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Bounds for the solutions of the complex Ginzburg–Landau equation in terms of the dispersion parameters

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Abstract

The diameter in the L^∞–norm of the global attractor of the complex Ginzburg–Landau equation $u_t = (1+\mathrm{i}\alpha)\Delta u + Ru - (1+\mathrm{i}\beta)|u|^{2\sigma}u$ is estimated by using weighted energy estimates for the solutions on the whole space $\mathbb{R}^d$. For all parameters d, σ, α, and β for which global existence is known we obtain the bound $C_\infty(d,\sigma,\alpha,\beta)R^{1/(2\sigma)}$. Upper estimates for C_∞ are given which are polynomial in (α,β) for $d\sigma < 2$ and exponential for $d\sigma = 2$.

1 Introduction

The complex Ginzburg–Landau equation (cGL) serves as a model in many areas of physics:

$$u_t = (1 + \mathrm{i}\alpha)\Delta u + Ru - (1 + \mathrm{i}\beta)|u|^{2\sigma}u. \tag{1}$$

Here $\alpha, \beta \in \mathbb{R}$ are the dispersion parameters and $R > 0$ is the instability parameter, which is usually associated to the Rayleigh number or the Reynolds number in a given physical system.

It is well–known that the dynamics of this system is chaotic for certain parameter regimes, in particular in the modulationally unstable domain $\alpha\beta < 0$. In fact, cGL shows features like turbulence when approaching the inviscid limit of the nonlinear Schrödinger equation, i.e., $(\alpha, \beta) \to \infty$. In the case $d\sigma > 2$ this is analogous to the convergence of the Navier–Stokes equation towards the Euler equation as the viscosity tends to 0, see [BC*90,DGL94]. It is also possible to define intrinsic dissipation length scales [BGO96] as in turbulence. These length scales as well as many other quantities, e.g. the Hausdorff

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dimensions of the attractor (see [GhH87]), depend essentially on good upper bounds for the diameter of the attractor.

Our aim is to present new estimates on the size of the attractor in dependence on $(d, \sigma, \alpha, \beta)$ and R. In most previous work cGL was considered on the underlying physical domain $\Omega = (0,1)^d$ with periodic boundary conditions. Notice that every cube can be considered if the parameter R is redefined by the invariance of cGL under the scaling

$$(t, x, R, u) \mapsto (\ell^2 t, \ell x, \ell^{-2} R, \ell^{-1/\sigma} u). \tag{2}$$

For the spatially periodic case estimates on the Hausdorff dimension of the attractor are given in [GhH87,DG*88], and estimates on the diameter of the attractor in certain norms are derived in [BC*90,DGL94,BGO96]. Analyticity or Gevrey regularity for the solutions was obtained in [DoT93,LeO96].

To date global existence and uniqueness of strong solutions is established if the dispersion parameters (α, β) lie in the set $\mathcal{P}(d, \sigma)$. For $d\sigma \le 2$ we have $\mathcal{P}(d, \sigma) = \mathbb{R}^2$, and for $d\sigma > 2$

$$\mathcal{P}(d,\sigma) = \begin{cases} \mathcal{P}_1(d,\sigma) \cup \mathcal{P}_{\rm ms}(\sigma) & \text{if } \sigma > \frac{1}{2} \text{ and } d < 2 + \frac{2}{\sigma} \\ \mathcal{P}_1(d,\sigma) & \text{else} \end{cases}$$

with $\mathcal{P}_1(d,\sigma) = \{(\alpha,\beta) : |\alpha| < 2\frac{\sqrt{d\sigma-1}}{d\sigma-2}\}$ and $\mathcal{P}_{\rm ms}(\sigma) = \{(\alpha,\beta) : -(1+\alpha\beta) < |\alpha-\beta|\frac{\sqrt{2\sigma+1}}{\sigma}\}$. These methods and the generalization to unbounded domains in [Mie97] give global existence and uniqueness of strong solutions for the same parameter values but for general initial data in $L^\infty(\mathbb{R}^d)$, which particularly includes all spatially periodic solutions (for arbitrary box sizes).

Our aim is to derive the a–priori estimate

$$\limsup_{t\to\infty} \|u(t,\cdot)\|_{L^\infty(\mathbb{R}^d)} \le C_\infty(d,\sigma,\alpha,\beta) R^{1/(2\sigma)} \tag{3}$$

for all solutions u of cGL, where the prefactor C_∞ is given explicitly in terms of d, σ, and the dispersion parameters α and β. This very simple dependence on R was already noted in [Mie97] since it follows directly from Theorem 9 in [Col94]. There it is shown that the global attractor of cGL is bounded in $L^\infty(\mathbb{R}^d)$ for any R. Now, using the scaling (2) the desired result (3) is immediate if $C_\infty(d,\sigma,\alpha,\beta)$ is the value obtained for $R = 1$.

In previous work (cf. [DG*88,BC*90,BGO96]) for the periodic case estimates like (3) were obtained in the form $C_\infty R^{e(d,\sigma,\alpha,\beta)}$ where generally the exponent e was larger than $1/(2\sigma)$ (e.g. $e \sim |\alpha|/4$ in the critical case $d\sigma = 2$) and the dependence of C_∞ on (d,σ,α,β) was not explicit. Extensive numerical simulations in [LuD93,Wil96] for the periodic case with $\sigma = 2$ and $d = 1$ show clearly the growth in the form $R^{1/4}$ as in our estimate (3).

Our approach is based on *weighted energy estimates*, which were initiated in [CoE90] and further developed in [Col94,MiS95,Mie97]. Weigthed norms allow us to study cGL on $\mathbb{R}^d$ or other unbounded domains Ω. For simplicity we consider here only $\Omega = \mathbb{R}^d$, such that we can easily employ the scaling (2). For general unbounded domains a–priori estimates can be obtained using the methods in [Mie97]. In [TB*96] an alternative approach using semigroup theory is given to parabolic systems on $\mathbb{R}^d$. See also [GiV96,GiV97] for a treatment of cGL allowing for initial conditions which are locally integrable but may grow for $|x| \to \infty$.

The weighted energy estimates involve an integrable weight function ρ, which is taken to be

$$\rho(x) = \mathrm{e}^{-|x|}$$

for definiteness. The only important properties of the weight are the estimate $|\nabla \rho(x)| \leq \mu \rho(x)$ for $x \in \mathbb{R}^d$ and $\int_{\mathbb{R}^d} \rho(x)\, \mathrm{d}x = \rho_d < \infty$. Notice that $1/\mu$ introduces a length scale into our analysis such that our estimates are only optimal for certain R. Our choice of the weight fixes $\mu = 1$, and thus we will mainly restrict our attention to $R = 1+\alpha^2$.

The main advantages of using a weighted norm is seen already when performing the easy L^2–estimate. If u is a solution of cGL, then we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x} \int_{\mathbb{R}^d} \rho\, |u|^2 \,\mathrm{d}x = \int_{\mathbb{R}^d} \rho \left(-|\nabla u|^2 + R|u|^2 - |u|^{2+2\sigma} \right) \,\mathrm{d}x + \mathrm{Re}\Big[(1+\mathrm{i}\alpha) \int_{\mathbb{R}^d} u^* \nabla \rho \cdot \nabla u \,\mathrm{d}x \Big].$$

Since all our integrals will extend over the whole physical space $\mathbb{R}^d$ we will omit the domain of integration further on. Comparing to the classical energy estimates without a weight function we obtain the additional term involving $\nabla \rho$, which is however of lower order since $|\nabla \rho|$ can be estimated by ρ itself. Thus estimating this integral using the Cauchy–Schwarz inequality we easily obtain

$$\begin{aligned} &\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}x} \textstyle\int \rho\, |u|^2 \,\mathrm{d}x \leq \int \rho\, (-|\nabla u|^2 + \sqrt{1+\alpha^2} |\nabla u||u| + R|u|^2 - |u|^{2+2\sigma}) \,\mathrm{d}x \\ &\leq \textstyle\int \rho\, (\Gamma |u|^2 - |u|^{2+2\sigma}) \,\mathrm{d}x \leq \int \rho\, \sigma \Gamma (\Gamma^{1/\sigma} - |u|^2) \,\mathrm{d}x \leq \sigma \Gamma^{1+1/\sigma} \rho_d - \sigma \Gamma \int \rho |u|^2 \,\mathrm{d}x, \end{aligned}$$

where $\Gamma = R + (1+\alpha^2)/4$, and the third inequality follows by estimating the integrand pointwise.

In analogy to [BGO96] we define, for $r \geq 0$ and $n \in \mathbb{N}$, the functionals

$$G_r = \int \rho\, |u|^{2(r\sigma+1)} \,\mathrm{d}x, \quad J_n = \int \rho\, |\nabla^n u|^2 \,\mathrm{d}x, \quad H_\infty = \|u\|_\infty = \operatorname*{ess\,sup}_{x \in \mathbb{R}^d} |u(x)|$$

as well as the uniform functionals

$$\mathcal{G}_r = \sup_{y \in \mathbb{R}^d} \int T_y \rho\, |u|^{2(r\sigma+1)} \,\mathrm{d}x, \quad \mathcal{J}_n = \sup_{y \in \mathbb{R}^d} \int T_y \rho\, |\nabla^n u|^2 \,\mathrm{d}x.$$

Here $T_y \rho$, $y \in \mathbb{R}^d$, denotes the translated weight $\rho(\cdot - y)$, and $\mathcal{J}_n < \infty$ is equivalent to $\nabla^n u$ being uniformly in $L^2_{\mathrm{loc}}(\mathbb{R}^d)$ (see [MiS95,GiV96,Mie97,GiV97]). Our aim is to derive estimates on G_r and J_n which allow us to give a–priori bounds on the longtime behavior of the solutions. In particular, we let

$$\overline{G}_r = \limsup_{t \to \infty} G_r(t), \qquad \overline{J}_n = \limsup_{t \to \infty} J_n(t), \qquad \overline{H}_\infty = \limsup_{t \to \infty} H_\infty(t),$$

and similarly for $\mathcal{G}_r$ and $\mathcal{J}_n$. Since all these quantities are evaluated along solutions of cGL, we do not indicate the dependence on u itself.

For example, our above weighted L^2–estimate gives $\frac{1}{2}\dot{G}_0 \le \sigma\Gamma(\Gamma^{1/\sigma} - G_0)$, and thus Gronwall's estimate implies $G_0(t) \le \mathrm{e}^{-2\sigma\Gamma t}\, G_0(0) + \Gamma^{1/\sigma}\rho_d$. Since this estimate remains valid when the weight ρ is translated we immediately obtain

$$\overline{G}_0 \le \overline{\mathcal{G}}_0 \le \Gamma^{1/\sigma} = \Big[R + (1{+}\alpha^2)/4\Big]^{1/\sigma}.$$

The desired L^∞–estimate is obtained by deriving similar energy estimates for the functionals G_r, $r > 0$, and J_n, $n \in \mathbb{N}$. Combining the approaches in [Mie97] and [BGO96] we find estimates of the form

$$\tfrac{1}{2}\dot{J}_n \le RJ_n - J_{n+1} + \sqrt{1{+}\alpha^2}J_n^{1/2}J_{n+1}^{1/2} + c_1(n)\sqrt{1{+}\beta^2}J_{n+1}^{n/(n+1)}G_{n+1}^{1/(n+1)}.$$

Compared to the case with weight $\rho \equiv 1$ in [BGO96], again only one additional term appears, namely the one involving $\sqrt{1{+}\alpha^2}$ which arises from $\mathrm{Re}(1{+}\mathrm{i}\alpha)\int \rho\nabla^n u^* \cdot \nabla^n(\Delta u)\,\mathrm{d}x$ through integration by parts.

Our final result (see Corollaries 6 and 9 and Theorem 10) is that $C_\infty(d,\sigma,\alpha,\beta)$ can be bounded polynomially in $|\alpha| + |\beta|$ whenever we are in the subcritical regime $d\sigma < 2$ or in the modulationally stable parameter set, i.e., $(\alpha,\beta) \in \mathcal{P}_{\mathrm{ms}}(\sigma)$. In the critical case $d\sigma = 2$ with $d = 1$ or 2 the bound C_∞ is roughly of the form $[(1{+}\alpha^4)(1{+}\beta^2)]^{(1+d|\alpha|)/4}$.

Finally we want to warn the reader that our estimates are only upper bounds for the diameter of the attractor in the L^∞–norm. We have managed to find the correct scaling behavior with respect to R, but there is no indication that the scaling behavior in the dispersion parameters is optimal. Another warning is in place for the interpretation of our results for the spatially periodic case. Of course there are situations where the traditional estimates in [BC*90,BGO96] are better, since our approach is optimized for the limit $R \to \infty$. Roughly we can say that our estimates give better upper bounds whenever R is substantially larger than $1{+}\alpha^2$.

There is only one series of numerical simulations which might allow for a comparison with our analytical results on the dependence on the dispersion parameters. In [LuD93] numerical bounds were obtained for the case $d = 1$ and $\sigma = 2$, and in the last simulation $R = 20$ was fixed while $\alpha = -\beta$ ranged from 25 to 175. For that regime the numerical bound $C_\infty(1,2,\alpha,-\alpha)(20)^{1/4} \approx 51\,\mathrm{e}^{0.0082\alpha}$ was found, and the exponential growth rate 0.0082 is much smaller than the one from our upper analytical bound, namely $\frac{3}{2}\log\alpha$. This growth rate is in fact even larger than the one obtained from the standard energy estimates, namely $\frac{1}{4}\log R \approx 0.749$ (see [LuD93]). This is because of the fact that in the considered regime R is much less than $1{+}\alpha^2$.

2 Weighted estimates

Our analysis follows closely that of [BGO96], we only have to be careful when partial integration is used. Then, derivatives of the weight come into play, and we use as an essential feature that $|\nabla\rho(x)| \le \rho(x)$ for all $x \neq 0$. For instance, we obtain

$$\begin{aligned}\textstyle\int \rho \nabla^n u^* \cdot \nabla^n(\Delta u)\,\mathrm{d}x &= -\textstyle\int \rho |\nabla^{n+1} u|^2\,\mathrm{d}x - \int (\nabla\rho \otimes \nabla^n u^*) \cdot \nabla^{n+1} u\,\mathrm{d}x \\ &\le -J_{n+1} + J_n^{1/2} J_{n+1}^{1/2}.\end{aligned} \tag{4}$$

where the term involving $\nabla\rho$ is estimated by the Cauchy–Schwarz inequality. We immediately find the basic differential (in)equalities

$$\tfrac{1}{2(r\sigma+1)}\dot{G}_r = RG_r - G_{r+1} + \mathrm{Re}\left[(1+\mathrm{i}\alpha)\int \rho |u|^{2r\sigma} u^* \Delta u\,\mathrm{d}x\right], \tag{5}$$

and, by using (4),

$$\tfrac{1}{2}\dot{J}_n \le RJ_n - J_{n+1} + \widehat{\alpha} J_n^{1/2} J_{n+1}^{1/2} - \mathrm{Re}\left[(1+\mathrm{i}\beta)\int \rho\, \nabla^n\left(|u|^{2\sigma} u\right) \cdot \nabla^n u^*\,\mathrm{d}x\right], \tag{6}$$

where $\widehat{\alpha} = \sqrt{1+\alpha^2}$ and similarly $\widehat{\beta} = \sqrt{1+\beta^2}$. From (5) we derive the following a–priori bounds for G_r.

Lemma 1 *If $\lambda_{\alpha,r} = r\sigma + 1 - r\sigma\sqrt{1+\alpha^2}$ is positive, then we have*

$$\tfrac{1}{2(r\sigma+1)}\dot{G}_r \le \tfrac{\sigma\widetilde{R}}{r\sigma+1}\left[\widetilde{R}^{r+1/\sigma}\rho_d - G_r\right], \tag{7}$$

where $\widetilde{R} = R + \widehat{\alpha}^2/(4\lambda_{\alpha,r})$ and $\rho_d = \int \rho\,\mathrm{d}x$.

PROOF: We integrate by parts the integral term in (5) to obtain

$$\begin{aligned}&\textstyle\int \rho |u|^{2r\sigma} u^* \Delta u\,\mathrm{d}x = -\int |u|^{2r\sigma} u^* \nabla\rho \cdot \nabla u\,\mathrm{d}x \\ &-\textstyle\int \rho\left[(r\sigma+1)|u|^{2r\sigma}|\nabla u|^2 + r\sigma|u|^{2(r\sigma-1)}(u^*)^2 \nabla u \cdot \nabla u\right]\,\mathrm{d}x.\end{aligned}$$

Estimating the first term on the right–hand side we arrive at

$$\tfrac{1}{2(r\sigma+1)}\dot{G}_r \le RG_r - G_{r+1} + \widehat{\alpha} G_r^{1/2} K_r^{1/2} - Q_{\alpha,r}$$

with $K_r = \int \rho |u|^{2r\sigma} |\nabla u|^2\,\mathrm{d}x$, $M_{\alpha,r} = M_{\alpha,r}^* = \left(\begin{smallmatrix} r\sigma+1 & (1+\mathrm{i}\alpha)r\sigma \\ (1-\mathrm{i}\alpha)r\sigma & r\sigma+1 \end{smallmatrix}\right)$, and

$$Q_{\alpha,r} = \tfrac{1}{2}\int \rho |u|^{2(r\sigma-1)} \sum_{j=1}^{d} \begin{pmatrix} u^*\,\partial_j u \\ u\,\partial_j u^* \end{pmatrix} \cdot M_{\alpha,r} \begin{pmatrix} u\,\partial_j u^* \\ u^*\,\partial_j u \end{pmatrix}\,\mathrm{d}x. \tag{8}$$

The smallest eigenvalue of $M_{\alpha,r}$ is exactly $\lambda_{\alpha,r}$, as given in the statement of the lemma, such that $-Q_{\alpha,r} \le -\lambda_{\alpha,r} K_r$. Taking the supremum over $K_r \ge 0$ we find

$$\tfrac{1}{2(r\sigma+1)}\dot{G}_r \le \widetilde{R}\,G_r - G_{r+1} = \int \rho |u|^{2(r\sigma+1)}[\widetilde{R} - |u|^{2\sigma}]\,\mathrm{d}x.$$

Estimating the integrand pointwise by $\rho\frac{\sigma\widetilde{R}}{r\sigma+1}[\widetilde{R}^{r+1/\sigma} - |u|^{2(r\sigma+1)}]$ we obtain (7). □

Theorem 2 *If $\lambda_{\alpha,r} = r\sigma + 1 - r\sigma\sqrt{1+\alpha^2}$ is positive, then we have*

$$\mathcal{G}_r(t) \leq \mathrm{e}^{-2\sigma\widetilde{R}\,t}\,\mathcal{G}_r(0) + \widetilde{R}^{r+1/\sigma}\rho_d(1-\mathrm{e}^{-2\sigma\widetilde{R}\,t}) \ \text{ and } \ \overline{\mathcal{G}}_r \leq \widetilde{R}^{r+1/\sigma}\rho_d. \tag{9}$$

PROOF: From (7) and Gronwall's estimate we have

$$G_r(t) \leq \mathrm{e}^{-2\sigma\widetilde{R}\,t}\,G_r(0) + \widetilde{R}^{r+1/\sigma}\rho_d(1-\mathrm{e}^{-2\sigma\widetilde{R}\,t}).$$

However, this estimate also holds if ρ is replaced by $T_y\rho$ for any $y \in \mathbb{R}^d$. Thus, taking the supremum over $y \in \mathbb{R}^d$ estimate (9) is established, and the last assertion follows by letting $t \to \infty$. □

Next we derive inequalities for the derivatives. To this end we have to estimate the integral term in (6).

Lemma 3 *For each $d, n \in \mathbb{N}$ and $\sigma \in \mathbb{N} \cup ((n-1)/2, \infty)$ there is a constant c_1 such that*

$$\int \rho \left| \nabla^n(|u|^{2\sigma}u)\cdot\nabla^n u^* \right| \mathrm{d}x \leq c_1 J_{n+1}^{n/(n+1)}\,G_{n+1}^{1/(n+1)}. \tag{10}$$

PROOF: In the case of $\rho \equiv 1$ and a bounded domain of integration the result is well known (see Lemma 4 and Corollary 7 of [BGO96]) and follows from Leibniz' rule, Hölder's inequality, and Gagliardo–Nirenberg estimates. Our case follows by partitioning $\mathbb{R}^d$ into the unit cubes $Q_p = \prod_{j=1}^d [p_j, p_j+1)$, $p \in \mathbb{Z}^d$. Since, on each Q_p the result holds with $\rho \equiv 1$ it also hold on Q_p if we use the weighted norms. This follows as $\max\{\,\rho(x) \ : \ x \in Q_p\,\} \leq \mathrm{e}^{\sqrt{d}}\min\{\,\rho(x) \ : \ x \in Q_p\,\}$. Adding up all (weighted) inequalities over $p \in \mathbb{Z}^d$ and applying Hölder's inequality the assertion is proved. □

From the estimates before the lemma we see that the instability parameter R naturally appears together with $\widehat{\alpha}^2 = 1+\alpha^2$. Thus, for all the following calculations we set R equal to $\widehat{\alpha}^2$. This will simplify the estimates considerably; in fact, all the estimates below can be done without this assumption but it turns out that, within the present level of accuracy, the choice $R = \widehat{\alpha}^2$ always gives the best results. This choice of R is forced since we have fixed a spatial length scale through the weight ρ. Equivalently, we could set $R = 1$ and use the weight $\widetilde{\rho}(x) = \mathrm{e}^{-\widehat{\alpha}|x|}$.

The differential inequality (6) for J_n takes the form

$$\tfrac{1}{2}\dot{J}_n \leq \widehat{\alpha}^2 J_n - J_{n+1} + \widehat{\alpha} J_n^{1/2} J_{n+1}^{1/2} + c_1\widehat{\beta} J_{n+1}^{n/(n+1)} G_{n+1}^{1/(n+1)}. \tag{11}$$

In order to combine Lemmata 1 and 3 we have to lower the index $r = n+1$ for G_r. This can be done by Gagliarda–Nirenberg estimates on the cost of increasing the power of J_{n+1}. Since these estimates are dimension dependent, we do them separately for $d = 1$ and $d = 2$. The task is done as soon as we have an a–priori bound for the L^∞–norm of the solutions. Because of the Sobolev embedding theorem we hence must control $\mathcal{J}_1$ for $d = 1$ and $\mathcal{J}_2$ for $d = 2$.

3 The dimension $d = 1$

The differential inequality (11) for J_1 involves the term $J_2^{1/2} G_2^{1/2}$, which can be controlled by the following generalization of the Gagliarda–Nirenberg estimate.

Lemma 4 *There is a constant c_2 such that for each $r \in [0,2]$ with $r \geq 1/2 - 1/\sigma$ we have*

$$G_2 \leq c_2 (J_2 + G_0)^\gamma \mathcal{G}_r^\delta \quad \text{with } \gamma = \tfrac{(2-r)\sigma}{4+3r\sigma} \text{ and } \delta = \tfrac{4+6\sigma}{4+3r\sigma}. \tag{12}$$

Proof: Note that the assumptions on r imply $\gamma \in [0,1]$ and $\delta + \gamma > 1$. As in the proof of Lemma 3 we decompose $\mathbb{R}^d$ into unit cubes Q_p and apply the standard Gagliarda–Nirenberg estimates with the same constant C_{GN} on each cube. Estimating the weight from above and below and summing up we obtain

$$G_2 \leq C_{\mathrm{GN}}\, \mathrm{e}^{\sqrt{d}} \sum_{p \in \mathbb{Z}^d} \Big(\int_{Q_p} \rho [|\nabla^2 u|^2 + |u|^2] \,\mathrm{d}x \Big)^\gamma \Big(\int_{Q_p} \rho |u|^{2(r\sigma+1)} \,\mathrm{d}x \Big)^{1-\gamma} \Big(\int_{Q_p} |u|^{2(r\sigma+1)} \,\mathrm{d}x \Big)^{\delta+\gamma-1}$$

Estimating the last factor by $\mathcal{G}_r^{\delta+\gamma-1}$ and applying Hölder's inequality to the remaining sum gives $G_2 \leq c_2 J_2^\gamma G_r^{1-\gamma} \mathcal{G}_r^{\delta+\gamma-1}$ which implies the desired result. $\square$

By partial integration we easily find $J_1 \leq G_0^{1/2} (J_1^{1/2} + J_2^{1/2})$ which implies

$$J_1 \leq \mathcal{G}_0 + 2\mathcal{G}_0^{1/2} J_2^{1/2}. \tag{13}$$

Inserting (12) into (11) with $n = 1$ gives $\frac{1}{2}\dot{J}_1 \leq -bJ_1 + \widetilde{M}(b, r, \mathcal{G}_0, \mathcal{G}_r, J_1, J_2)$ with $\widetilde{M} = (\widehat{\alpha}^2 + b) J_1 - J_2 + \widehat{\alpha} J_1^{1/2} J_2^{1/2} + \widehat{\beta} c_1 c_2^{1/2} J_2^{1/2} (J_2^{\gamma/2} + \mathcal{G}_0^{\gamma/2}) \mathcal{G}_r^{\delta/2}$. Since $\widetilde{M}$ is increasing in J_1 we can use (13) and find

$$\tfrac{1}{2}\dot{J}_1 \leq -bJ_1 + M(b, r, \mathcal{G}_0, \mathcal{G}_r) \;\text{ with } M = \sup_{J_2 \geq 0} \widetilde{M}(b, r, \mathcal{G}_0, \mathcal{G}_r, \mathcal{G}_0 + 2\mathcal{G}_0^{1/2} J_2^{1/2}, J_2).$$

The function under the supremum is a sum of the term $-J_2$ and positive terms with factors J_2^κ, $\kappa = 1/2,\ 3/4$, and $(1+\gamma)/2$. As long as $\gamma \in [0,1)$ elementary applications of Young's inequality and $\gamma + \delta \leq \delta/(1-\gamma)$ give

$$M(b, r, \mathcal{G}_0, \mathcal{G}_r) \leq c_5 \Big[(b^2 + \widehat{\alpha}^4) \mathcal{G}_0 + \tfrac{1}{1-\gamma} \widehat{\beta}^{2/(1-\gamma)} \mathcal{G}_{0,r}^{\delta/(1-\gamma)} \Big],$$

where $\mathcal{G}_{0,r} = \max\{\mathcal{G}_0, \mathcal{G}_r\}$.

Theorem 5 *Let $d = 1$ and assume either $\sigma \in (0,2]$ or $\sigma > 2$ and $\widehat{\alpha} < \sigma/(\sigma-2)$. Then, the solutions of cGL exist globally. For $R = \widehat{\alpha}^2$ we have the bound $\overline{\mathcal{J}}_1 \leq C_{(1)}(1, \sigma, \alpha, \beta)$ with*

$$C_{(1)}(1, \sigma, \alpha, \beta) = c_6 \Big[\widehat{\alpha}^{2+2/\sigma} + \tfrac{1}{\sqrt{2-\sigma}} (\widehat{\alpha}^{\nu(\sigma)} \widehat{\beta}^2)^{1/(2-\sigma)} \Big] \quad \text{where } \nu(\sigma) = 5 - \sigma + \tfrac{2}{\sigma}$$

for $\sigma \in (0,2)$. If $\sigma = 2$, then for each $\varepsilon \in (0,1)$ there exists a constant $c_6(\varepsilon)$ with

$$C_{(1)}(1, 2, \alpha, \beta) = c_6(\varepsilon) \widehat{\alpha}^{3/2} \Big[\widehat{\alpha}^2 \widehat{\beta} \Big]^{3/4 + \kappa(\alpha, \varepsilon)/2} \quad \text{where } \kappa(\alpha, \varepsilon) = \max\{\tfrac{1}{2}, \tfrac{2(\widehat{\alpha}-1)}{(1-\varepsilon)}\}.$$

PROOF: According to [DGL94,BGO96] we obtain global existence of solutions if and only if we can find a value r such that $\lambda_{\alpha,r} > 0$ (cf. Lemma 1) and γ in (12) is less than 1. For $\sigma < 2$ we may choose $r = 0$ and for $\sigma = 2$ any $r > 0$ suffices. For $\sigma > 2$ the condition of the theorem arises.

Since $\overline{J}_1 \leq M(b, r, \overline{\mathcal{G}}_0, \overline{\mathcal{G}}_r)/b$ we arrive at the following result by optimizing with respect to b:

$$\overline{J}_1 \leq c_6 \overline{\mathcal{G}}_0^{1/2} [\widehat{\alpha}^2 \overline{\mathcal{G}}_0^{1/2} + (1-\gamma)^{-1/2} \widehat{\beta}^{1/(1-\gamma)} \overline{\mathcal{G}}_{0,r}^{\delta/(2-2\gamma)}]. \tag{14}$$

The estimate for $\sigma < 2$ follows if $r = 0$ is used.

For $\sigma = 2$ we let $r = 1/\kappa(\alpha, \varepsilon)$ which implies $\lambda_{\alpha,r} \geq \varepsilon > 0$ and hence $\overline{\mathcal{G}}_r \leq c(\varepsilon)\widehat{\alpha}^{2r+1}$. This gives the desired result. $\square$

In the case $\sigma < 2$ the result can be sharpened by enlarging r if $|\alpha|$ is not too large. Thus, it is possible to decrease the exponent on $\widehat{\beta}$ by making it dependent on α. For simplicity and clearness we just gave a constant exponent. In the critical case $\sigma = 2$ this is no longer possible as the exponent on $\widehat{\beta}$ behaves like $|\alpha| - 1/4$ (in the limit $\varepsilon = 0$). Of course, these are only upper bounds on $\overline{J}_1$ and it is not clear whether this difference in the upper bounds is induced by the method only or it is really intrinsic to the solution behavior of cGL.

Since bounds on J_n provide bounds for the uniform quantity $\mathcal{J}_n$ by translating the weight function, we are now able to estimate the L^∞–norm.

Corollary 6 *For $d = 1$, $\sigma \in (0, 2]$, and general R we have $\overline{H}_\infty \leq C_\infty(1, \sigma, \alpha, \beta) R^{1/(2\sigma)}$, where*

$$C_\infty(1, \sigma, \alpha, \beta) \leq c_7 \Big[\widehat{\alpha}^{1/2} + (2-\sigma)^{-1/8} (\widehat{\alpha}^\nu \widehat{\beta}^2)^{1/(8-4\sigma)} \widehat{\alpha}^{-1/(2\sigma)}\Big] \quad \textit{and}$$
$$C_\infty(1, 2, \alpha, \beta) \leq c_7(\varepsilon) \widehat{\alpha}^{1/8} (\widehat{\alpha}^2 \widehat{\beta})^{\kappa(\alpha,\varepsilon)/4},$$

where $\nu(\sigma)$, ε and $\kappa(\alpha, \varepsilon)$ are as in Theorem 5.

PROOF: Using Theorem 2 ($\overline{\mathcal{G}}_0 \leq \widehat{\alpha}^{2/\sigma}$ for $R = \widehat{\alpha}^2$) and Theorem 5 the Sobolev estimate $\|u\|_\infty^4 \leq 2(\mathcal{J}_1 + \mathcal{G}_0)\mathcal{G}_0$ provides an estimate $\overline{H}_\infty \leq C(\sigma, \alpha, \beta)$ for the special case $R = \widehat{\alpha}^2$. Since the L^∞ norm scales like $R^{1/(2\sigma)}$ we find the desired result with $C_\infty = \widehat{\alpha}^{-1/\sigma} C$. $\square$

In the case $\sigma = 1$ this result was already obtained in [Mie97, Thm. 3.1] using a completely different method.

4 The dimension $d = 2$

We proceed as in the case $d = 1$ but have to use the appropriate Sobolev embeddings. Since $H^1(\mathbb{R}^2)$ does not embed into $L^\infty(\mathbb{R}^2)$ we have to control not only $\mathcal{J}_1$ but also $\mathcal{J}_2$.

Lemma 7 *There is a constant c_2 such that*

$$
\begin{aligned}
G_2 &\le c_2(J_2+G_0)^{\gamma_2}\mathcal{G}_r^{\delta_2} \quad \textit{with } \gamma_2=\tfrac{\sigma(2-r)}{2+r\sigma},\ \delta_2=\tfrac{2+2\sigma}{2+r\sigma},\ r\in[0,2],\ r\ge 1-1/\sigma,\\
G_3 &\le c_2(J_3+G_0)^{\gamma_3}\mathcal{G}_r^{\delta_3} \quad \textit{with } \gamma_3=\tfrac{\sigma(3-r)}{3+2r\sigma},\ \delta_3=\tfrac{3+6\sigma}{3+2r\sigma},\ r\in[0,3],\ r\ge 1-1/\sigma.
\end{aligned} \tag{15}
$$

Inserting (15) and (13) into (11) gives, as in the case $d=1$, an a–priori bound for J_1, as soon as r can be chosen such that $\gamma_2<1$ and $\lambda_{\alpha,r}>0$. This is always possible for $\sigma\le 1$ while for $\sigma>1$ the restriction $\widehat{\alpha}<\sigma/(\sigma-1)$ has to be satisfied.

Theorem 8 *For $d=2$ assume either $\sigma\in(0,1]$ or $\sigma>1$ and $\widehat{\alpha}\le\sigma/(\sigma-1)$. Then, the solutions of cGL exist globally. In the case $R=\widehat{\alpha}^2$ we have $\overline{\mathcal{J}}_n\le C_{(n)}(2,\sigma,\alpha,\beta)$ with the following estimates. There is $c_8>0$ such that for all $\sigma\in(0,1)$*

$$C_{(1)}(2,\sigma,\alpha,\beta)=\tfrac{c_8}{\sqrt{1-\sigma}}\Big[\widehat{\alpha}^{2/\sigma}\widehat{\beta}\Big]^{1/(1-\sigma)},\quad C_{(2)}(2,\sigma,\alpha,\beta)=\tfrac{c_8}{(1-\sigma)^{3/4}}\Big[\widehat{\alpha}^{1+3/\sigma}\widehat{\beta}^2\Big]^{1/(1-\sigma)}.$$

If $\sigma=1$, then for each $\varepsilon\in(0,1)$ there is $c_8(\varepsilon)$ such that

$$C_{(1)}(2,1,\alpha,\beta)=c_8(\varepsilon)\widehat{\alpha}^{5/2}\Big[\widehat{\alpha}^2\widehat{\beta}\Big]^{1/2+\kappa(\alpha,\varepsilon)},\quad C_{(2)}(2,1,\alpha,\beta)=c_8(\varepsilon)\widehat{\alpha}^{15/4}(\widehat{\alpha}^2\widehat{\beta})^{5/4+2\kappa}$$

with $\kappa(\alpha,\varepsilon)=\max\{\frac12,\frac{2(\widehat{\alpha}-1)}{1-\varepsilon}\}$.

Proof: Since (14) is still valid if δ and γ are replaced by δ_2 and γ_2, we obtain, by the choice of $r=0$ for $\sigma<1$ and $r=1/\kappa$ for $\sigma=1$, the estimates $\overline{\mathcal{J}}_1\le C_{(1)}(2,\sigma,\alpha,\beta)$.

Now we consider (11) with $n=2$, where $J_2\le J_1+2J_1^{1/2}J_3^{1/2}$ is used to eliminate J_2:

$$
\begin{aligned}
\tfrac12\dot J_2 &\le -bJ_2-J_3+c_8\Big[(\widehat{\alpha}^2{+}b)J_1+(\widehat{\alpha}^2{+}b)J_1^{1/2}J_3^{1/2}+\widehat{\alpha}J_1^{1/4}J_3^{3/4}+\widehat{\beta}J_3^{(2+\gamma_3)/3}\mathcal{G}_r^{\delta_3/3}\Big]\\
&\le -bJ_2+c_9\Big[(\widehat{\alpha}^4{+}b^2)J_1+\tfrac{1}{1-\gamma_3}(\mathcal{G}_r^{\delta_3/3}\widehat{\beta})^{3/(1-\gamma_3)}\Big].
\end{aligned}
$$

Note that $\gamma_2<1$ implies already $\gamma_3<1$ such that a bound for J_2 can be obtained if this is the case for J_1. Taking the limit $t\to\infty$ and optimizing with respect to b yields

$$\overline{J}_2\le c_{10}\Big[\widehat{\alpha}^2\overline{J}_1+\tfrac{1}{\sqrt{1-\gamma_3}}\overline{J}_1^{1/2}(\overline{\mathcal{G}}_r^{\delta_3/3}\widehat{\beta})^{3/(2-2\gamma_3)}\Big]. \tag{16}$$

Using the same choice of r as above the results for $C_{(2)}$ follow. □

This implies the following L^∞–bound.

Corollary 9 *For $d=2$, $\sigma\in(0,1]$ and general R we have $\overline{H}_\infty\le C_\infty(2,\sigma,\alpha,\beta)R^{1/(2\sigma)}$ with the following estimates. For $\sigma\in(0,1)$ and each $\delta\in(0,1)$ there is a constant $c_9(\delta)$ such that*

$$C_\infty(2,\sigma,\alpha,\beta)\le c_9(\delta)(1-\sigma)^{-1/4}\Big[\widehat{\alpha}^{2/\sigma}\widehat{\beta}\Big]^{1/(2-2\sigma)}\Big[(1-\sigma)\widehat{\alpha}^{4/\sigma}\Big]^{\delta/16}.$$

For $\sigma=1$ and each $\varepsilon,\delta\in(0,1)$ there is a constant $c_9(\varepsilon,\delta)$ such that

$$C_\infty(2,1,\alpha,\beta)\le c_{12}(\varepsilon,\delta)\widehat{\alpha}^{1/4}\Big[\widehat{\alpha}^2\widehat{\beta}\Big]^{1/4+\kappa(\alpha,\varepsilon)/2}\Big[\widehat{\alpha}^5\widehat{\beta}\Big]^{\delta/16}.$$

PROOF: The proof follows from the Sobolev estimate

$$H_\infty \le c(\delta)\mathcal{G}_0^{\delta/4}(\mathcal{J}_2+\mathcal{G}_0)^{\delta/4}(\mathcal{J}_1+\mathcal{G}_0)^{(1-\delta)/2} \tag{17}$$

and the scaling argument since the results in Theorem 8 are for $R=\widehat{\alpha}^2$. $\square$

5 The modulationally stable domain

For dispersion parameters in the modulationally stable domain

$$\mathcal{P}_{\mathrm{ms}}(\sigma)=\{\,(\alpha,\beta)\in\mathbb{R}^2 \,:\, -(1+\alpha\beta)<|\alpha-\beta|\sigma^*\,\} \qquad \text{with } \sigma^*=\frac{\sqrt{2\sigma+1}}{\sigma}$$

the above estimates can be improved considerably. For this purpose we consider the functional

$$F_1=J_1+\frac{a}{\sigma+1}G_1.$$

It turns out that for a suitable choice of a it is possible to derive a new a–priori bound for F_1 which relies on cancellation properties of the nonlinear terms in (5) and (6), see [DGL94]. In addition to $Q_{\alpha,1}(u)$ (see (8)) we use the expression

$$V(N,u)=\int \rho\begin{pmatrix}|u|^{2\sigma}u\\ \Delta u\end{pmatrix}\cdot N\begin{pmatrix}|u|^{2\sigma}\overline{u}\\ \Delta\overline{u}\end{pmatrix}\,\mathrm{d}x$$

which is real for $N=N^*$. Partial integration gives

$$\begin{aligned}0 &= \mathrm{Re}\,[(1+\mathrm{i}\gamma)\textstyle\int \mathrm{div}(\rho|u|^{2\sigma}\overline{u}\nabla u)\,\mathrm{d}x]\\ &= \mathrm{Re}\,[(1+\mathrm{i}\gamma)\textstyle\int |u|^{2\sigma}\overline{u}\,\nabla\rho\cdot\nabla u\;\mathrm{d}x]+Q_{\gamma,1}(u)+V(N_{1+\mathrm{i}\gamma},u)\end{aligned}$$

where $N_z=\frac{1}{2}\left(\begin{smallmatrix}0&\overline{z}\\ z&0\end{smallmatrix}\right)$. Since $Q_{\gamma,1}(u)\ge 0$ for $|\gamma|\le\sigma^*$, we immediately obtain

$$\widehat{\gamma}G_2^{1/2}J_1^{1/2}-V(N_{1+\mathrm{i}\gamma},u)\ge 0 \;\text{ for } |\gamma|\le\sigma^* \text{ with } \widehat{\gamma}=\sqrt{1+\gamma^2}. \tag{18}$$

In deriving a differential inequality for F_1 we introduce $\widetilde{J}_2=\int\rho|\Delta u|^2\,\mathrm{d}x$ and use the relation $\int\rho\nabla u^*\cdot\nabla(\Delta u)\,\mathrm{d}x=-\int(\nabla\rho\cdot\nabla u^*)\Delta u\,\mathrm{d}x-\widetilde{J}_2$ in place of (4). Adding (5) and (6) we have, for $R=\widehat{\alpha}^2$,

$$\tfrac{1}{2}\dot{F}_1\le\widehat{\alpha}^2(J_1+aG_1)+(\widehat{\alpha}\widetilde{J}_2^{1/2}+\widehat{\beta}G_2^{1/2})J_1^{1/2}+V(\widetilde{N},u) \tag{19}$$

with $\widetilde{N}=\frac{1}{2}\left(\begin{smallmatrix}-2a & 1+a+i(\beta-a\alpha)\\ 1+a-i(\beta-a\alpha) & -2\end{smallmatrix}\right)$. The term $-\widetilde{J}_2$ is represented via the lower diagonal entry of $\widetilde{N}$. Multiply (18) by $\mu\ge 0$ and add it to (19) to find, for any $\kappa\in(0,1)$ and $b>0$,

$$\tfrac{1}{2}\dot{F}_1\le\widehat{\alpha}^2(J_1+aG_1)+(\widehat{\alpha}\widetilde{J}_2^{1/2}+(\widehat{\beta}+\mu\widehat{\gamma})G_2^{1/2})J_1^{1/2}-(1-\kappa)(\widetilde{J}_2+aG_2)+V(\widehat{N},u), \tag{20}$$

with $\widehat{N} = \frac{1}{2}\begin{pmatrix} -2a\kappa & 1+a-\mu+\mathrm{i}(\beta-a\alpha+\mu\gamma) \\ 1+a-\mu-\mathrm{i}(\beta-a\alpha+\mu\gamma) & -2\kappa \end{pmatrix}$.

The set $\mathcal{P}_{\mathrm{ms}}(\sigma)$ is exactly defined by the condition that it is possible to find $\kappa \in [0,1)$, $a > 0$, $\mu \geq 0$ and $\gamma \in [-\sigma^*, \sigma^*]$, such that $\widehat{N}$ is negative semi–definite. Hence, the term $V(\widehat{N}, u)$ can be dropped. For simplicity, our future analysis will be restricted to the case where κ can be taken equal to 0. This reduces our analysis to the set

$$\mathcal{R}_{\sigma^*} = (-\infty, \sigma^*)^2 \cup (-\sigma^*, \infty)^2$$

which has the same asymptotes as $\mathcal{P}_{\mathrm{ms}}(\sigma)$. For $(\alpha, \beta) \in \mathcal{R}_{\sigma^*}$ we let

$$a = \frac{s\beta+\sigma^*}{s\alpha+\sigma^*}, \quad \mu = 1 + a, \quad \text{and } \gamma = \frac{\sigma^*(\beta-\alpha)}{s(\alpha+\beta)+2\sigma^*},$$

where $s = 1$ for $\alpha + \beta \geq 0$ and $s = -1$ else. Then, the off–diagonal entries of $\widehat{N}$ vanish and $\kappa = 0$ is allowed.

Using that (13) also holds when J_2 is replaced by $\widetilde{J}_2$, the quantity J_1 can be eliminated; and we obtain, with $\kappa = 0$, $\widehat{\gamma} \leq 1 + 1/\sigma$, and after taking the supremum over $\widetilde{J}_2 \geq 0$,

$$\begin{aligned} \tfrac{1}{2}\dot{F}_1 &\leq -bF_1 - aG_2 \\ &\quad + c\Big[(\widehat{\alpha}^4+b^2)G_0 + (\widehat{\alpha}^2+b)aG_1 + (\widehat{\beta}+\mu)G_0^{1/2}G_2^{1/2} + (\widehat{\beta}+\mu)^{4/3}G_0^{1/3}G_2^{2/3}\Big]. \end{aligned}$$

Eliminating G_1 via $G_1 \leq G_0^{1/2}G_2^{1/2}$ the right–hand side can be further estimated by taking the maximum with respect to G_2 and we arrive at

$$\tfrac{1}{2}\dot{F}_1 \leq -bF_1 + c\Big[\widehat{\alpha}^4(1+a) + b^2(1+a) + (\widehat{\beta}+a)^4/a^2\Big]G_0,$$

where $\mu = 1 + a$ was used. Optimizing with respect to b produces

$$\overline{F}_1 = \limsup_{t\to\infty} F_1(t) \leq c(1+a^{1/2})[\widehat{\alpha}^2(1+a^{1/2}) + (\widehat{\beta}+a)^2/a]\overline{G}_0. \tag{21}$$

This provides new estimates on $\overline{J}_1 \leq F_1$ and $\overline{G}_1 \leq (1+\sigma)\overline{F}_1/a$ which allow for improving the previous results, particularly in the case $d\sigma = 2$ where only exponential bounds could be derived.

Our definition of a is such that $a+1/a$ blows up as the boundary of $\mathcal{R}_{\sigma^*}$ is approached. However, for $(\alpha, \beta) \in \mathcal{R}_{\sigma^*-\varepsilon}$ we have $c_1(\varepsilon)\widehat{\beta}/\widehat{\alpha} \leq a \leq c_2(\varepsilon)\widehat{\beta}/\widehat{\alpha}$, and (21) reduces to

$$\max\{\overline{J}_1, \tfrac{a}{\sigma+1}\overline{G}_1\} \leq \overline{F}_1 \leq c(\varepsilon)\widehat{\alpha}^{(2+\sigma)/(2\sigma)}(\widehat{\alpha}+\widehat{\beta})^{3/2} \qquad \text{for } (\alpha, \beta) \in \mathcal{R}_{\sigma^*-\varepsilon}. \tag{22}$$

This implies the following L^∞–bound.

Theorem 10 *Let $d = 1, 2$, $\sigma > 1/2$, and $(\alpha, \beta) \in \mathcal{P}_{\mathrm{ms}}(\sigma)$. Then, the solutions of cGL exist globally and $\overline{H}_\infty \leq C_\infty(d, \sigma, \alpha, \beta)R^{1/(2\sigma)}$ with the following estimates. For each $\varepsilon \in (0, \sigma^*)$ and each $\delta \in (0, 1)$ there exists constants $c_1(\varepsilon)$ and $c_2(\varepsilon, \delta)$ such that for all $(\alpha, \beta) \in \mathcal{R}_{\sigma^*-\varepsilon}$*

$$\begin{aligned} &C_\infty(1, \sigma, \alpha, \beta) \leq c_1(\varepsilon)\min\{\widehat{\alpha}^{1/8}(\widehat{\alpha} + \widehat{\beta})^{3/8}, (\widehat{\alpha}^3/\widehat{\beta} + \widehat{\beta}^2)^{1/(2\sigma+4)}\} \text{ and} \\ &C_\infty(2, \sigma, \alpha, \beta) \leq c_2(\varepsilon, \delta)[\widehat{\alpha}(\widehat{\alpha} + \widehat{\beta})^3]^{1/4+\delta(3+\sigma)}. \end{aligned}$$

PROOF: For $d = 1$ we have $H_\infty \le C \min\{[(\mathcal{J}_0+\mathcal{G}_0)\mathcal{G}_0]^{1/4}, [(\mathcal{J}_1+\mathcal{G}_0)\mathcal{G}_1]^{1/(2\sigma+4)}\}$. Using $\overline{\mathcal{G}}_0 \le c\hat{\alpha}^{2/\sigma}$ and (22) gives the desired result for $R = \hat{\alpha}^2$. The case for general R is obtained by rescaling.

For $d = 2$ we note that our bounds on J_1 and G_1 imply via (16) with $r = 1$ a polynomial bound on J_2, for instance the rough estimate $\overline{J}_2 \le c(\hat{\alpha}+\hat{\beta})^{14+4\sigma}$. The result is achieved by employing (17) and rescaling. □

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