otherwise. Consequently we have the L^2 estimate

$$E(t) \leq \max\{E(0), E_*\} \stackrel{\mathrm{def}}{=} E^*, \tag{10}$$

as long as solutions exist, and, if the solution exists globally in time, we further have

$$\limsup_{t\to\infty} E(t) \leq E_*\,. \tag{11}$$

To estimate the H^1 norm of $u(t)$ we need to bound $G_u(t)$ from below using Eq. (10):

$$2\tau G_{\mathrm{sat}} \geq \frac{2\tau G_{\mathrm{sat}}E_{\mathrm{sat}}}{E_{\mathrm{sat}} + E(t)} = \tau G_u(t) \geq \frac{2\tau G_{\mathrm{sat}}E_{\mathrm{sat}}}{E_{\mathrm{sat}} + E^*} \stackrel{\mathrm{def}}{=} \frac{1}{\gamma}\,. \tag{12}$$

We rewrite Eq. (6) in integral form

$$u(t) = \mathcal{U}_{\tau G_u}(t,t_0)u(t_0) + \int_{t_0}^{t_1} \mathcal{U}_{\tau G_u}(t,s)N(u(s))ds\,, \tag{13}$$

where $\mathcal{U}_g$ denotes the semigroup obtained by solving the time dependent linear evolution equation

$$u_t = \left(g(t) + i\frac{D}{2}\right)u_{xx}, \quad u(s,x) = u_0(x) \tag{14}$$

and

$$N(u) = \Big[G_u(t) - \rho + i\alpha|u|^2 - \sigma_f B_f(u) - \sigma_s B_s(u)\Big]\, u\,. \tag{15}$$

We construct $u(t) = \mathcal{U}_g(t,s)u_0$ from the Fourier transform of Eq. (14):

$$\begin{aligned} &\hat{u}(t,k) = \phi(t,s;k)\hat{u}_0(k) \quad \text{with} \\ &\phi(t,s;k) = \exp\left\{-\left[\int_s^t g(\sigma)\,d\sigma + i\frac{D}{2}(t-s)\right]k^2\right\}. \end{aligned} \tag{16}$$

Also from Fourier transformation we have $\|u\|_{H^r}^2 = \int_{\mathbb{R}}(1+k^2)^r|\hat{u}|^2dk$, and so

$$\begin{aligned} \|\mathcal{U}_g(t,s)u\|_{H^r}^2 &= \int_{\mathbb{R}}(1+k^2)^r|\phi(t,s;k)\hat{u}_0(k)|^2dk \\ &\le \left(\sup_{k\in\mathbb{R}}(1+k^2)^{\frac{r-r'}{2}}|\phi(t,s;k)|\right)^2\int_{\mathbb{R}}(1+k^2)^{r'}|\phi(t,s;k)\hat{u}_0(k)|^2dk \\ &\le \left(\sup_{k\in\mathbb{R}}(1+k^2)^{\frac{r-r'}{2}}|\phi(t,s;k)|\right)^2\|u\|_{H^{r'}}^2\,. \end{aligned} \tag{17}$$

Recalling that from Eq. (12) that $g(t) = \tau G_u(t) \ge 1/\gamma$ and using the fact that

$$\sup_{k\in\mathbb{R}}\left\{|k|^r\exp\left[-\tfrac{(t-s)k^2}{\gamma}\right]\right\} \le C(r)\left(\frac{t-s}{\gamma}\right)^{-r/2}, \tag{18}$$

inequality (17) yields the following estimates

$$\|\mathcal{U}_{\tau G_u}(t,s)\|_{H^{r_1}\to H^{r_2}} \le C(r_1,r_2)\left(\frac{\gamma}{t-s}\right)^{\frac{r_1-r_2}{2}}; \tag{19}$$

$$\|\partial_x[\mathcal{U}_{\tau G_u}(t,s)u_0]\| \le \left(\frac{\gamma}{2e(t-s)}\right)^{1/2}\|u_0\|\,, \tag{20}$$

with $C(r,r) = 1$.

Recalling Eqs. (7) and (12), we see that $N(u)$ may be bounded as follows:

$$\begin{aligned} \|N(u)\| &\le \|[G_u - \rho - \sigma_f B_f(u) - \sigma_s B_s(u)]u\| + \|\alpha|u|^2u\| \\ &\le \mu\|u\| + \alpha\left(\int_{\mathbb{R}}|u|^6dx\right)^{1/2}, \end{aligned} \tag{21}$$

where, using $G_u \in [0, 2G_{\text{sat}}]$, $B_{s,f}(u) \in [0,1]$, and $\sigma_{s,f} \ge 0$, we have $\mu = \max\{\rho + \sigma_f + \sigma_s, 2G_{\text{sat}} - \rho\}$. The final term of Eq. (21) may be bounded using a particular case of the Gagliardo-Nirenberg estimate:

$$\|u\|_\infty^2 \le 2\|u\|\|u_x\|\,,$$

which follows by noting the identity

$$|u(x)|^2 = \int_{y=-\infty}^{x}\frac{d}{dy}\left(\exp(\alpha(y-x))|u(y)|^2\right)dy\,;\ \forall\alpha > 0, u \in H^1(\mathbb{R}),$$

and estimating the integral and letting $\alpha \to 0$. Thus we have

$$\left(\int_{\mathbb{R}} |u|^6 dx\right)^{1/2} \leq \|u\|_\infty^2 \|u\| \leq 2\|u_x\| \|u\|^2 = 2\|u_x\| E(t)\,. \tag{22}$$

Henceforth, we write the norm $\|u_x\| = D(t)$, so that, from inequalities (21) and (22) we have

$$\|N(u)\| \leq \mu E^{1/2}(t) + 2\alpha D(t) E(t). \tag{23}$$

Differentiating Eq. (13) with respect to x and taking L^2 norms, we have

$$D(t) \leq \|\partial_x[\mathcal{U}_{\tau G_u}(t,t_0)u_0]\| + \int_{t_0}^t \|\partial_x[\mathcal{U}_{\tau G_u}(t,s)N(u(s))]\| ds\,,$$

and using Eqs. (20) and (23) and the earlier L^2 bound $E(t) \leq E^*$, this yields

$$D(t) \leq \left(\frac{\gamma E^*}{2e(t-t_0)}\right)^{1/2} + \int_{t_0}^t \left(\frac{\gamma}{2e(t-s)}\right)^{1/2} [\mu (E^*)^{1/2} + 2\alpha D(s) E^*] ds\,. \tag{24}$$

This then allows us to prove the following proposition.

Proposition 1. *If u solves Eq. (6) with $\|u(t)\|^2 \leq E^*$ on $[0,T]$, then*

$$\|u_x(t)\| = D(t) \leq \left(\frac{\gamma E^*}{2et}\right)^{1/2} + \psi(\gamma,\mu,\alpha,E^*), \tag{25}$$

where ψ is independent of u and $T > 0$. In particular, if u exists globally then

$$\limsup_{t\to\infty} D(t) \leq \left(\frac{2\mu E_*}{\tau\rho}\right)^{1/2} + \frac{4\alpha E_*^{3/2}}{\tau\rho}, \tag{26}$$

with $E_* = (2G_{\rm sat}/\rho - 1)^+ E_{\rm sat}$ and $\mu = \max\{\rho + \sigma_f + \sigma_s, 2G_{\rm sat} - \rho\}$.

Proof. We consider the function $d(t,t_0) = D(t) - (\gamma E^*/[2e(t-t_0)])^{1/2}$, which by direct substitution into Eq. (24) satisfies

$$\begin{aligned} d(t,t_0) \;&\leq \int_{t_0}^t \left(\frac{\gamma}{2e(t-s)}\right)^{1/2} \left[\mu (E^*)^{1/2} + 2\alpha E^* \left(\left(\frac{\gamma E^*}{2e(s-t_0)}\right)^{1/2} + d(s,t_0)\right)\right] ds \\ &= \mu[2\gamma E^*(t-t_0)/e]^{1/2} + \pi\alpha\gamma (E^*)^{3/2}/e + \alpha (2\gamma/e)^{1/2} E^* \int_{t_0}^t \frac{d(s,t_0)}{(t-s)^{1/2}} ds. \end{aligned}$$

Observe that, if

$$\tilde{d} \geq \left[\mu(2\gamma E^*/e)^{1/2} + \alpha(8\gamma/e)^{1/2} E^* \tilde{d}/e\right] \Delta + \pi\alpha\gamma(E^*)^{3/2}/e, \tag{27}$$

then $d(t,t_0) \leq \psi$ for $t \in [t_0, t_0+\Delta^2]$, and that Eq. (27) is satisfied by, for example, $\tilde{d} = 2\alpha\gamma(E^*)^{3/2}$ and Δ sufficiently small. Via the definition of $d(t,t_0)$, this implies that

$$D(t) \leq \left(\frac{\gamma E^*}{2e(t-t_0)}\right)^{1/2} + \tilde{d} \quad \text{for } t \in (t_0, t_0+\Delta^2]. \tag{28}$$

To obtain the bound (25), simply let $\psi = \tilde{d} + (\gamma E^*/(2e\Delta^2)^{1/2}$. Then, (25) follows by choosing $t_0 = (t - \Delta^2)^+$.

Estimate (26) follows from Eq. (28) with $t - t_0 = \Delta^2$ by optimizing $\tilde{d}$ and Δ and finally (for $t \to \infty$) replacing E^* by E_* (cf. Eq. (11)) and γ by $\gamma_* = (E_{\text{sat}} + E_*)/(2G_{\text{sat}}E_{\text{sat}}) = 1/(\tau \min\{2G_{\text{sat}}, \rho\})$. To see this, let $a_1 = \mu(2\gamma_* E_*/e)^{1/2}$, $a_2 = \alpha(8\gamma_*/e)^{1/2}E_*$, $a_3 = \pi\alpha\gamma_*(E_*)^{3/2}/e$ and $a_4 = (\gamma_* E_*/(2e))^{1/2}$, so that

$$\tilde{d} \geq (a_1 + a_2\tilde{d})\Delta + a_3 \quad \text{and} \quad D(t) \leq a_4/\Delta + \tilde{d}\,.$$

Minimize $\tilde{d}$ as $\tilde{d} = (a_1\Delta + a_3)/(1 - a_2\Delta)$, and then choose $\Delta = \sqrt{a_4}/(\sqrt{a_1 + a_2a_3} + a_2\sqrt{a_4})$ to minimize the upper bound on $D(t)$. Using these values we find

$$\begin{aligned} D(t_1) &\leq 2\sqrt{(a_1 + a_2a_3)a_4} + (a_3 + a_2a_4) \leq 2(\sqrt{a_1a_4} + a_3 + a_2a_4) \\ &\leq (2\mu\gamma_* E_*)^{1/2} + 4\alpha\gamma_* E_*^{3/2}\,. \end{aligned}$$

This is the required estimate (26) if we use $E_* = 0$ for $2G_{\text{sat}} \leq \rho$ and $\gamma_* = 1/(\tau\rho)$ for $2G_{\text{sat}} > \rho$. $\square$

3. Existence and uniqueness

We shall first solve the integral form Eq. (13) of Eq. (6) locally in time using a contraction argument. This will require local Lipschitz estimates on the nonlinear part $N(u)$ of Eq. (15) as a mapping from $H^1(\mathbb{R})$ into $L^2(\mathbb{R})$ and local boundedness with values in $H^1(\mathbb{R})$. More precisely, we assume there exists a monotone function $l : [0, \infty) \to [0, \infty)$ such that

$$\|N(u) - N(v)\| \leq l(\|u\|_1 + \|v\|_1)\|u - v\|_1 \text{ and } \|N(u)\|_1 \leq l(\|u\|_1)\,, \tag{29}$$

where $\|f\|_1 = (\|f\|^2 + \|f_x\|^2)^{1/2}$ denotes the H^1 norm. As we shall see in the next section, this will require modification of the original functions $B_f(u)$ and $B_s(u)$ of Eq. (5) as proposed by Kutz *et al* [1].

Note that, via the term $\tau G_u(t)u_{xx}$, Eq. (6) is quasilinear, and so the straightforward arguments of Henry [7] do not suffice, but since $\tau G_u(t)$ is simply a scalar factor, we need not appeal to general quasilinear theory.

To begin, we require an estimate on the linear operator in Eq. (13)

Proposition 2. *Assume $g_1, g_2 \in C^0([0,T], \mathbb{R})$ with $g_j(t) \geq 1/\gamma > 0$. Then, for $r_1, r_2 \geq 0$ there exists a constant $C = C(\gamma, r_1, r_2)$ such that for all s, t with $0 \leq s < t \leq T$ we have*

$$\|\mathcal{U}_{g_1}(t,s) - \mathcal{U}_{g_2}(t,s)\|_{H^{r_1} \to H^{r_2}} \leq C\|g_1 - g_2\|_\infty \begin{cases} \min\{1, (t-s)^{\frac{r_1 - r_2}{2}}\} \text{ for } r_1 \geq r_2, \\ \max\{1, (t-s)^{\frac{r_1 - r_2}{2}}\} \text{ for } r_1 \leq r_2. \end{cases} \tag{30}$$

Proof. Let $G_j(t) = \int_s^t g_j(\xi)d\xi$, so that $|G_1(t) - G_2(t)| \leq (t-s)\|g_1 - g_2\|_\infty$ and we may assume without loss of generality that $G_1(t) \geq G_2(t)$. From Eq. (16), the difference $\mathcal{U}_{g_1} - \mathcal{U}_{g_2}$ is defined in Fourier space by

$$\exp\left(-i\frac{D}{2}(t-s)k^2\right)\left(\exp(-G_1(t)k^2) - \exp(-G_2(t)k^2)\right),$$

the modulus of which is bounded by

$$\exp\left(-G_2(t)k^2\right)\left[1 - \exp(-(G_1(t) - G_2(t))k^2)\right]. \tag{31}$$

Using $1 - \mathrm{e}^{-\alpha} \leq \alpha$ for $\alpha > 0$, Eq. (31) is in turn bounded by

$$\exp\left[-\tfrac{(t-s)k^2}{\gamma}\right]\|g_1 - g_2\|_\infty (t-s)k^2. \tag{32}$$

Multiplying by $(1+k^2)^{(r_2-r_1)/2}$ (cf. Eq. (17)), taking the supremum over $k \in I\!R$ and using Eq. (18), we find that Eq. (32) is bounded above as specified in Eq. (30). □

For initial data $u_0 \in H^1(R)$, the desired solution of Eq. (13) is a fixed point of the mapping

$$u \mapsto \mathcal{F}(u) \quad \text{with } \mathcal{F}(u)(t) = \mathcal{U}_{G_u}(t,0)u_0 + \int_0^t \mathcal{U}_{G_u}(t,s)N(u(s))\,ds. \tag{33}$$

We shall show that, under an appropriate norm $|||\cdot|||_M$ and for $T > 0$ sufficiently small, $\mathcal{F}$ is a contraction on a subset of

$$B_0(T) = \left\{u \in C^0([0,T], H^1(I\!R)) \mid \|u(t)\|_1 \leq 1 + \|u_0\|_1\right\}.$$

Theorem 3. *Consider the evolution equation*

$$\begin{aligned} u_t &= \left(\tau G_u(t) + i\frac{D}{2}\right)u_{xx} + N(u), \quad \textit{with} \\ G_u(t) &= 2G_{\text{sat}}E_{\text{sat}}/(E_{\text{sat}} + \|u\|^2) \quad \textit{and} \\ N(u) &= [G_u(t) - \rho + i\alpha|u|^2 - \sigma_f B_f(u) - \sigma_s B_s(u)]u, \end{aligned} \tag{34}$$

where $\rho, G_{\text{sat}}, E_{\text{sat}}, \tau, \sigma_s, \sigma_f$ are positive, $D, \alpha \in I\!R$, $B_f(u)(x,t), B_s(u)(x,t) \in [0,1]$ and B_f, B_s are such that $N(u) : H^1(I\!R) \to L^2(I\!R)$ is locally Lipschitz as defined in Eq. (29). Then, for each $u_0 \in H^1(I\!R)$ there is a unique solution of Eq. (34) existing globally in time and satisfying

$$\limsup_{t\to\infty}\|u(t)\|^2 \leq E_* \quad \textit{and} \quad \limsup_{t\to\infty}\|u_x(t)\|^2 \leq \left(\tfrac{2\mu E_*}{\tau\rho}\right)^{1/2} + \tfrac{4\alpha E_*^{3/2}}{\tau\rho},$$

with $E_* = (2G_{\text{sat}}/\rho - 1)^+ E_{\text{sat}}$ and $\mu = \max\{\rho + \sigma_f + \sigma_s, 2G_{\text{sat}} - \rho\}$.

Proof. We first observe that $\mathcal{F}$ maps $B_0(T)$ into itself for $T < T_0 = 1/l(\|u_0\|_1)$. This simply follows from Eqs. (29) and (33), for we have $\|\mathcal{F}(u)(t)\|_1 \leq \|u_0\|_1 + t \cdot l(\|u_0\|_1)$.

To obtain the contraction estimates, we use Eq. (30) with suitable choices of r_1, r_2 and appeal to Lipschitz continuity of G_u:

$$\|G_u(t) - G_v(t)\|_\infty \le L\|u(t) - v(t)\|_0 .$$

Temporarily letting $m_{r_1,r_2}(t-s)$ denote the (time-dependent) bounds of inequality (30), we therefore have

$$\begin{aligned}
\|\mathcal{F}(u)(t) - \mathcal{F}(v)(t)\|_{r_2} &\le \|(\mathcal{U}_{G_u}(t,0) - \mathcal{U}_{G_v}(t,0))u_0\|_{r_2} \\
&\quad + \int_0^t \|\mathcal{U}_{G_u}(t,s)N(u(s)) - \mathcal{U}_{G_v}(t,s)N(v(s))\|_{r_2} ds \\
&\le \|\mathcal{U}_{G_u}(t,0) - \mathcal{U}_{G_v}(t,0)\|_{H^1\to H^{r_2}} \|u_0\|_1 \\
&\quad + \int_0^t \|\mathcal{U}_{G_u}(t,s) - \mathcal{U}_{G_v}(t,s)\|_{H^1\to H^{r_2}} \|N(u(s))\|_1 \, ds \\
&\quad + \int_0^t \|\mathcal{U}_{G_v}(t,s)\|_{H^0\to H^{r_2}} \|N(u(s)) - N(v(s))\|_0 \, ds \\
&\le C m_{1,r_2}(t) \sup_{s\in[0,t]} \|u(s) - v(s)\|_0 \\
&\quad + \hat{C} \int_0^t m_{1,r_2}(t-s)\, ds \sup_{s\in[0,t]} \|u(s) - v(s)\|_0 \\
&\quad + \tilde{C} \int_0^t m_{0,r_2}(t-s)\, ds \sup_{s\in[0,t]} \|u(s) - v(s)\|_1 ,
\end{aligned}$$

where the constants $C, \hat{C}, \tilde{C}$ only depend on $\|u_0\|_1$, since $u, v \in B_0(T)$ and N satisfies (29). Reinserting the appropriate expression for $m_{r_1,r_2}(t)$ and setting $r_1, r_2 = 0, 1$ each in turn, we obtain the two estimates:

$$\|\mathcal{F}(u)(t) - \mathcal{F}(v)(t)\|_0 \le C_0\sqrt{t}(1+t) \sup_{s\in[0,t]} \|u(s) - v(s)\|_0 + \hat{C}_0 t \sup_{s\in[0,t]} \|u(s) - v(s)\|_1, \tag{35}$$

$$\|\mathcal{F}(u)(t) - \mathcal{F}(v)(t)\|_1 \le C_1(1+t) \sup_{s\in[0,t]} \|u(s) - v(s)\|_0 + \hat{C}_1\sqrt{t} \sup_{s\in[0,t]} \|u(s) - v(s)\|_1. \tag{36}$$

We now choose the norm $|||u|||_M = \sup_{s\in[0,t]}(M\|u(s)\|_0 + \|u(s)\|_1)$ and, using Eqs. (35)-(36), compute that

$$|||\mathcal{F}(u) - \mathcal{F}(v)|||_M \le \left(C_0\sqrt{T}(1+T) + MT\hat{C}_0 + \frac{C_1}{M}(1+T) + \sqrt{T}\hat{C}_1 \right) |||u - v|||_M .$$

Setting $M = 1/\sqrt{T}$, it follows that $\mathcal{F}$ is a contraction mapping provided $T > 0$ is taken sufficiently small. This establishes local existence and uniqueness of solutions. Global existence then follows from Proposition 1 and the a priori estimates obtained earlier, since we may continue the solution on the intervals $[nT, (n+1)T]$ inductively. This concludes the proof of the theorem. $\square$

4. A modified SBR model

As noted above, the terms $B_f(u), B_s(u)$ defined in Eq. (5) do not appear to lead to a Lipschitz continuous nonlinear function $N(u)$. In this section we present alternative definitions, close to those of Eq. (5), which result in a nonlinear part satisfying the hypotheses of Theorem 3. The remaining terms in $N(u)$, the linear part and $i\alpha|u|^2u$, are clearly locally Lipschitz, as required in Eq. (29).

Consider first the fast response term, which we regularize to read

$$B_{f,\delta}(u) = 1 - \frac{|u|^2}{\delta + \|u\|_\infty^2}\,,\ \delta \geq 0. \tag{37}$$

For $\delta = 0$, this reduces to the definition as given in Eq. (5), and for any $\delta > 0$ the mapping $u \mapsto B_{f,\delta}(u)u$ is locally Lipschitz continuous from $H^1(I\!R)$ into $L^2(I\!R)$:

$$\|B_{f,\delta}(u)u - B_{f,\delta}(v)v\| \leq \frac{2}{\delta}(\|u\|_\infty^2 + \|v\|_\infty)\|u-v\| + \frac{1}{\delta^2}(\|u\|_\infty^4 + \|v\|_\infty^4)\|u-v\|_\infty.$$

The denominator of the slow response term can be similarly regularized, but the (discontinuous) Heaviside function in the exponential weight term of Eq. (5) must also be modified. The family of functions

$$B_{s,\delta k}(u) = 1 - \frac{\int_{-\infty}^x |u(\xi)|^2 d\xi}{\delta + \|u\|^2}\exp\left[-\frac{(x-x_k)^+}{X_d}\right], \tag{38}$$

with

$$x_k = \frac{\int_{I\!R}\phi_k(x)|u(x)|^2dx}{\int_{I\!R}|u|^2dx} \quad \text{and} \quad \phi_k(x) = \begin{cases} x & \text{for } |x| \leq k \\ k\,\mathrm{sign}(x) & \text{for } |x| \geq k \end{cases}, \tag{39}$$

provides a locally Lipschitz regularization. When $k = \infty$, Eq. (39) defines the "center-of-mass" of the function $u(x)$, which in the case of (symmetric) pulse-type functions with a single maximum, coincides with that maximum. Since we are interested only in solutions decaying rapidly at infinity, Eq. (39) with $k < \infty$ sufficiently large provides a good estimate of the mass center. For non-symmetric functions, of course, x_k will not necessarily coincide with the maximum value. This raises an interesting modelling issue.

When regarded as a reflection boundary condition as in Eq. (4), the original expression in Eq. (5) for $B_s(u)$ switches on a decaying weight function as the incoming pulse's peak encounters the SBR. As shown in Kutz *et al* (Ref. [1], Figures 3-4; also see Figure 1 below), this has the effect of reducing attenuation following the peak, which is appropriate to the saturable dynamics, but the formulation applies only to functions with unique maxima. Moreover, it is not strictly causal, since the SBR must "know" when the maximum has been achieved, although the averaged non-local Eq. (6) is a well-posed, causal problem for classes of functions having a single maximum. In any event, one can reasonably suggest alternative weight functions, including that defined in Eqs. (38) and (39), which have a similar effect of reducing attenuation after half of (or some other fraction of) the energy in the pulse has passed the SBR, and which have the advantage of applying to a broader class of functions. We therefore propose $B_{s,\delta k}(u)$ of Eqs. (38) and (39), which is locally Lipschitz for all $\delta > 0$ and $k < \infty$, as plausible replacements for $B_s(u)$ of Eq. (5).

To compare the effects of the regularized functions $B_{f,\delta}(u)$, $B_{s,\delta k}(u)$ with those of the original model of Kutz *et al* [1], we follow the convention of that paper and plot the change in reflectivity due to an incident pulse of the form $u^-(x,t) = \eta \operatorname{sech} wx \exp(i\theta t)$ (cf. Sec. 5 below). We have

$$\Delta_f = 1 - f(|u^-|, \sigma_s = \sigma_l = 0) = \sigma_f \left(1 - \frac{|u^-|^2}{\delta + \|u\|_\infty^2}\right) = \sigma_f \left(1 - \frac{\eta^2 \operatorname{sech}^2 wx}{\delta + \eta^2}\right)$$

$$\Delta_s = 1 - f(|u^-|, \sigma_f = \sigma_l = 0) = \sigma_s \left(1 - \frac{\int_{-\infty}^x |u^-|^2 dx}{\delta + \|u\|^2} \exp\left[-\frac{(x - x_k)^+}{X_d}\right]\right)$$

$$= \sigma_s \left(1 - \frac{\eta^2 (1 + \tanh wx)}{w(\delta + 2\eta/w^2)} \exp\left[-\frac{x^+}{X_d}\right]\right).$$

In Figure 1 we plot the quantities

$$\sigma_f - \Delta_f = \frac{\operatorname{sech}^2 wx}{1 + \delta/\eta^2} \quad \text{and} \quad \sigma_s - \Delta_s = \frac{(1 + \tanh wx)}{2 + \delta w/\eta^2} \exp\left[-\frac{x^+}{X_d}\right] \tag{40}$$

for several values of δ and (including $\delta = 0$, the case of Eq (5)) and taking $\eta = w = 1, X_d = 10$. Note that, for symmetric functions such as u^- above, the expression in Eq. (40) and Δ_s in general, is independent of the parameter k.

5. Linear stability and an exact pulse solution

The *a priori* L^2 estimates (8)-(9) show that the trivial solution $u \equiv 0$ is globally attracting provided $2G_{\text{sat}} < \rho$. A simple stability computation on the linearized equation

$$u_t = \left(2\tau G_{\text{sat}} + i\frac{D}{2}\right) u_{xx} + (2G_{\text{sat}} - \rho_\sigma) u,$$

with $\rho_\sigma \overset{\text{def}}{=} \rho + \sigma_f + \sigma_s$, yields the dispersion relation

$$\lambda(k) = -\left(2\tau G_{\text{sat}} + i\frac{D}{2}\right) k^2 + (2G_{\text{sat}} - \rho_\sigma),$$

where we sought solutions of the form $u = \exp(\lambda t + ikx)$. This then implies that $u \equiv 0$ is linearly stable if and only if

$$2G_{\text{sat}} < \rho_\sigma = \rho + \sigma_f + \sigma_s,$$

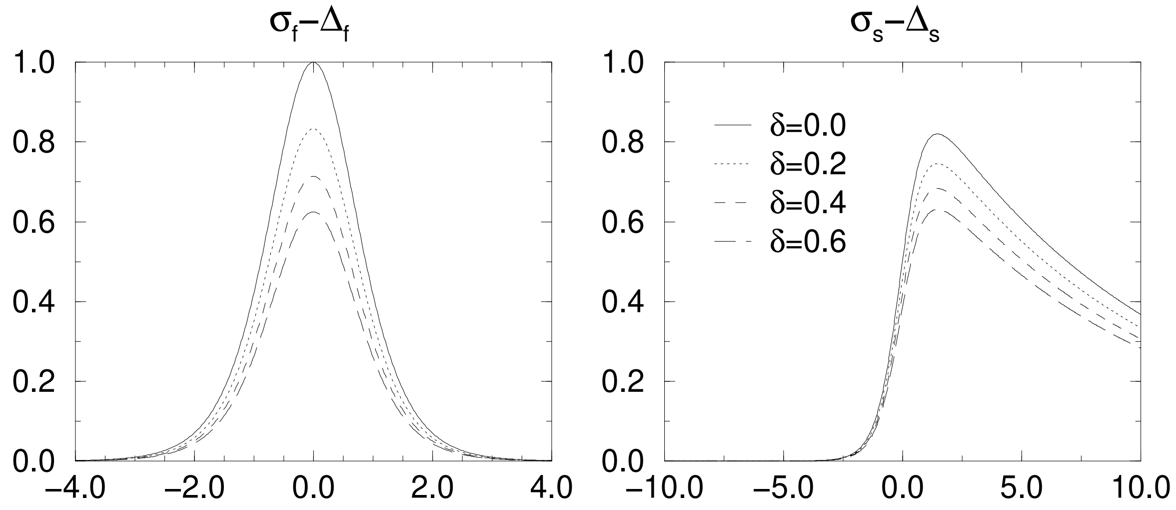


Figure 1. Reflectivity of SBR due to a hyperbolic secant incident pulse with the regularization δ. In the left figure, the reflectivity due to the instantaneous reflection is given while in the right figure the integrated response is given.

and that the "critical" mode occurs at $k = 0$. We conjecture that, for a general $B_f(u), B_s(u)$ with $B_f(0) = B_s(0) \equiv 1$, pulse-type solutions bifurcate at $2G_{\text{sat}} = \rho_\sigma$. This may be verified in the special case of Eq. (5) with $\sigma_s = 0$, for then, as shown in Kutz *et al* [1], an exact "chirped soliton" solution of the form

$$u(x,t) = \eta(\operatorname{sech}\omega x)^{1+iA}\exp(i\theta t) \tag{41}$$

can be found, where η, ω, A, and θ are real parameters.

Before doing this calculation we derive a necessary condition for the stability of any pulse solution $u_p(x,t) = U(x)\exp(i\theta t)$. Consider a solution in the form $u(x,t) = u_p(x,t) + v(x,t)$ where $v(x,0)$ is assumed to be very small and to have a compact support far away from the center of the pulse. Thus, up to small perturbations from the tails of the pulse, the linearization in v is the constant coefficient problem

$$v_t = (\tau G_U + i\frac{D}{2})v_{xx} + (G_U - \rho_\sigma)v,$$

where $G_U = 2G_{\text{sat}}E_{\text{sat}}/(E_{\text{sat}} + \int_{I\!R}|U(x)|^2dx)$. As above the linear dispersion relation gives

$$\lambda(k) = -\left(\tau G_U + i\frac{D}{2}\right)k^2 + (G_U - \rho_\sigma),$$

and stability can only hold if

$$G_U < \rho_\sigma. \tag{42}$$

Note that $G_U < 2G_{\text{sat}}$ for nontrivial pulses such that the trivial solution may be unstable while a nontrivial pulse is stable. This is a special feature of our model which is due to the nonlocality of the PDE.

We now give a bifurcation analysis for pulse solutions for our model in the case $\sigma_s = 0$. However, we only consider pulses in the form given in Eq. (41). (Using methods in [11, 12] it should be possible to show the existence of other pulse solutions such as multipulses.) Substitution of Eq. (41) into Eq. (6), with the modified definition of $B_{f,\delta}(u)$ of Eq. (37), yields a system of four equations which must be solved for the parameters η, ω, A, and θ. Using the abbreviation $g = G_u = 2G_{\text{sat}}E_{\text{sat}}/(E_{\text{sat}} + \|u\|^2)$, this system may be reduced to two coupled equations

$$2\alpha\eta^2\left[\tau g(1-A^2) - DA\right] + (g - \rho_\sigma)Q(g, A) = 0 \tag{43}$$

$$\left[\alpha(\delta + \eta^2)\left(2\tau g(A^2-2) + 3DA\right) + \sigma_f Q(g, A)\right]Q(g, A) = 0 \tag{44}$$

$$\text{where } Q(g, A) = 6\tau gA + D(2-A^2), \tag{45}$$

for the unknowns A and η, where g is considered as a parameter. The desired values ω and θ may be found from

$$\begin{aligned}\omega^2 &= 2\alpha\eta^2/Q(g, A)\\ \theta &= \omega^2\left[2\tau gA + \frac{D}{2}(1 - A^2)\right],\end{aligned}$$

and the gain G_{sat} consistent with this solution is

$$g = \frac{2G_{\text{sat}}E_{\text{sat}}}{E_{\text{sat}} + \|u\|^2} = \frac{2G_{\text{sat}}}{1 + \frac{2\eta^2}{\omega E_{\text{sat}}}}.$$

This shows that solutions can only exist when $\alpha Q(g, A) > 0$. Moreover,

$$G_{\text{sat}} = \frac{g}{2}\left(1 + \sqrt{\frac{2Q(g, A)}{\alpha}}\,\frac{|\eta|}{E_{\text{sat}}}\right). \tag{46}$$

Note that, unlike in Kutz *et al* [1], here we do *not* assume that the solution is saturated in equilibrium ($E_{\text{sat}} = \|u\|^2 = 2\eta^2/\omega$), rather we take g as a given parameter and seek η, ω, A, θ as well as G_{sat} as functions of g. The remaining parameters $D, \alpha, \rho_\sigma, \tau$, etc. are kept fixed. This enables us to study the whole branch of chirped solutions.

From Eqs. (43)+(44) we see that $(\eta, A) = (0, A_{1,2}(g))$ form two branches of trivial solutions if $Q(g, A_{1,2}(g)) = 0$. From these branches no bifurcations occur. However, for $g = \rho_\sigma$ there are two other trivial solutions $(\eta, A) = (0, A_{3,4})$ where $A_{3,4}$ solve $\alpha\delta(2\tau\rho_\sigma(A^2-2) + 3DA) + \sigma_f Q(\rho_\sigma, A) = 0$. These solutions can be continued by the implicit function theorem. If $\alpha Q(\rho_\sigma, A_j) > 0$, then we find a solution branch with

$$g = G_u = \tilde{g}_j(\eta) = \rho_\sigma + \frac{2\alpha\left(\tau\rho_\sigma(A_j^2 - 1) + DA_j\right)}{Q(\rho_\sigma, A_j)}\eta^2 + \mathcal{O}(\eta^4). \tag{47}$$

Inserting this expansion into Eq. (46) we find $G_{\rm sat}$ as a function of η,

$$G_{\rm sat} = \widetilde{G}(\eta) = \frac{\rho_\sigma}{2}\left(1 + \sqrt{\frac{2Q(\rho_\sigma, A_j)}{\alpha}\,\frac{|\eta|}{E_{\rm sat}}}\right) + \mathcal{O}(\eta^2).$$

This shows that any bifurcating branch is supercritical, namely $G_{\rm sat} > \rho_\sigma/2$. However, we cannot easily conclude stability from this fact, as is usually done in bifurcation theory when either Liapunov-Schmidt or center manifold methods are employed. In the present case, the bifurcation of the pulses has to be understood in the sense of bifurcation out of the purely continuous spectrum of the trivial state $u \equiv 0$. Moreover, our necessary condition (42) now reads $g < \rho_\sigma$, and our expansion (47) gives instability whenever $\tau\rho_\sigma(A_j^2 - 1) + DA_j > 0$, since $\alpha Q(\rho_\sigma, A_j) > 0$. We hope to study the exact stability properties of these solutions in subsequent work.

In Figure 2 we show the resulting super-critical bifurcation curve generated by plotting the growth of the L^2 norm ($\|u\| = \eta\sqrt{2/\omega} \sim \eta^{1/2}$ for small η) versus the bifurcation parameter $G_{\rm sat}$. We assumed the physically reasonable parameter values: $D = \alpha = 1, E_{\rm sat} = 2, \rho = 0.2, \sigma_f = \tau = \delta = 0.1, \sigma_s = 0$. Note that the bifurcation occurs at the predicted value of $G_{\rm sat} = \rho_\sigma/2 = 0.15$ and that the growth of the L^2 norm is bounded from above by the *a priori* bound $\|u\|^2 \leq E_* = (2G_{\rm sat}/\rho - 1)^+ E_{\rm sat}$ of Eq. (9). Moreover, the curve lies in the region $G_u < \sigma_\rho$, and hence the pulses may be stable, as one expects from the "physical" behavior of the SBR, which attenuates small amplitude regions away from the center of the pulse.

6. Numerical simulations

We simulate the governing equation (6) using a fourth-order Runge-Kutta method in time and a filtered pseudospectral method in space [8]. This procedure combines the advantages of split-step methods [9] and explicit Runge-Kutta methods [10] giving a relatively simple fourth-order scheme with improved numerical stability properties.

Since the asymptotic L^2 and H^1 estimates derived in the foregoing sections are independent of the initial conditions, we consider pulse formation starting from a noisy initial field. We assume the following parameter values: $\alpha = 1, E_{\rm sat} = 2, G_{\rm sat} = 0.5, \tau = \sigma_f = \sigma_s = \delta = 0.1, \rho = 0.2, X_d = 5$. For illustrative purposes, we consider solutions in both the focusing ($D = 1$) and defocusing ($D = -1$) parameter regimes. In the focusing case ($D = 1, \sigma_s > 0$), it is reasonable to expect a localized nonlinear bound state similar to Eq. (41) to persist as an attractor when $\sigma_s > 0$ due to the underlying soliton dynamics of the NLS equation. However, in the defocusing case ($D = -1, \sigma_s > 0$), localized bound states are no longer solutions to the NLS equation. Despite this, localized bound states are shown to be supported. In Figure 3 we show the evolution dynamics of both the focusing and defocusing regimes for the parameters given above. Note that in both

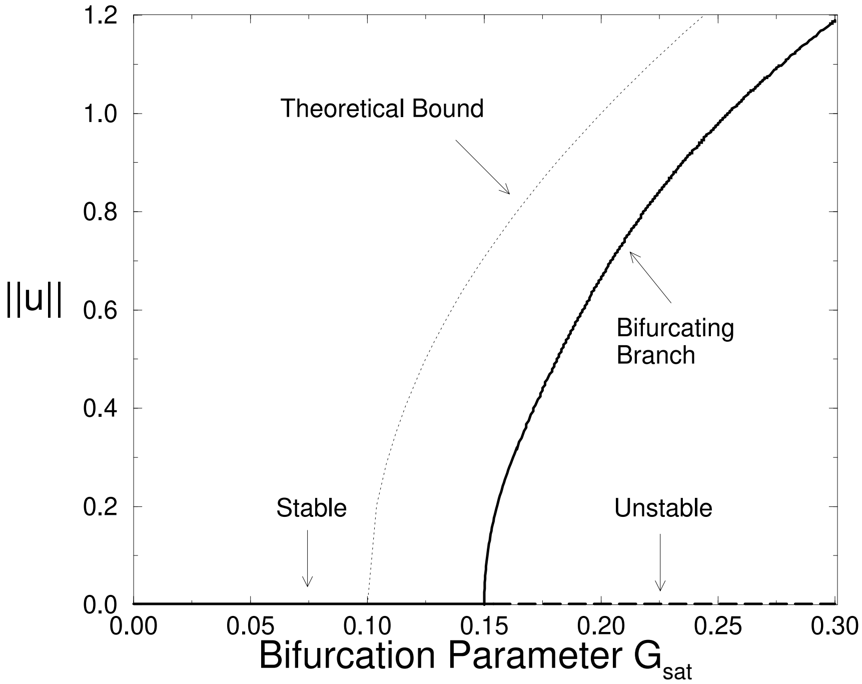
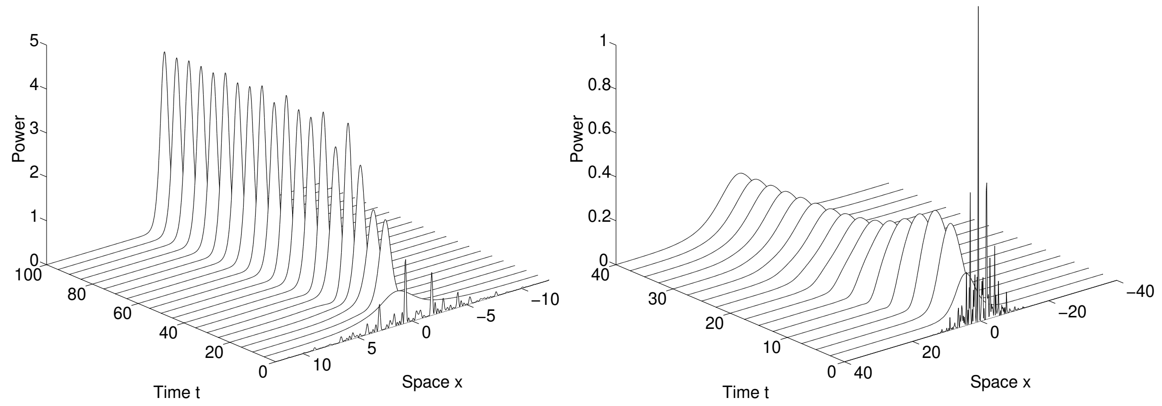


Figure 2. Super-critical bifurcation curve generated from numerical solutions of Eqs. (43) – (45). The physically reasonable parameter values used are $D = \alpha = 1, E_{\mathrm{sat}} = 1, \rho = 0.2, \sigma_f = \tau = \delta = 0.1, \sigma_s = 0.0$. The bifurcating branch of solutions is below the theoretically predicted bound $E_* = (2G_{\mathrm{sat}}/\rho - 1)^+ E_{\mathrm{sat}}$ given in Eq. (9).

regimes, and starting from identical noisy initial data, the evolution equation (6) gives rise to stable, localized solutions, with widths similar to those predicted by the theory in the case $\sigma_s = 0$ (direct comparisons for a slightly different case are made in Kutz *et al* [1]).

We can also keep track of the L^2 and H^1 norm evolutions as the pulse evolves to the attracting pulse solutions of Figure 3. In Figure 4 we depict the evolutions of $\|u\|$ and $\|u_x\|$ for both the focusing and defocusing cases of Figure 3. Note that in each case, initial transient oscillations settle onto fixed values for the L^2 and H^1 norms. For values of G_{sat} below the super-critical bifurcation point of the last section, the L^2 and H^1 norms both decay to zero exponentially.

In Figure 5 we show the evolution of the same initial data as considered in Figures 3 and 4 with its associated L^2 and H^1 dynamics for $G_{\mathrm{sat}} = 1$. In this case, we find that the solution given by Eq. (41) is unstable for sufficiently large values of the bifurcation parameter G_{sat}. Figure 5a shows that the evolution gives rise to two distinct propagating pulses. This structure persists for long times. However, we only show a limited portion



(a) $D = 1$ (b) $D = -1$

Figure 3. Evolution of the power ($|u|^2$) for an arbitray (noisy) initial condition (identical for both (a) and (b)) given the parameter values $\alpha = 1, E_{\rm sat} = 2, G_{\rm sat} = 0.5, \rho = 0.3, \tau = \sigma_f = \sigma_s = \delta = 0.1$. In (a), the pulse evolution is depicted for the focusing regime. In (b), the defocusing regime is shown. Note the convergence of the initial data onto the attracting state given by Eq. (41).

of the evolution (up to $t = 100$) due to the computational expense of the long run times. Note that the H^1 norm continues to oscillate even when the L^2 norm has nearly settled to a fixed value. The range of $G_{\rm sat}$ for which the solution (41) is stable is currently under investigation.

7. Conclusion

We have proved existence and uniqueness of solutions for a class of non-local, damped, quasi-linear modifications of the nonlinear Schrödinger equation. Such equations arise in an asymptotic limit of models for a laser with a saturable Bragg reflector (see Kutz *et al* [1]). Our proof uses *a priori* L^2 and H^1 estimates, and is completed by showing that the corresponding integral equation defines a contraction mapping. For this, the non-local nonlinear terms must satisfy Lipshitz estimates, which require mild regularization of certain terms in the original model of Kutz *et al* [1].

We also discuss linear stability of the trivial solution and bifurcations to a special class of exact solutions in a limiting case, and we illustrate our results by numerical simulations, which reveal stable pulse type solutions for both focussing and defocussing equations, over a considerable parameter range.

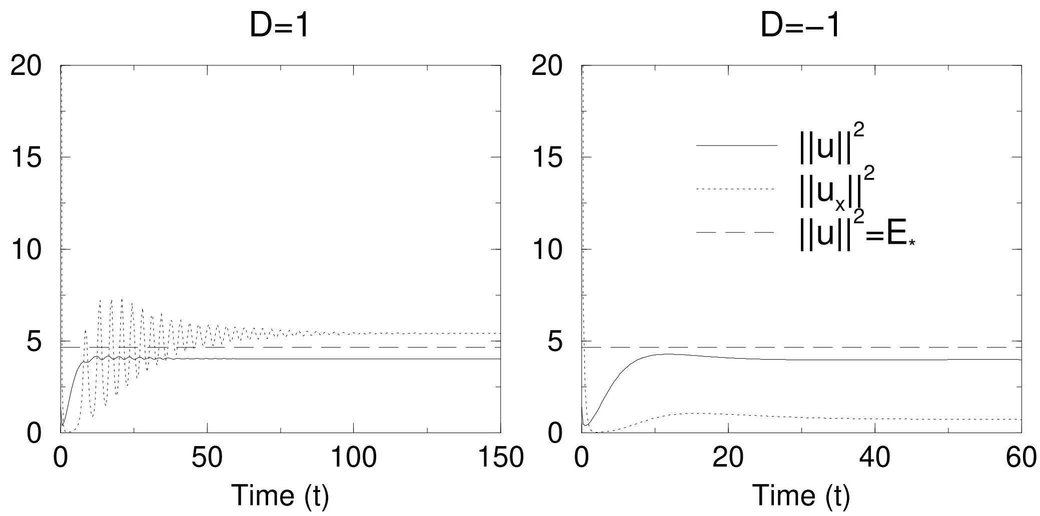
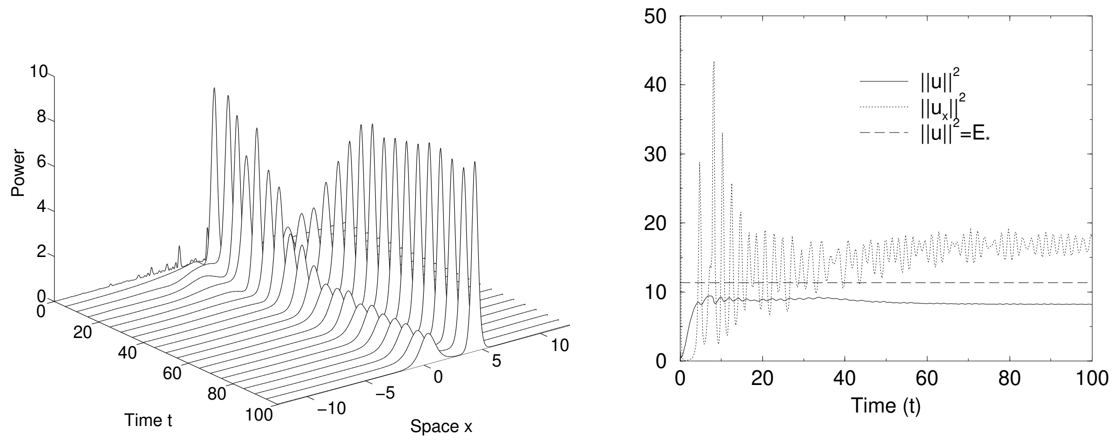


Figure 4. Evolution of the $||u||^2$ and $||u_x||^2$ norms for the arbitrary (noisy) initial condition given in Figure 3 with the parameter values $\alpha = 1, E_{\text{sat}} = 2, G_{\text{sat}} = 0.5, \rho = 0.3, \tau = \sigma_f = \sigma_s = \delta = 0.1$. Initially, the $||u_x||^2$ norm is $\approx$100 due to the noisy initial data. The L^2 norm stays below the theoretically predicted bound $||u||^2 = E_* = (2G_{\text{sat}}/\rho{-}1)E_{\text{sat}}$, and the solution convergences onto a steady-state.



(a) (b)

Figure 5. Evolution of the power ($|u|^2$) and $||u||^2$ and $||u_x||^2$ norms for an arbitrary (noisy) initial condition given the parameter values $D = \alpha = 1, E_{\text{sat}} = 2, G_{\text{sat}} = 1, \rho = 0.3, \tau = \sigma_f = \sigma_s = \delta = 0.1$. In (a), the pulse evolution is depicted with pulse splitting occuring. In (b), the corresponding norm evolutions are given showing slowly decaying oscillatory behavior of the $||u_x||^2$ norm while the L^2 norm settles to a fixed value.

Acknowledgments

We would especially like to thank Keren Bergman, Wayne Knox and Michael Weinstein for invaluable discussions related to the modelling efforts in this work. A. Mielke acknowledges partial support from Deutsche Forschungsgemeinschaft under Grant no. Mi 459/2–2 and is grateful to the Program of Applied and Computational Mathematics at Princeton University for the hospitality during the stay in which this reasearch was initiated. P. Holmes and J. N. Kutz acknowledge support from a National Science Foundation University-Industry Postdoctoral Fellowship (DMS-9407473). P. Holmes acknowledges additional support from the Department of Energy (DE-FG02-95ER.25238.A000).

References

[1] Kutz J N, Collings B C, Bergman B, Tsuda S, Cundiff S, Knox W, Holmes P and Weinstein M 1997 Modelocking pulse dynamics in a fiber laser with a saturable Bragg reflector *J. of the Opt. Soc. of Am. B* (to appear)

[2] Tsuda S, Knox W H, Zyskind J L, Cunningham J E, Jan W Y and Pathak R 1996 Paper CFD2, Conference on Lasers and Electro-Optics, Anaheim, CA

[3] Tsuda S, Knox W H, de Souza E A, Jan W Y and Cunningham J E 1995 Low-loss intracavity AlAs/AlGaAs saturable Bragg reflector for femtosecond modelocking in solid-state lasers *Opt. Lett.* **20** 1406

[4] Collings B C, Stark J B, Tsuda S, Knox W H, Cunningham J E, Jan W Y, Pathak R and Bergman K 1996 Saturable Bragg reflector self-starting passive mode locking of a Cr^{4+}:YAG laser pumped with a diode-pumped Nd:YVO_4 laser *Opt. Lett.* **21** 1171

[5] Newell A C and Moloney J V 1992 *Nonlinear Optics* (Redwood City: Addison-Wesley)

[6] Agrawal G P 1989 *Nonlinear Fiber Optics* (San Diego: Academic Press, Inc.)

[7] Henry D 1981 *Geometric Theory of Semilinear Parabolic Equations* Springer Lecture Notes in Mathematics **840** (New York: Springer-Verlag)

[8] Hou T Y, Lowengrub J S and Shelley M J 1994 Removing the stiffness from interfacial flows with surface tension *J. Comput. Phys.* **114** 312

[9] Fleit M D and Fleck J A 1978 Light propagation in graded-index optical fibers *Appl. Opt.* **17** 3990

[10] Taha T R and Ablowitz M J 1984 Analytical and numerical aspects of certain nonlinear evolution equations. II. Numerical, nonlinear Schrödinger equation *J. Comput. Phys.* **55** 203

[11] Maier-Paape S and Kapitula T 1996 Spatial dynamics of time-periodic solutions for the Ginzburg-Landau equation. *Zeits. angew. Math. Physik (ZAMP)* **47** 265-305.

[12] Afendikov A and Mielke A 1997 Multibump solutions for the stationary complex Ginzburg-Landau equation in one dimension. In preparation.