Quasiconvexity at the boundary and a simple variational formulation of Agmon's condition

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1. Introduction

Consider a hyperelastic body subjected to dead loads which is in an equilibrium state u_0, i.e. the deformation u_0 solves the Euler–Lagrange equations associated to the energy functional

$$I(u) = \int_\Omega W\big(\nabla u(x)\big)\,dx - \int_\Omega f(x)\cdot u(x)\,dx - \int_\Gamma g(x)\cdot u(x)\,da.$$

It is of interest to know whether u_0 is a weak or strong local minimizer. A major aim in the calculation of variations is to provide necessary and sufficient conditions for these minimizing properties. The one–dimensional case $\Omega \subset \mathbb{R}$ was essentially solved by Weierstraß (see [GH96, pp.332]); and there is some work for higher dimensions in the case that u has to satisfy Dirichlet boundary conditions on all of $\partial\Omega$, see [Si89, Zh90]. The present work is devoted to the case that there is a part Γ of $\partial\Omega$ where u is allowed to vary freely. This is motivated from many applications, as for instance the buckling problem in nonlinear elasticity, where on some part of the boundary only traction conditions (Neumann boundary conditions) are prescribed.

Meyers [Me65] showed that Morrey's quasiconvexity of W at $F_0 = \nabla u_0(x_0)$ for interior points x_0 is necessary for a deformation to be a strong local minimizer. Ball and Marsden [BM84] generalized this necessary condition to points x_0 in the free boundary part Γ. This condition involves in addition to the energy density W and the gradient F_0 also the normal vector ν at $x_0 \in \Gamma \subset \partial\Omega$. More precisely, W is called quasiconvex at the boundary in (F_0, ν) if there exists $q \in \mathbb{R}^n$ such that

$$\int_{B_\nu} W\big(F_0 + \nabla\varphi(x)\big)\,dx \geq \int_{B_\nu} W(F_0)\,dx + \int_{\Gamma_\nu} q\cdot\varphi(x)\,da$$

for every $\varphi \in W^{1,\infty}_{\Gamma_\nu}(B_\nu, \mathbb{R}^n)$ where $B_\nu = \{y \in \mathbb{R}^n : |y| \leq 1,\ y\cdot\nu < 0\}$ and the subscript Γ_ν for the Sobolev space indicates that the functions φ vanish at the boundary part $\partial B_\nu \backslash \big(\partial B_\nu \cap \{\, y : y\cdot\nu = 0\,\}\big)$. For a general discussion of this concept and the notion of polyconvexity at the boundary we refer to [Sp96].

Here we restrict our attention solely to quadratic functionals. For the general case the expansion $W(F_0 + B) = W(F_0) + S_0 : B + C_0[B]:B + \mathcal{O}(|B|^3)$ with $S_0 = \partial W(F_0)/\partial F$ and $C_0 = \partial^2 W(F_0)/\partial^2 F$ shows that a necessary condition for weak and strong local minimizers is the nonnegativity of the quadratic form

$$Q(\varphi) = \int_{B_\nu} C_0\big[\nabla\varphi(x)\big]{:}\nabla\varphi(x)\,dx, \quad \varphi \in W^{1,2}_{\Gamma_\nu}(B_\nu, \mathbb{R}^n).$$

The relevance of this fact was first investigated by H.C. Simpson and S.J. Spector in [SS87] and further developed in [SS89]. They also extensively discuss necessary and sufficient algebraic conditions on the coefficients of C_0 such as the Legendre-Hadamard condition, the complementing condition, and Agmon's condition. Our work proceeds in the same spirit and allow us to provide simpler conditions guaranteeing positivity of Q.

Clearly, it is necessary that C_0 is rank–one convex (also called Legendre-Hadamard condition): $C_0[a\otimes b]{:}(a\otimes b) \geq \gamma|a|^2|b|^2$ with $\gamma \geq 0$. In the strict case $\gamma > 0$, Agmon's condition is exactly the additional condition to make these two conditions necessary and sufficient for the nonnegativity of Q (see also [SS91]). In [SS89] it is shown that in the case $\gamma = 0$ the additional condition (S3) given below is needed.

Our aim is to replace Agmon's condition by a simpler one which is purely algebraic and, moreover, is variational. This means we have to prove positive semidefiniteness of certain Hermitian matrices. For given C_0 and ν our condition can be formulated using the matrices M, N_τ, and P_τ defined via

$$Ma := C_0[a\otimes\nu]\nu, \quad N_\tau a := C_0[a\otimes\tau]\nu, \quad P_\tau a := C_0[a\otimes\tau]\tau.$$

Our first main result reads as follows:

The quadratic functional Q is nonnegative if and only if for all $\tau \in \mathbb{R}^n$ which are perpendicular to ν there exists a complex $H(\tau) \geq 0$ such that

$$E_{\tau,H(\tau)} = \begin{pmatrix} M & -iN_\tau + H(\tau) \\ iN_\tau^T + H(\tau) & P_\tau \end{pmatrix} \geq 0. \tag{1.1}$$

If all $H(\tau) \geq 0$ can be chosen such that $E_{\tau,H(\tau)} > 0$ for all τ, then Q is coercive, that is there exists $\delta > 0$ such that $Q(\phi) \geq \delta \int_{B_\nu} |\nabla\varphi|^2 dx$ for all $\phi \in W^{1,2}_{\Gamma_\nu}(B_\nu, \mathbb{R}^n)$.

We continue to use the notation $X \geq 0$ and $Y > 0$ to indicate that $X, Y \in \mathbb{C}^{n\times n}$ are hermitian, X positive semidefinite, and Y positive definite.

Note that the algebraic structure of condition (1.1) is very direct. In contrast, Agmon's condition means solving the differential equation $-M\ddot{z} + i(N_\tau + N_\tau^T)\dot{z} + P_\tau z = -\alpha^2 z$ and showing that all solutions bounded on $[0,\infty)$ and satisfying $M\dot{z}(0) - iN_\tau z(0) = 0$ are trivial. In Section 3 we give all definitions properly and show that rank-one convexity as well as the third condition in [SS89] (see (S3) below) are easy consequences of $E_{\tau,H(\tau)} \geq 0$. The usage of Agmon's condition is avoided completely as we stay in the framework of variational problems. The matrix $H(\tau)$ can be thought of as playing the role of a null Lagrangian, cf. eqn. (4.3) below.

The second part of our main theorem is the relation of (1.1) to Riccati inequalities and equations for hermitian matrices which are well studied in control theory, see [Sc91, LR95] for standard references. If M is invertible then $E_{\tau,H(\tau)} \geq 0$ is equivalent to

$$\mathcal{R}_\tau(H(\tau)) \leq 0, \quad \text{where} \quad \mathcal{R}_\tau(X) = (X + iN_\tau^T)M^{-1}(X - iN_\tau) - P_\tau.$$

An surprising property of Riccati inequalities is that $\mathcal{R}_\tau(X) \leq 0$ has a solution $X \geq 0$ if and only if the Riccati equation $\mathcal{R}_\tau(Y) = 0$ has a solution $Y \geq 0$. This fact is known in control theory and was found independently in [Sp96] using variational methods, see Lemma 4.3. Thus, condition (1.1) is in fact equivalent to solving the algebraic equation $\mathcal{R}_\tau(Y) = 0$ and testing whether any of the solutions Y satisfies $Y \geq 0$.

In Section 5 we apply our criterion to several quadratic forms which are of interest in linearized elasticity theory. First we give a complete characterization that the elasticity tensor C_0 of a general isotropic material in the plane is quasiconvex at the boundary with $\nu = (1,0)^T$. Moreover, we provide an example where the conditions on the coefficients of C_0 vary continuously if ν is varied.

For isotropic materials in $\mathbb{R}^n$ with Lamé constants λ and μ we recover a result from [SS87]. Quasiconvexity at the boundary to any ν holds if and only if $\mu \geq 0$ and $\lambda + \mu \geq 0$. This condition is strictly stronger than rank-one convexity ($\mu \geq 0$, $\lambda + 2\mu \geq 0$) and, for $n \geq 3$, strictly weaker than convexity ($\mu \geq 0$, $n\lambda + 2\mu \geq 0$). Our analysis of the anisotropic energy density

$$W(F) = \frac{\mu}{2} F{:}F + h(\det F) + \frac{\omega}{2}|Fe_1|^2 + \frac{\omega}{2}|F^T e_1|^2,$$

considered in [SS89], manifests the fact that our conditions are simpler to analyze. In fact, we are able to show that the conditions for quasiconvexity at the boundary are different if ν is chosen parallel or perpendicular to the direction of anisotropy e_1. Finally we present an example which shows that in finite elasticity quasiconvexity at the boundary is a condition on W which should not be asked for all (F_0, ν). For the well-accepted stored energy density $W(F) = 1/2\, F{:}F + h(\det F)$ (with h convex and $h(\lambda) \to +\infty$ for $\lambda \to 0+$) we show that W is not quasiconvex at the boundary in $(\sqrt{\lambda}\mathbb{I}, \nu)$ whenever λ is sufficiently small.

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2. Preliminaries. Local minimizers.

Let $\mathbb{K}^{n\times n}$ be the space of all $n \times n$ matrices over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. We write $F{:}G = \mathrm{tr}(FG^T)$, where G^T denotes the transpose of the matrix G and tr is the trace. By $\mathbb{R}^{n\times n}_+$ we shall mean the subset of $\mathbb{R}^{n\times n}$ consisting of all matrices with positive determinant.

In the following, Ω will denote a domain (i.e. nonempty, open and connected subset) of $\mathbb{R}^n$ whose boundary $\partial\Omega$ is piecewise continuously differentiable. Furthermore, we suppose that Γ is a nonempty relatively open subset of $\partial\Omega$. We denote by $L^p(\Omega, \mathbb{R}^n)$ and $W^{1,p}(\Omega, \mathbb{R}^n)$ with $1 \leq p \leq \infty$ the usual Lebesgue and Sobolev spaces of $\mathbb{R}^n$ valued functions. The closure of $C_0^\infty(\Omega, \mathbb{R}^n)$ in $W^{1,p}(\Omega, \mathbb{R}^n)$ is denoted by $W_0^{1,p}(\Omega, \mathbb{R}^n)$.

We identify Ω with the reference configuration of a hyperelastic homogeneous n-dimensional body whose stored energy function is W. Throughout the whole paper we will assume that $W : \mathbb{R}^{n\times n} \to \mathbb{R} \cup \{+\infty\}$ is continuous and that W restricted to $\mathbb{R}^{n\times n}_+$ takes finite values. The derivatives

$$S(F_0) = \frac{\partial W}{\partial F}(F_0) \in \mathbb{R}^{n\times n}, \quad C(F_0) = \frac{\partial^2 W}{\partial F^2}(F_0) \in \mathrm{Lin}(\mathbb{R}^{n\times n}, \mathbb{R}^{n\times n})$$

are called the *Piola-Kirchhoff stress tensor* and the *elasticity tensor*. We may regard $C(F_0)$ as a bilinear form on $\mathbb{R}^{n\times n}$ via $C(F_0)[G, H] = C(F_0)[G]{:}H$.

By a *deformation* we mean a map $u : \Omega \to \mathbb{R}^n$ such that $u \in W^{1,p}(\Omega, \mathbb{R}^n)$ and the associated gradient satisfies $\nabla u(x) \in \mathbb{R}^{n\times n}_+$ for almost every $x \in \Omega$. For a deformation u the total energy is given by

$$I(u) = \int_\Omega W\big(\nabla u(x)\big)\, dx - \int_\Omega f(x)\cdot u(x)\, dx - \int_\Gamma g(x)\cdot u(x)\, da, \tag{2.1}$$

where "$\cdot$" denotes the usual inner product of $\mathbb{R}^n$, $f \in L^q(\Omega, \mathbb{R}^n)$ the external body force density, and $g \in L^q(\Gamma, \mathbb{R}^n)$ the surface traction density ($\frac{1}{p} + \frac{1}{q} = 1$).

We are interested in deformations u_0 that minimize the total energy locally on the set of admissible functions $u_0 + W^{1,p}_\Gamma(\Omega, \mathbb{R}^n)$, where

$$W^{1,p}_\Gamma(\Omega, \mathbb{R}^n) := \big\{\varphi \in W^{1,p}(\Omega, \mathbb{R}^n) \,:\, \varphi = 0 \text{ on } \partial\Omega\backslash\Gamma \text{ in the sense of trace}\big\}.$$

Precisely, a deformation $u_0 \in W^{1,p}(\Omega, \mathbb{R}^n)$ is said to be a *weak* (resp. *strong*) *local minimizer* of I if there exists a $\varepsilon > 0$ such that $I(u_0 + \varphi) \geq I(u_0)$ for every $\varphi \in W^{1,p}_\Gamma(\Omega, \mathbb{R}^n)$ and $\|\varphi\|_{W^{1,\infty}} < \varepsilon$ (resp. $\|\varphi\|_{L^\infty} < \varepsilon$). Obviously every strong local minimizer is a weak local minimizer.

First we recall briefly some definitions and results that are related closely to local minimizers. In this paper we are interested in necessary conditions which apply at an free boundary point. For conditions which are necessary in interior points see e.g. [Ba77].

We now introduce a special class of domains. By a *standard boundary domain with normal* $\nu \in \mathbb{R}^n$ we mean a bounded domain $D_\nu \subset \mathbb{R}^n$ that satisfies the following condition: D_ν is contained in the half-space $H_{\nu,a} := \{x \in \mathbb{R}^n \,:\, x \cdot \nu < a\}$ for some $a \in \mathbb{R}$ and the $n-1$ dimensional interior Γ_ν of $\partial D_\nu \cap \partial H_{\nu,a}$ is nonempty.

Definition 2.1. *The energy density W is called* quasiconvex at the boundary at $(F_0, \nu) \in \mathbb{R}^{n\times n} \times \mathbb{R}^n$ *if there exists a vector $q \in \mathbb{R}^n$ such that*

$$\int_{D_\nu} W\big(F_0 + \nabla\varphi(x)\big)\, dx \geq \int_{D_\nu} W(F_0)\, dx + \int_{\Gamma_\nu} q \cdot \varphi(x)\, da \tag{QB}$$

holds for every standard boundary domain $D_\nu \subset \mathbb{R}^n$ with normal vector ν and for every $\varphi \in W^{1,\infty}_{\Gamma_\nu}(D_\nu, \mathbb{R}^n)$.

Remark 2.2. A scaling argument shows that quasiconvexity at the boundary is independent of the domain, i.e. if (QB) holds for a given standard boundary domain, then (QB) holds for every such standard boundary domain.

The condition of quasiconvexity at the boundary is necessary for a strong local minimizer and applies at a free boundary point, see [BM84].

Theorem 2.3. *Let u_0 be a strong local minimizer of the functional I. Then, for every $x_0 \in \Gamma$ with normal ν the stored energy function W is quasiconvex at the boundary at $\nabla u_0(x_0)$ with normal ν.*

Remark 2.4. More precisely, Ball and Marsden [BM84] have shown that the vector $q \in \mathbb{R}^n$ that appears in (QB) can be chosen as $q = g(x_0)$. Moreover, if W is differentiable at $\nabla u_0(x_0)$ one can only choose $q = \frac{\partial W}{\partial F}\big(\nabla u_0(x_0)\big)\nu(x_0) = g(x_0)$.

In [SS89] the following sufficient conditions for a weak local minimizer in mixed boundary problems is derived.

Theorem 2.5. *Let u_0 be a weak local minimizer of I. Suppose that the stored energy function W restricted to $\mathbb{R}^{n\times n}_+$ is C^2 and put $C_0 = \frac{\partial^2 W}{\partial F^2}\big(\nabla u_0(x_0)\big)$ for $x_0 \in \overline{\Omega}$. Then, for $x_0 \in \Gamma$ with normal $\nu \in \mathbb{R}^n$ the inequality*

$$Q(\varphi) = \int_{D_\nu} C_0\big[\nabla\varphi(x)\big]{:}\nabla\varphi(x)\,dx \geq 0 \tag{2.2}$$

is satisfied for every standard boundary domain $D_\nu \subset \mathbb{R}^n$ with normal ν and every $\varphi \in W^{1,\infty}_{\Gamma_\nu}(D_\nu, \mathbb{R}^n)$, i.e. the quadratic form induced by C_0 is quasiconvex at the boundary with normal ν.

The aim of this paper is to give a simple algebraic condition in terms of positive definite matrices involving the coefficients of C_0 which is equivalent to condition (2.2).

Remark 2.6. In the quadratic case quasiconvexity at the boundary does not depend on the point $F_0 \in \mathbb{R}^{n\times n}$. Therefore, we say that C_0 is quasiconvex at the boundary with normal ν if the inequality (2.2) holds.

Remark 2.7. One should note that all the above statements remain valid for variation integrals of the type

$$I(u) = \int_\Omega f\big(x, u(x), \nabla u(x)\big)\,dx - \int_\Gamma g\big(x, u(x)\big)\,da,$$

where the integrands $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R} \cup \{+\infty\}$ and $g : \Gamma \times \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ are continuous functions and m, n arbitrary.

3. Quasiconvexity at the boundary in the quadratic case

In this section we present new pointwise conditions on the elasticity tensor C_0 that are necessary and sufficient for the validity of inequality (2.2). For vectors $\nu, \tau \in \mathbb{R}^n$ we define, as in [SS89], matrices $M, N_\tau, P_\tau \in \mathbb{R}^{n\times n}$ by

$$Ma := C_0[a\otimes\nu]\nu, \quad N_\tau a := C_0[a\otimes\tau]\nu, \quad P_\tau a := C_0[a\otimes\tau]\tau \tag{3.1}$$

for every $a \in \mathbb{R}^n$. Here "$\otimes$" denotes the tensorial product between two vectors, i.e. $(a\otimes b)c = (b\cdot c)a$. The symmetry of C_0 implies that M and P_τ are symmetric. If M is invertible we define the *Riccati form*

$$\mathcal{R}_\tau(X) := (X + iN_\tau^T)M^{-1}(X - iN_\tau) - P_\tau. \tag{3.2}$$

We say that the *Riccati inequality* $\mathcal{R}_\tau(X) \le 0$ resp. the *Riccati equation* $\mathcal{R}_\tau(X) = 0$ has a solution if there exists a hermitian matrix $X \ge 0$ sucht that $-\mathcal{R}_\tau(X) \ge 0$ resp. $\mathcal{R}_\tau(X) = 0$. For a hermitian matrix $H = \overline{H}^T \in \mathbb{C}^{n\times n}$ we denote *positive definiteness* by $H > 0$ and positive semidefiniteness by $H \ge 0$. This means $Hv \cdot \overline{v} > 0$ resp. $Hv \cdot \overline{v} \ge 0$ for $0 \ne v \in \mathbb{C}^n$. Notice that our definition of scalar product does not involve a complex conjugation.

The main result of this paper shows that for the quadratic functional Q quasiconvexity at the boundary is equivalent to the existence of a solution of the Riccati equation resp. Riccati inequality.

Main Theorem. *Let* C_0, Q, *and* ν *as in Theorem 2.5, and* M, N_τ, P_τ, *and* $\mathcal{R}_\tau$ *as in* (3.1) *and* (3.2).
(i) The following statements are equivalent:
(M1): *the quadratic form* Q *is quasiconvex at the boundary with normal* ν*;*
(M2): *for each* $\tau \in \mathbb{R}^n$ *perpendicular to* ν *there exists an* $H \ge 0$ *such that*

$$E_{\tau,H} = \begin{pmatrix} M & -iN_\tau + H \\ iN_\tau^T + H & P_\tau \end{pmatrix} \ge 0.$$

(ii) If $M \in \mathbb{R}^{n\times n}$ *is invertible, then (M1) or (M2) are equivalent to the following statements: for each* $\tau \in \mathbb{R}^n$ *perpendicular to* ν *there exists a solution* $X \ge 0$
(M3): *of the Riccati inequality* $\mathcal{R}_\tau(X) \le 0$*;*
(M4): *of the Riccati equation* $\mathcal{R}_\tau(X) = 0$.
(iii) If all H *in (M2) can be chosen such that* $E_{\tau,H} > 0$, *then the quadratic form* Q *is coercive, that is there is* $\delta > 0$ *such that* $Q(\varphi) \ge \delta \int_{D_\nu} |\nabla\varphi|^2 dx$ *for all* $\varphi \in W^{1,2}_{\Gamma_\nu}(D_\nu, \mathbb{R}^n)$.

A proof of the main theorem is given at the end of the next section.

Remark 3.1. A scaling argument shows that it suffices to take those vectors $\tau \in \mathbb{R}^n$ with $\tau \cdot \nu = 0$ which have norm 1.

Remark 3.2. The equivalence of the statements (M3) and (M4) is well known in control theory (see [LR95]). There the proof involves monotone iteration techniques. The proof of our main theorem is based on methods of calculus of variations and is quite different from the analysis used in control theory.

Remark 3.3. If C_0 is rank-two convex, i.e. $C_0[A, A] \ge 0$ for every rank-two matrix $A \in \mathbb{R}^{n\times n}$, then C_0 is quasiconvex at the boundary with every normal ν. This follows immediately form (M2) with the trivial hermitian matrix $H = 0$.
We do not know whether the rank-two convexity of C_0 is a necessary condition for C_0 to be quasiconvex at the boundary with normal ν for every $\nu \in \mathbb{R}^n$ and $|\nu| = 1$.

A general discussion of positivity of the quadratic form Q was initiated by H.C. Simpson and S.J. Spector in [SS87] and [SS89], where also the connections to the complementing condition (also called Lopatinsky-Shapiro condition) are given. Our result is closely related to [SS89], where it is shown that C_0 is quasiconvex at the boundary with normal ν if and only if the following three statements are fulfilled:

(S1) C_0 is rank-one convex, i.e. $C_0[a\otimes b, a\otimes b] \geq 0$ for each $a, b \in \mathbb{R}^n$;
(S2) (C, ν) satisfies **Agmon's condition**, i.e. for every τ perpendicular to ν and for every $\alpha > 0$ the only bounded solution $z \in C^2\big([0,\infty), \mathbb{C}^n\big)$ of

$$-M\ddot{z}(t) + i(N_\tau + N_\tau^T)\dot{z}(t) + P_\tau z(t) = -\alpha^2 z(t), \quad M\dot{z}(0) - iN_\tau z(0) = 0$$

is the trivial solution $z = 0$;
(S3) If $C_0[a\otimes\nu, a\otimes\nu] = 0$ for some $a \in \mathbb{R}^n$ then $C_0[a\otimes\nu] = 0$.

Strict inequality in (S1) is also referred to as the *strong ellipticity* for C_0. Conditions (S1) and (S3) can be checked easily. But in general it is rather difficult to verify Agmon's condition (S2). Our condition (M2) seems much easier to formulate and to check in concrete problems. Agmon's condition (S2) is an algebraic condition since solving the constant coefficient equation is a purely algebraic problem. However, this algebraic structure is rather intricate. Our equivalent formulation (M2) shows the algebraic nature directly and, moreover, is formulated within the theory of variational problems in finite dimensions, namely that of positive definite quadratic forms.

We start by showing that conditions (S1) and (S3) are easy consequences of (M2). Given $a, b \in \mathbb{R}^n$ we write $b = \beta\nu + \tau$ where $\nu{\cdot}\tau = 0$. A straightforward calculation shows

$$C_0[a\otimes b, a\otimes b] = \begin{pmatrix} M & -iN_\tau \\ iN_\tau^T & P_\tau \end{pmatrix} \begin{pmatrix} a \\ i\beta a \end{pmatrix} \cdot \begin{pmatrix} a \\ -i\beta a \end{pmatrix} = E_{\tau,H} \begin{pmatrix} a \\ i\beta a \end{pmatrix} \cdot \begin{pmatrix} a \\ -i\beta a \end{pmatrix} \geq 0,$$

which is condition (S1). The last equality follows since $a \in \mathbb{R}^n$.

Now assume $C_0[a\otimes\nu, a\otimes\nu] = 0$ for some $a \in \mathbb{R}^n$. This is equivalent to $Ma = 0$, and because of $C_0[a\otimes\nu](\alpha\nu + \tau) = \alpha Ma + N_\tau^T a$ we have to show $N_\tau^T a = 0$ in order to establish (S3). From $E_{\tau,H}(a,b)^T \cdot (a,b)^T \geq 0$ for all $b \in \mathbb{C}^n$ and $Ma \cdot a = 0$ we conclude $(H + iN_\tau^T)a = 0$. Dotting this identity with a gives $Ha \cdot a = 0$ since the first term is real and the second term is purely complex. Since $H \geq 0$ we conclude $Ha = 0$ which gives $N_\tau^T a = iHa = 0$. This is the desired result (S3).

4. Proof of the main theorem

The first step in deriving the pointwise conditions in the main theorem is to reduce the analytic inequality (2.2) in the n–dimensional half space $\big\{y{\in}\mathbb{R}^n : (y{-}x_0){\cdot}\nu \leq 0\big\}$ to an analytic inequality in the one–dimensional half–line.

For any $\tau \in \mathbb{R}^n$ we define the quadratic functional $J_\tau : W^{1,2}\big((0,\infty), \mathbb{C}^n\big) \to \mathbb{R}$ by

$$J_\tau(z) = \int_0^\infty \Big\{ M\dot{z}(t) \cdot \overline{\dot{z}(t)} - 2\mathrm{Re}\big(iN_\tau z(t) \cdot \overline{\dot{z}(t)}\big) + P_\tau z(t) \cdot \overline{z(t)} \Big\}\, dt. \tag{4.1}$$

The functional J_τ is called *coercive* if there exists $\alpha > 0$ such that

$$J_\tau(z) \geq \alpha \|z\|_{1,2}^2 = \alpha \int_0^\infty \{|\dot{z}(t)|^2 + |z(t)|^2\}\, dt$$

for all $z \in W^{1,2}\big((0,\infty), \mathbb{C}^m\big)$. The following result, which is based on Fourier transform in all directions τ perpendicular to ν, is due to [Ag62] and [DF63] (see also [Sp96]).

Theorem 4.1. *The quadratic form Q induced by C_0 is quasiconvex at the boundary with normal ν if and only if the functional J_τ is nonnegative for every $\tau \in \mathbb{R}^n$ perpendicular to ν.*

The quadratic form Q is coercive if and only if J_τ is coercive for every $\tau \in \mathbb{R}^n$ perpendicular to ν.

The crucial idea is now to interpret the non-negativity of J_τ as a variational problem in the following sense: J_τ is nonnegative if and only if for every vector $v \in \mathbb{C}^n$ the infimum on the set $X_v := \{z \in W^{1,2}((0,\infty)) \,:\, z(0) = v\}$ is nonnegative. The advantage of this interpretation is that we can apply well known results concerning quadratic variational integrals. For the sake of simplicity we will omit the index τ further on.

For coercive functionals we have the following preliminary result.

Lemma 4.2. *If J is coercive, then the symmetric matrices M and P are positive definite.*

Proof: Let $a \in \mathbb{C}^n$ be arbitrary. Given $\rho > 0$ we define a function $z_\rho \in W^{1,2}((0,\infty),\mathbb{C}^m)$ by $z_\rho(t) = a\sqrt{\rho}e^{-t/\rho}$. If we insert z_ρ in (4.1) we conclude with the coerciveness of J

$$Ma\cdot\overline{a} - 2\rho\mathrm{Re}(iNa\cdot\overline{a}) + \rho^2 Pa\cdot\overline{a} \geq \alpha(1+\rho^2)a\cdot\overline{a}$$

for some $\alpha > 0$. Letting $\rho \to 0$ resp. $\rho \to \infty$ we get the desired result. $\square$

By a standard result of the calculus of variations, for a coercive functional J there exists a unique minimizer z_v on X_v, i.e. $J(z_v) = \inf\{J(z) \,:\, z \in X_v\}$. Due to a regularity result of Morrey [Mo66, Theorem 1.10.1] we have $z_v \in C^2([0,\infty),\mathbb{C}^n)$. Therefore, z_v solves the Euler-Lagrange equations corresponding to J

$$-M\ddot{z}(t) + i(N+N^T)\dot{z}(t) + Pz(t) = 0 \tag{4.2}$$

with the boundary condition $z(0) = v$.

Next we show that the minimizer z_v has an exponential representation with a generating matrix which is independent of v.

Lemma 4.3. *Let J be coercive. Then the following statements hold:*

(i) *Let $z \in W^{1,2}((0,\infty),\mathbb{C}^n) \cap C^2([0,\infty),\mathbb{C}^n)$ be a solution of (4.2) with $z(0) = v \in \mathbb{C}^n$. Then $z = z_v$, i.e. z is the unique minimizer of J on X_v.*

(ii) *There exists $H > 0$ such that for every $v \in \mathbb{C}^n$ the exponential representation $z_v(t) = e^{tM^{-1}(iN-H)}v$, $t \in [0,\infty)$, holds.*

Proof: (i) The function $z_0 := z - z_v$ is contained in $W^{1,2}((0,\infty),\mathbb{C}^n) \cap C^2([0,\infty),\mathbb{C}^n)$ and solves (4.2) with the initial value $z_0(0) = 0$. Multiplying the Euler-Lagrange equations by $\overline{z_0}$ and after integration by part we get $J(z_0) = 0$. By the coerciveness of J we deduce $z_0 = 0$.

(ii) By linearity of (4.2) and the uniqueness result in (i) we have $z_{\beta_1 v_1 + \beta_2 v_2} = \beta_1 z_{v_1} + \beta_2 z_{v_2}$ for $v_1, v_2 \in \mathbb{C}^n$ and $\beta_1, \beta_2 \in \mathbb{C}$. Therefore, there exists a map $U : [0,\infty) \to \mathbb{C}^{n\times n}$ such that $z_v(t) = U(t)v$ for $v \in \mathbb{C}^n$ and $t \in [0,\infty)$. The continuity of z_v implies that U is continuous. Moreover, we have

$$U(0) = \mathbb{I} \quad \text{and} \quad U(t+s) = U(t)U(s) \quad \text{for every} \quad t,s \geq 0,$$

where $\mathbb{I}$ is the identity matrix. The first relation follows from $z_v(0) = v = U(0)v$. The second relation deduces from the fact that (4.2) is independent of time implying that $z_v(t+s) = z_{z_v(s)}(t)$. Hence, by a standard result of semigroup theory in finite dimensions there is a matrix $A \in \mathbb{C}^{n\times n}$ such that $U(t) = e^{tA}$ for $t \geq 0$.

Using integration by parts we find

$$J(U(\cdot)v) = Hv \cdot \overline{v} \quad \text{with } H = -M\dot{U}(0) + iNU(0).$$

The integral term vanishes due to the fact that $U(\cdot)v$ is a solution of the Euler-Lagrange equation (4.2). With $U(t) = e^{tA}$ we have $H = -MA + iN$. Moreover, $H > 0$ since J is coercive. By Lemma 4.1 M is positive definite and thus $A = M^{-1}(iN - H)$, which is the desired result. $\square$

The central part of our proof of the main theorem is to derive pointwise conditions on the elasticity tensor C_0 that are necessary and sufficient for the coerciveness of the functional J. The full result then follows by approximating nonnegative functional by coercive ones.

Theorem 4.4. *The following statements are equivalent:*

(i) *J is coercive;*

(ii) *there exist $\alpha > 0$ and $\widetilde{H} > 0$ such that $M - \alpha\mathbb{I}$ is positive definite and $\widetilde{H}$ solves the Riccati equation*

$$(\widetilde{H} + iN^T)(M - \alpha\mathbb{I})^{-1}(\widetilde{H} - iN) - (P - \alpha\mathbb{I}) = 0;$$

(iii) *there is a matrix $H > 0$ such that*

$$E_H = \begin{pmatrix} M & -iN + H \\ iN^T + H & P \end{pmatrix} > 0.$$

Proof: We show that the implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i)$\Rightarrow$ (ii) hold. In the following we set $M_\alpha := M - \alpha\mathbb{I}$ and $P_\alpha := P - \alpha\mathbb{I}$.

(ii) $\Rightarrow$ (iii): Since M_α is symmetric and positive definite it follows from the square-root theorem (see, e.g. [Gu81]) that there exists a symmetric, positive definite matrix K with $M_\alpha = K^2$. With $H = \widetilde{H}$ a simple calculation yields

$$\begin{aligned} E_H \begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} \overline{a} \\ \overline{b} \end{pmatrix} &= \begin{pmatrix} M_\alpha & -iN{+}H \\ iN^T{+}H & P_\alpha \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} \overline{a} \\ \overline{b} \end{pmatrix} + \alpha\big(|a|^2 + |b|^2\big) \\ &= \big[P_\alpha - (iN^T{+}H)M_\alpha^{-1}(-iN{+}H)\big] b \cdot \overline{b} + |Ka{+}K^{-1}(-iN{+}H)b|^2 + \alpha\big(|a|^2 + |b|^2\big). \end{aligned}$$

On the right side the first term vanishes due to the assumed matrix identity in (ii) and the second term is nonnegative. This gives the desired result.

(iii) $\Rightarrow$ (i): We define the quadratic form $n : W^{1,2}\big((0,\infty), \mathbb{C}^m\big) \to \mathbb{R}$ by

$$n(z) = 2\int_0^\infty \mathrm{Re}\big(Hz(t) \cdot \overline{\dot{z}(t)}\big)\, dt + Hz(0) \cdot \overline{z(0)}. \tag{4.3}$$

Integration by part shows that $n(z)=0$ for every $z\in W^{1,2}\big((0,\infty),\mathbb{C}^m\big)$. Therefore we have

$$J(z)=\int_0^\infty \begin{pmatrix} M & -iN+H\\ iN^T+H & P\end{pmatrix}\begin{pmatrix}\dot z(t)\\ z(t)\end{pmatrix}\cdot\begin{pmatrix}\overline{\dot z(t)}\\ \overline{z(t)}\end{pmatrix}dt+Hz(0)\cdot\overline{z(0)}.$$

Since $H>0$ and $E_H>0$ the functional J is coercive.

(i) $\Rightarrow$ (ii): For coercive J there exists $\alpha>0$ such that the quadratic form $J_\alpha(z):=J(z)-\alpha\|z\|^2_{W^{1,2}}$ remains coercive. Then by Lemma 4.1 M_α is positive definite. According to Lemma 4.3 the unique minimizer z_v has the form $z_v(t)=e^{tM^{-1}(iN-H)}v$. Moreover z_v solves (4.2) which gives the desired matrix identity. $\square$

The function $n(z)$ defined in (4.3) is called null Lagrangian since it vanishes for all z. From the construction of the matrix $E_{\tau,H}$ it is clear that H plays exactly the role of the null Lagrangian on the algebraic level of hermitian, positive semidefinite matrices.

Now we are in the position to prove the main theorem.

Proof of the main theorem.

(M1) $\Rightarrow$ (M2): Let $\tau\in\mathbb{R}^n$ be perpendicular to ν. For every $k\in\mathbb{N}$ we define a quadratic functional J_τ^k on $W^{1,2}\big((0,\infty),\mathbb{C}^n\big)$ by $J_\tau^k(z):=J_\tau(z)+k^{-1}\|z\|^2_{1,2}$. In view of Lemma 4.1 the functional J_τ^k is coercive. Therefore, by Theorem 4.4 for every $k\in\mathbb{N}$ there exists $H_k>0$ such that

$$\begin{pmatrix} M+k^{-1}\mathbb{I} & -iN_\tau+H_k\\ iN_\tau^T+H_k & P_\tau+k^{-1}\mathbb{I}\end{pmatrix}>0. \tag{4.4}$$

Since the sequences $\big(M+k^{-1}\mathbb{I}\big)_{k\in\mathbb{N}}$ and $\big(N_\tau+k^{-1}\mathbb{I}\big)_{k\in\mathbb{N}}$ are bounded in $\mathbb{C}^{n\times n}$, we conclude that the sequence $\big(H_k\big)_{k\in\mathbb{N}}$ is bounded in $\mathbb{C}^{n\times n}$. Therefore, there is a subsequence $\big(H_{k_j}\big)_{j\in\mathbb{N}}$ that converges to $H_\tau^*\geq 0$. Letting $j\to\infty$ in (4.4) we conclude that $\begin{pmatrix} M & -iN_\tau+H_\tau^*\\ iN_\tau^T+H_\tau^* & P_\tau\end{pmatrix}$ is positive semidefinite.

(M2) $\Rightarrow$ (M1): Analogous to the proof of the implication (iii) $\Rightarrow$ (i) of Theorem 4.4 we can deduce that the J_τ is nonnegative for every $\tau\in\mathbb{R}^n$ perpendicular to ν. Hence, by Lemma 4.1 we obtain the implication.

(M2) $\Leftrightarrow$ (M3): If we set $Q_\tau:=-iN_\tau+H$ the equivalence follows from the identity

$$\begin{pmatrix} M^{-\frac12} & 0\\ -Q_\tau^H M^{-1} & \mathbb{I}\end{pmatrix}\begin{pmatrix} M & Q_\tau\\ Q_\tau^H & P_\tau\end{pmatrix}\begin{pmatrix} M^{-\frac12} & -M^{-1}Q_\tau\\ 0 & \mathbb{I}\end{pmatrix}=\begin{pmatrix}\mathbb{I} & 0\\ 0 & -\mathcal{R}_\tau(H)\end{pmatrix}.$$

(M1) $\Rightarrow$ (M4): As above we define for every $\tau\in\mathbb{R}^n$ perpendicular to ν the coercive functional $J_\tau^k(z):=J_\tau(z)+k^{-1}\|z\|^2_{1,2}$. Then there is a number k_0 such that $M+k^{-1}\mathbb{I}$ is invertible for every $k\geq k_0$. By Lemma 4.4 there exist $\widetilde H_k\geq 0$, $k\geq k_0$, such that the identities

$$P_\tau+k^{-1}\mathbb{I}=\big(i(N_\tau^T+k^{-1}\mathbb{I})+\widetilde H_k\big)\big(M+k^{-1}\mathbb{I}\big)^{-1}\big(-i(N_\tau+k^{-1})+\widetilde H\big)$$

hold. The sequence $\big(\widetilde H_k\big)$ is bounded in $\mathbb{C}^{n\times n}$ and thus there is a subsequence which converges to $\widetilde H^*\geq 0$. Continuity shows $\mathcal{R}_\tau(\widetilde H^*)=0$, as desired.

(M4) $\Rightarrow$ (M2): This can be proved analogous to (ii) $\Rightarrow$ (iii) of Theorem 4.4.

The result on the coercivity follows directly combining Theorems 4.1 and 4.4. Thus the proof of the main theorem is complete. $\square$

5. Applications

Our theory allows us to deduce certain sharp results for quasiconvexity at the boundary for general anisotropic quadratic forms. We first give a two-dimensional result which is, however, extremely useful also for higher dimensional problems since the condition of quasiconvexity at the boundary only involves the two directions ν and τ. In particular, we will give examples where the dependence of the conditions on ν becomes clear.

All our applications are based on the following result which handles the case of a general quadratic form which is invariant under reflection with respect to the normal and the tangential direction ν and τ.

Theorem 5.1. *Let $W : \mathbb{R}^{2\times 2} \to \mathbb{R}$ be the quadratic form defined by*

$$W(G) = M_1 G_{11}^2 + P_1 G_{12}^2 + M_2 G_{21}^2 + P_2 G_{22}^2 + 2N_1 G_{11} G_{22} + 2N_2 G_{12} G_{21}, \tag{5.1}$$

where $N_i, M_i, P_i \in \mathbb{R}$ with $i = 1, 2$. Then, W is quasiconvex at the boundary with normal $\nu = (1, 0)^T$ if and only if the following four conditions are satisfied:

$$M_1 \geq 0, M_2 \geq 0, P_1 \geq 0, P_2 \geq 0; \tag{5.2}$$

$$N_1^2 \sqrt{M_2 P_1} + N_2^2 \sqrt{M_1 P_2} \leq \sqrt{M_1 P_2} \sqrt{M_2 P_1} \left(\sqrt{M_1 P_2} + \sqrt{M_2 P_1}\right); \tag{5.3}$$

$$\text{If } M_1 P_2 = 0 \text{ then } N_2^2 \leq M_2 P_1 \text{ and } N_1 = 0; \tag{5.4}$$

$$\text{If } M_2 P_1 = 0 \text{ then } N_1^2 \leq M_1 P_2 \text{ and } N_2 = 0. \tag{5.5}$$

Remark: For W given in (5.1) the conditions for convexity are

$$M_1, M_2, P_1, P_2 \geq 0, \quad N_1^2 \leq M_1 P_2, \quad \text{and} \quad N_2^2 \leq M_2 P_1.$$

Knowles and Sternberg [KS77] (see also [Da91], [Mi91]) showed that W is rank-one convex if and only if M_1, M_2, P_1, P_2 are nonnegative and

$$|N_1 + N_2| \leq \sqrt{M_1 P_2} + \sqrt{M_2 P_1}.$$

Therefore, quasiconvexity at the boundary is a condition which is strictly between convexity and rank-one convexity. Fixing positive M_1, M_2, P_1, and P_2 the corresponding sets of (N_1, N_2) are depicted in Figure 1. We obtain a rectangle for convexity, an ellipse for quasiconvexity at the boundary, and a strip for rank-one convexity. All three sets have the two points $\pm(\sqrt{M_1 P_2}, \sqrt{M_2 P_1})$ on their boundary.

Proof: First, we prove that the algebraic conditions are necessary. With $\tau = (0, 1)^T$ the above notation gives

$$M = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}, \quad P_\tau = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} \text{ and } N_\tau = \begin{pmatrix} 0 & N_1 \\ N_2 & 0 \end{pmatrix}.$$

According to (M2) we know that M and P_τ are positive semidefinite, which gives condition (5.2). Now we consider two cases:

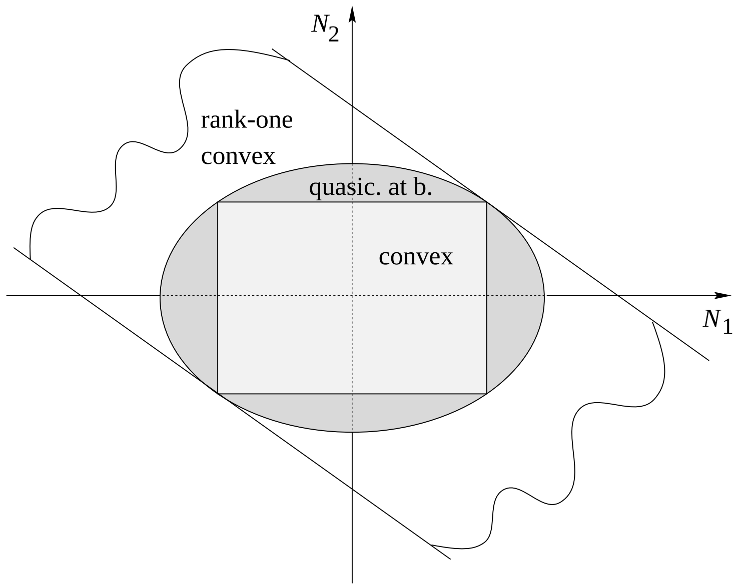

FIGURE 1. Parameter regions for convexity, quasiconvexity at the boundary, and rank-one convexity.

CASE 1: M and P_τ are regular. By the main theorem there is $H = \begin{pmatrix} H_1 & H_3+iH_4 \\ H_3-iH_4 & H_2 \end{pmatrix} \geq 0$, $H_i \in \mathbb{R}$, such that $\mathcal{R}_\tau(H) = 0$. A direct calculation yields

$$(5.6) \qquad H_1 = \sqrt{M_1P_1 - \frac{M_1}{M_2}\left(\frac{N_1+N_2}{1+\gamma}\right)^2}, \; H_2 = \gamma\frac{M_2}{M_1}H_1, \; H_4 = \frac{N_1-\gamma N_2}{1+\gamma}, \; H_3 = 0,$$

with $\gamma = \sqrt{\frac{M_1P_2}{M_2P_1}}$. By the positive definiteness of H we have (5.3).

CASE 2: M or P_τ are singular. Then for $\varepsilon > 0$ the matrices $M(\varepsilon) := M + \varepsilon\mathbb{I}$ and $P_\tau(\varepsilon) := P_\tau + \varepsilon\mathbb{I}$ are regular and we can apply Case 1. Letting $\varepsilon \to 0$ we get (5.4) resp. (5.5).

We now prove sufficiency. If M_1, M_2, P_1, P_2 are positive, we choose H_1, H_2, H_4 as in (5.6). By the inequality (5.3) the hermitian matrix $H = \begin{pmatrix} H_1 & iH_4 \\ -iH_4 & H_2 \end{pmatrix}$ is positive semidefinite. A straightforward calculations yields $\mathcal{R}_\tau(H) = 0$ and hence W is quasiconvex at the boundary with $\nu = e_1$.

Now, consider the case $M_1P_2 = 0$. If we set $H = 0$ and make use of $N_2^2 \leq M_2P_1$ and $N_1 = 0$ we conclude $\begin{pmatrix} M & -iN_\tau \\ iN_\tau^T & P_\tau \end{pmatrix} \geq 0$. Thus, by (M2) W is quasiconvex at the boundary, and (5.4) is established. The case (5.5) is analogous. $\square$

Example 5.2. We apply our result to the elasticity tensor of a general two-dimensional isotropic material. The stored-energy density W is frame indifferent if $W(RF) = W(F)$ for all rotations R. The material is called isotropic when the material is invariant under internal rotations, i.e. $W(FR) = W(F)$ for all rotations R. It is well known that for an isotropic material the stored-energy density depends only on the singular values λ_1, λ_2 of F, which are the square roots of the eigenvalues of F^TF. Thus, we have $W(F) = \Phi\big(\lambda_1(F), \lambda_2(F)\big)$, where Φ is a symmetric function. Let $F = \lambda_1 e_1 \otimes e_1 + \lambda_2 e_2 \otimes e_2$, i.e. λ_i

are the principal stretches with the principal strain axes e_i. In this case [Ba84] (see also [Da91]) has shown that the elasticity tensor $C_0 := C(F)$ can be expressed in terms of Φ by

$$C_0[G,G] = \sum_{i,j=1}^{2} \Phi_{,ij} G_{ii} G_{jj} + \sum_{i\neq j} \big(\alpha G_{ij}^2 + \beta G_{ij} G_{ji}\big),$$

where $\alpha = \frac{\lambda_1 \Phi_{,1} - \lambda_2 \Phi_{,2}}{\lambda_1^2 - \lambda_2^2}$ and $\beta = \frac{\lambda_2 \Phi_{,1} - \lambda_1 \Phi_{,2}}{\lambda_1^2 - \lambda_2^2}$ for $\lambda_1 \neq \lambda_2$. Here $\Phi_{,i}$ denotes $\frac{\partial \Phi}{\partial \lambda_i}$ etc., and all derivatives of Φ are evaluated at (λ_1, λ_2).

For simplicity we assume $\Phi_{,11} > 0, \Phi_{,22} > 0, \alpha > 0$. By Theorem 5.1 we obtain that C_0 is quasiconvex at the boundary with normal $\nu = e_1$ if and only if the inequality

$$\Phi_{,12}^2\, \alpha + \beta^2 \sqrt{\Phi_{,11}\Phi_{,22}} \leq \alpha\, \sqrt{\Phi_{,11}\Phi_{,22}}\big(\alpha + \sqrt{\Phi_{,11}\Phi_{,22}}\big)$$

is satisfied.

Example 5.3. We show the dependence of quasiconvexity at the boundary on the normal in a neighborhood of a fixed vector ν. Consider

$$C_0[G] = G + p \begin{pmatrix} 0 & G_{21} \\ G_{12} & 0 \end{pmatrix}$$

where $p \in \mathbb{R}$. For $\nu = (1,0)^T$ and $\tau = (0,1)^T$ notation (3.1) gives $M_0 = P_0 = \mathbb{I}$ and $N_0 = p\, e_2 \otimes e_1$. From Theorem (5.1) we conclude that C_0 is quasiconvex at the boundary with normal $\nu = e_1$ if and only if $|p| \leq \sqrt{2}$. For $p = \sqrt{2}$ a simple calculation shows that

$$H_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & -i \\ i & -1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \pm 1 - i \\ \pm 1 + i & 0 \end{pmatrix}$$

are all hermitian solutions of the Riccati equation $\mathcal{R}_0(X) = (X + iN^T)(X - iN) - P = 0$. However, H_0 is the only solution with $X \geq 0$. The linearization of $\mathcal{R}_0$ at H_0 is given via

$$\mathcal{L}\, G = D_X \mathcal{R}_0(H_0)[G] = G(H_0 - iP_0) + (H_0 + iP_0^T) G$$

and is invertible because its four eigenvalues are $(2 + i\, n)/\sqrt{2}$ with $n = -2, 0, 0, 2$.

Our goal is to study the dependence of the condition of quasiconvexity at the boundary for $p \approx \sqrt{2}$ and for $\nu_\varphi = (\cos\varphi, \sin\varphi)^T$, $\tau_\varphi = (-\sin\varphi, \cos\varphi)^T$ with $\varphi \approx 0$. Hence we have to discuss the Riccati equation

$$\mathcal{R}_{p,\varphi}(X) = (X + iN_\varphi^T) M_\varphi^T (X - iN_\varphi) - P_\varphi = 0, \tag{5.7}$$

with $M_\varphi = \begin{pmatrix} 1 & -p\sin\varphi\cos\varphi \\ -p\sin\varphi\cos\varphi & 1 \end{pmatrix}$, $P_\varphi = \begin{pmatrix} 1 & p\sin\varphi\cos\varphi \\ p\sin\varphi\cos\varphi & 1 \end{pmatrix}$, $N_\varphi = \begin{pmatrix} 0 & -p\sin^2\varphi \\ p\cos^2\varphi & 0 \end{pmatrix}$. Since $\mathcal{R}_{p,\varphi}(X) \to \infty$ for $|X| \to \infty$ we conclude by standard compactness and continuity arguments that for $(p,\varphi) \approx (\sqrt{2}, 0)$ solutions of (5.7) can only exist in small neighborhoods of the four know solutions of $\mathcal{R}_{\sqrt{2},0}(X) = 0$. Thus, solutions $H \geq 0$ can only occur near H_0, and the invertibility of $\mathcal{L}$ allows us to to apply the implicit function theorem. Then the unique hermitian solution of (5.7) near H_0 has the expansion

$$H_{p,\varphi} = H_0 + (p - \sqrt{2}) H_* + \varphi H_1 + \varphi^2 H_2 + \mathcal{O}\big(|p - \sqrt{2}|^2 + |p - \sqrt{2}||\varphi| + |\varphi|^3\big)$$

with $H_1 = 0$, $H_* = \frac{1}{2\sqrt{2}}\left(\begin{smallmatrix} 2 & i \\ -i & 0 \end{smallmatrix}\right)$ and $H_2 = -\sqrt{2}\mathbb{I}$. Thus, in a neighborhood of $(p,\varphi) = (\sqrt{2},0)$ the positive semidefiniteness of $H_{p,\varphi}$ is determined by

$$\det H_{p,\varphi} = -\frac{1}{2}\big((p-\sqrt{2}) + 4\varphi^2\big) + \mathcal{O}\big(|p-\sqrt{2}|^2 + |p-\sqrt{2}\varphi^2 + \varphi^4\big) \geq 0.$$

Solving $\det H_{p,\varphi} = 0$ for $p = p_{\max}(\varphi)$ we find $p_{\max}(\varphi) = \sqrt{2} - 4\varphi^2 + \mathcal{O}(\varphi^4)$.

Thus, for small $|\varphi|$ the quadratic functional Q induced by C_0 is quasiconvex at the boundary with respect to ν_φ for $|p| \leq p_{\max}(\varphi)$. This exhibits nicely the dependence of quasiconvexity at the boundary on the normal vector.

Example 5.4. (*Isotropic material*) We consider an isotropic hypereelastic body at equilibrium in a stress-free reference configuration. The elasticity tensor $C_0 := C(\mathbb{I})$ for such a body is given by

$$C_0[G] = \lambda(G{:}\mathbb{I})\mathbb{I} + \mu(G + G^T).$$

By isotropy we may assume $\nu = e_1$ and $\tau = e_2$. This leads to the matrices

$$M = \mu\mathbb{I} + (\lambda{+}\mu)e_1{\otimes}e_1, \quad P = \mu\mathbb{I} + (\lambda{+}\mu)e_2{\otimes}e_2, \quad N = \lambda e_1{\otimes}e_2 + \mu e_2{\otimes}e_1.$$

We immediately see that the equations for $e_3, \ldots, e_n$ decouple such that E_H in (M2) has block structure if H has block structure. Obviously $\mu \geq 0$ is necessary and sufficient for the lower part to be semidefinite. The equations arising from the first two equations are now in the form of Theorem 5.1. A simple calculation gives the necessary and sufficient condition

$$\lambda + \mu \geq 0 \quad \text{and} \quad \mu \geq 0 \tag{5.8}$$

for quasiconvexity at the boundary. Because of the isotropy there is no dependence on the normal direction. This recovers the result given in [SS87], Thm. 6. The condition is stronger than quasiconvexity (in the interior) which only needs $\lambda + 2\mu \geq 0$ and $\mu \geq 0$. And, for $n \geq 3$, it is weaker than convexity which holds for $n\lambda + 2\mu \geq 0, \mu \geq 0$.

Example 5.5. (*Anisotropic material*) Following [SS89] we consider an anisotropic hyperelastic material with a stored-energy functions of the form

$$W(F) = \frac{\mu}{2}F{:}F + h(\det F) + \frac{\omega}{2}|Fe_1|^2 + \frac{\omega}{2}|F^Te_1|^2,$$

where $\lambda, \mu, \omega \in \mathbb{R}$ and h is C^2. A simple calculation shows that $W(FR) = W(F)$ for all rotations R with $Re_1 = e_1$, i.e. the anisotropic direction of W is e_1. The elasticity tensor in the reference configuration is given by

$$C_0[G] := C(\mathbb{I})[G] = 2\mu E + \lambda(E{:}\mathbb{I})\mathbb{I} + 2\omega(Ee_1 \otimes e_1 + e_1 \otimes Ee_1), \tag{5.9}$$

where $\lambda = h''(1) - \mu$ and $E = \frac{1}{2}(G + G^T)$.

First, we derive necessary and sufficient conditions such that the C_0 is quasiconvex at the boundary with normal $\nu = e_1$. Since this is the anisotropic direction it is suffices to verify (M4) for $\tau = e_n = (0, \dots, 0, 1)^T$. Then by (3.1) and (5.9) we have

$$M_{||} = (\mu+\omega)\mathbb{I} + (\mu+\lambda+3\omega)e_1 \otimes e_1,\ \ P_{||} = \mu\mathbb{I} + (\lambda+\mu)e_n \otimes e_n + \omega e_1 \otimes e_1,$$
$$N_{||} = (\mu+\omega)e_n \otimes e_1 + \lambda e_1 \otimes e_n.$$

By the special structure of $M_{||}, P_{||}, N_{||}$ there is again a decoupling of the components associated to $e_2, \dots, e_{n-1}$. For these we again obtain the condition $\mu \geq 0$ as in the isotropic case. It remains to study the two-dimensional system associated to $(\nu, \tau) = (e_1, e_n)$. We obtain the matrices $\widetilde{M} = \left(\begin{smallmatrix} 2\mu+\lambda+4\omega & 0 \\ 0 & \mu+\omega \end{smallmatrix}\right)$, $\widetilde{P} = \left(\begin{smallmatrix} \mu+\omega & 0 \\ 0 & 2\mu+\lambda \end{smallmatrix}\right)$ and $\widetilde{N} = \left(\begin{smallmatrix} 0 & \lambda \\ \mu+\omega & 0 \end{smallmatrix}\right)$. In view of Theorem 5.1 the elasticity tensor C_0 is quasiconvex at the boundary with normal $\nu = e_1$ if and only if

$$\mu \geq 0,\ 2\mu{+}\lambda \geq 0,\ \mu{+}\omega \geq 0,\ 2\mu{+}\lambda{+}4\omega \geq 0,\ \mu(\mu{+}\lambda) + \omega(2\mu{+}\lambda) \geq 0.$$

Now, choose the different normal $\nu = e_2$, which is perpendicular to the direction of anisotropy e_1. We derive a necessary condition for quasiconvexity by considering $\tau = e_n$. The associated matrices are

$$M^\perp = \mu\mathbb{I} + \omega e_1{\otimes}e_1 {+} (\mu{+}\lambda)e_2{\otimes}e_2, \quad P_\tau^\perp = \mu\mathbb{I} + \omega e_1{\otimes}e_1 + (\mu{+}\lambda)e_n{\otimes}e_n,$$
$$N_\tau^\perp = \mu e_n{\otimes}e_2 + \lambda e_2{\otimes}e_n.$$

The decoupled system associated to the directions (e_2, e_n) leads to the necessary condition $\mu{+}\lambda \geq 0$.

Thus, we find an open set for (μ, λ, ω) such that C_0 is quasiconvex at the boundary for $\nu = e_1$ but not for $\nu = e_2$, for instance we may take $(\mu, \lambda, \omega) = (2, -3, 3)$.

Example 5.6. Our last example shows that for an important class of nonlinear materials (see e.g. [BM84], eqn. (3.14)) the condition of quasiconvexity at the boundary fails in $(\sqrt{\lambda}\,\mathbb{I}, e_1)$ for sufficiently small positive λ. With $n = 2$ we consider the isotropic stored-energy density

$$W(F) = \frac{1}{2}F{:}F + h(\det F),$$

where h is a strictly convex C^2 function and satisfies $h(t) \to +\infty$ for $t \to 0^+$ resp. for $t \to \infty$. Then, the elasticity tensor is given by

$$C(\sqrt{\lambda}\,\mathbb{I})[G] = G - h'(\lambda)G^T + \big[h'(\lambda) + \lambda h''(\lambda)\big](G{:}\mathbb{I})\mathbb{I}.$$

With the notation of (5.1) we have $M_1 = P_2 = 1 + \lambda h''$, $M_2 = P_1 = 1$, $N_1 = h' + \lambda h''$ and $N_2 = -h'$, where h' resp. h'' are evaluated at λ. Hence, Thereom 5.1 shows that the elasticity tensor $C(\sqrt{\lambda}\,\mathbb{I})$ is quasiconvex at the boundary with normal $\nu = e_1$ if and only if the inequality

$$\lambda h''\big(h'^2{+}2h'{-}3\big) + 2(h'^2{-}1) \leq 0.$$

holds. This condition is not satisfied for very small $\lambda > 0$, and therefore deformations under extreme compression must are unstable.

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