Mathematical Analysis of Sideband Instabilities with Application to Rayleigh–Bénard Convection

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Dedicated to the memory of Juan Simo

Abstract

We introduce a new method for the analysis of sideband instabilities which are important for periodic patterns appearing in systems close to the instability threshold. The method relies on a two–fold application of the Liapunov–Schmidt reduction procedure, a first application to the nonlinear bifurcation problem and a second application to the linear spectral problem. We obtain rigorous result on the spectrum of the associated linearization in spaces allowing for general sideband perturbations by treating the sideband vector and the spectral parameter as small bifurcation parameters.

We apply the theory to the small roll solutions in Rayleigh–Bénard convection and derive domains in Rayleigh, Prandtl and wave number space where the rolls are unstable. We recover the Eckhaus, zigzag, and the skew–varicose instabilities obtained earlier by formal methods.

RUNNING TITLE: Sideband instabilities for convection rolls

1 Introduction

The development of modern bifurcation theory led to the understanding of many nonlinear phenomena in the natural sciences, notably problems involving the spontaneous formation of patterns from a homogeneous state. While the bifurcation of these new patterns is often very well understood mathematically, there is still a need for tools in order to study the associated stability problems. Here we are concerned with problems which are posed on physically extended domains which generate patterns of much smaller wave length. Classical bifurcation and stability theory, e.g. the Liapunov–Schmidt reduction or the center manifold technique, can easily treat the stability of such patterns with respect to the perturbations which have the same type of periodicity as the basic pattern. However, many problems involve instabilities with respect to more general perturbations, such as quasiperiodic or localized ones.

It was Wiktor Eckhaus [Eck65] who gave a first systematic approach to such sideband instabilities for problems which have one unbounded direction such as the Couette–Taylor problem between infinitely long cylinders. From this starting point a vast body of research emanated in order to understand more complicated situations in hydrodynamic stability theory, e.g. involving mean flow effects or two unbounded spatial directions, cf. [Bu71], [SdP78], [BuB84], [Be94], and for a recent review see [NPL93, DP94]. Yet, all of these theories are lacking a sound mathematical basis, since they are built on formal methods involving the method of amplitude equations or the three (or higher) mode interaction theory.

First rigorous approaches to the Eckhaus instability were given in [MT84, CE90], where the one–dimensional Swift–Hohenberg problem was considered. A more general mathematical approach to sideband instabilities was developed in [BM95, BM96], which can detect instabilities of small bifurcating periodic solution for problems on infinitely long cylindrical domains. The method relies on the theory of spatial center manifolds in which the elliptic nonlinear problem as well as the eigenvalue problem are reduced to ordinary differential equations. In [BM95] a first rigorous proof of the Benjamin–Feir instability of gravity surface waves on a fluid layer of finite depth could be derived.

Here we introduce a new method, which was already outlined in [Mi95], where the sideband instabilities of the roll solutions in the two–dimensional Swift–Hohenberg equation were studied. This methods will be generalized here and applied in detail to the sideband instabilities of convection rolls in the Rayleigh–Bénard problem with stress–free boundary conditions. In contrast to [BM95] we base our theory on the Liapunov–Schmidt reduction (LSR) rather than the spatial center manifold reduction. The key observation is that the spectral problem obtained from linearization at the given steady state can also be reduced by the LSR to a finite dimensional eigenvalue problem which then contains all the needed information on possible unstable sidebands.

To be more specific let us consider a system of PDEs for a vector–valued function

$u(t,x,\widetilde{x})$ on the domain $\mathbb{R}^d\times\Sigma$ where $x\in\mathbb{R}^d$, $d\geq 1$, denotes the unbounded variable whereas $\widetilde{x}\in\Sigma\subset\mathbb{R}^m$, $m\geq 0$, denotes the bounded cross–sectional variables, namely

$$M\partial_t u=\mathcal{N}(\mu,\partial_x,u)\stackrel{\text{def}}{=}A_\mu(\partial_x)u+N(\mu,u). \tag{1.1}$$

Here $A_\mu(\partial_x)$ is an elliptic operator on $\mathbb{R}^d\times\Sigma$ where only the derivatives with respect to the unbounded variables x are indicated for later convenience. For the present purposes it suffices to assume that M is a constant bounded linear operator, but generalizations are possible. Moreover, $N(\mu,u)=\mathcal{O}(|u|^2)$, μ is a vector of parameters with $|\mu|$ small, and $u\equiv 0$ is the trivial homogeneous solution.

The bifurcation of steady, spatially periodic pattern can be treated by considering the time–independent problem on a function space Z with given periodicity in $x\in\mathbb{R}^d$. The basic assumption is that the bifurcation problem $\mathcal{N}(\mu,\partial_x,u)=0$ can be reduced by the LSR, i.e. $A_0(\partial_x):Z\to Y$ has a finite–dimensional kernel Z_0 and co–range Y_0, such that the projections $Q:Z\to Z_0$ and $P:Y\to Y_0$ are bounded. Let $u=u_0+u_1$ with $u_0=Qu$ and consider:

$$\begin{aligned} P\mathcal{N}(\mu,\partial_x,u_0+u_1) &= 0,\\ (I-P)\mathcal{N}(\mu,\partial_x,u_0+u_1) &= 0. \end{aligned} \tag{1.2}$$

The basic idea is to solve the second equation in (1.2) by the implicit function theorem for $u_1=\mathcal{U}(\mu,u_0)$ in order to obtain the finite dimensional bifurcation problem

$$0=n(\mu,u_0)\stackrel{\text{def}}{=}P\mathcal{N}(\mu,\partial_x,u_0+\mathcal{U}(\mu,u_0)). \tag{1.3}$$

Using this algebraic problem in Z_0 the existence of nontrivial patterns can be studied. We refer to [CH88] and [GS85], Ch. VII, for the general theory of the LSR.

Assume now that we have found a family $(\mu,u_\mu)=(\mu,u_{\mu 0}+\mathcal{U}(\mu,u_{\mu 0}))$ of steady states of (1.1). To study the stability of u_μ we consider the linearization of the full problem (1.1) and assume that the perturbation is of the form $v(t,x)=e^{\lambda t}w(x)$. We obtain

$$M(\lambda w)=A_\mu(\partial_x)w+DN(\mu,u_\mu)w. \tag{1.4}$$

This resolvent problem again has to be solved in some function space which should include general enough perturbations, e.g. $w\in L^\infty(\mathbb{R}^d,L^2(\Sigma))$. It is obvious that u_μ is (linearly) unstable if there exists (λ,w) with $\operatorname{Re}\lambda>0$ and a nontrivial bounded w.

Following the formal approaches to sideband instability (see e.g. [Bu71, BuB84, Be94]) it is natural to look for perturbations in the form $v(t,x)=e^{\lambda t+i\sigma\cdot x}W(x)$ with $\sigma\in\mathbb{R}^d$ and $W\in Z$. Here σ is called the sideband vector and it will be small in our analysis. Hence, in general v is quasiperiodic in $x\in\mathbb{R}^d$ and need not lie in Z. The problem for W reads

$$0=\mathcal{B}(\mu,\sigma,\lambda)W\stackrel{\text{def}}{=}[D_u\mathcal{N}(\mu,i\sigma+\partial_x,u_\mu)-\lambda M]\,W \tag{1.5}$$

where $\mathcal{B}(\mu,\sigma,\lambda)$ is a bounded linear operator from Z into Y. Note the appearance of $i\sigma+\partial_x$, which also explains why we display the differential with respect to the unbounded variable. Since $\mathcal{B}(0,0,0) = A_0(\partial_x)$ it is obvious that nontrivial solutions of (1.5) can be obtained via LSR in the form $W = W_0 + W_1$ with $W_1 = \mathcal{W}(\mu,\sigma,\lambda)W_0$; and (1.5) is equivalent to the *reduced spectral problem*

$$0 = b(\mu,\sigma,\lambda)W_0 \stackrel{\text{def}}{=} P\mathcal{B}(\mu,\sigma,\lambda)\left[W_0 + \mathcal{W}(\mu,\sigma,\lambda)W_0\right] \qquad (1.6)$$

for small (μ,σ,λ) and the same P as in (1.2). Instability of the steady periodic pattern (μ,u_μ) of (1.1) is shown as soon as we find (σ,λ) such that $\det b(\mu,\sigma,\lambda) = 0$ and $\operatorname{Re}\lambda > 0$. This reduction of the spectral problem (1.4) leads to the *Principle of Reduced Instability* (cf. [BM95, BM96] and [Mi95]) since it is clear that any unstable solution (i.e. $\operatorname{Re}\lambda > 0$) of the reduced spectral problem (1.6) leads to instability of u_μ in the full problem (1.1). However, the opposite conclusion need not be valid, since our reduction requires λ and σ to be small, such that instabilities with large λ or σ are not detected.

Thus, the question of sideband instability has been reduced to a sequence of two related LSRs: (i) the nonlinear bifurcation problem and (ii) the linear spectral problem. The functional analytic set up in both reductions is the same (i.e. Z_0, Z_1, Y_0, Y_1 do not change), in fact in Theorem 2.2 we show

$$b(\mu,0,0) = D_{u_0}n(\mu,u_{\mu 0}) \qquad \text{and } b(0,\sigma,\lambda) = P(A_0(i\sigma+\partial_x) - \lambda M)Q.$$

There are several papers on the so–called *Principle of Reduced Stability* which states that the stability of a steady state $u_\mu \in Z$ of the problem $\partial_t u = \mathcal{N}(\mu,u)$ can be determined from the stability of the steady state $u_{\mu 0} \in Z_0$ of the problem $\partial_t u_0 = n(\mu,u_0)$, cf. [GS85], Ch. I, Thm. 4.1. However, in general the desired result does not hold (see [KL83]) and a very subtle theory was developed to derive sufficient condition for the validity of the principle [Va84, Re89, Da91]. Our principle of reduced instability works the other way round. We construct a slightly more complicated reduced stability problem (involving λ nonlinearly) which contains the exact stability information.

The main part of this paper is the application of the abstract theory to the Bénard problem for a fluid layer between two horizontal plates at distance π, that is, for the flow in the domain $\mathcal{Q}_\infty = \mathbb{R}^2 \times (0,\pi)$. Let (u,θ,p) be nondimensionalized velocity, temperature, and pressure, $\mathrm{Ra} = \mathcal{R}^2$ the Rayleigh and $\mathcal{P}$ the Prandtl number; then the problem reads

$$\left.\begin{aligned} \partial_t u + (u\cdot\nabla)u + \nabla p - \Delta u - \mathcal{R}\theta e_3 &= 0 \\ \mathcal{P}(\partial_t\theta + (u\cdot\nabla)\theta) - \Delta\theta - \mathcal{R}u_3 &= 0 \\ -\operatorname{div} u &= 0 \end{aligned}\right\} \quad \text{in } \mathcal{Q}_\infty,$$
$$\partial_{x_3}u_1 = \partial_{x_3}u_2 = u_3 = \theta = 0 \quad \text{on } \partial\mathcal{Q}_\infty.$$

The bifurcation parameter is $\mathcal{R} - \mathcal{R}_0$ with the critical Rayleigh number $\mathrm{Ra}_{\mathrm{crit}} = \mathcal{R}_0^2 = 27/4$. The critical wave number is $k_0 = 1/\sqrt{2}$. We have assumed stress–free boundary conditions which enables us to perform all calculations in the reduction process analytically. Of course the problem with no–slip boundary conditions (i.e. $u = 0$ on $\mathcal{Q}_\infty$) is also tractable with the method developed here, however, it is not known how to give the relevant eigenfunctions explicitly. Before going into the analysis we give a definition of instability for the Bénard problem, which is somewhat tricky since the evolution problem cannot be posed in $L^\infty(\mathcal{Q}_\infty)$, as we shall see in Section 3.

The existence of steady Bénard convection rolls is classical and we give, for completeness, an independent proof in which important information for the stability question is prepared. For all $\mathcal{R} > \widetilde{\mathcal{R}}(k,0) = (1+k^2)^{3/2}/k$ (with $|k-k_0|$ and $\mathcal{R} - \widetilde{\mathcal{R}}(k,0)$ sufficiently small) there exists a solution which is two–dimensional ($u_2 \equiv 0, \partial_{x_2}(u,\theta,p) \equiv 0$) and has period $2\pi/k$ in the x_1–direction:

$$
z_{\rho,k}(x_1,x_3) = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \theta \\ p \end{pmatrix} = \rho \begin{pmatrix} \sqrt{2}\cos(kx_1)\cos x_3 \\ 0 \\ \sin(kx_1)\sin x_3 \\ \sqrt{3}\sin(kx_1)\sin x_3 \\ -3\sin(kx_1)\cos x_3 \end{pmatrix} + \mathcal{O}(\rho^2)
$$

where $k \approx k_0 = 1/\sqrt{2}$ and ρ is defined such that $\mathcal{R} = (1+k^2)^{3/2}/k + \sqrt{3}\mathcal{P}^2\rho^2/16 + \mathcal{O}(\rho^4)$.

Rescale the periodicity variable $y = kx_1$ with $y \in S^1 = \mathbb{R}/_{2\pi\mathbb{Z}}$. Our construction in Section 4 is built on the basic spaces

$$
\begin{aligned}
&Y = L^2(\Omega) \times H^1(\Omega) \ \text{ and} \\
&Z = \{\, (u,\theta,p) \in [H^2(\Omega)]^4 \times H^1(\Omega) \ : \ \partial_{x_3}u_1 = \partial_{x_3}u_2 = u_3 = \theta = 0 \text{ on } \partial\Omega \,\},
\end{aligned}
$$

where $\Omega = S^1 \times (0,\pi)$ is the bounded periodicity domain.

The linear operator $A_0 : Z \to Y$ of the steady problem at $(\mathcal{R},k) = (\mathcal{R}_0,k_0)$ is symmetric with respect to the $L^2(\Omega)^5$ scalar product and has a five–dimensional kernel which is spanned by $\xi_1,\ldots,\xi_5$ given by

$$
\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \sqrt{2}\cos y\cos x_3 \\ 0 \\ \sin y \sin x_3 \\ \sqrt{3}\sin y\sin x_3 \\ -3\sin y\cos x_3 \end{pmatrix}, \begin{pmatrix} \sqrt{2}\sin y\cos x_3 \\ 0 \\ -\cos y\sin x_3 \\ -\sqrt{3}\cos y\sin x_3 \\ 3\cos y\cos x_3 \end{pmatrix}.
$$

Note that ξ_1 and ξ_2 describe mean flow effects in x_1 and x_2 direction while ξ_3 corresponds to the trivial addition of a constant to the pressure.

Letting $z_0 = \sum \eta_j\xi_j \in Z_0$, the LSR of the steady problem leads to

$$
\widetilde{n}(\mathcal{R},k,\eta) = \begin{pmatrix} g_1(\mathcal{R},k,\eta_1^2,r)\eta_1 \\ 0 \\ 0 \\ g_4(\mathcal{R},k,\eta_1^2,r)\eta_4 - g_5(\mathcal{R},k,\eta_1^2,r)\eta_1\eta_5 \\ g_5(\mathcal{R},k,\eta_1^2,r)\eta_1\eta_4 + g_4(\mathcal{R},k,\eta_1^2,r)\eta_5 \end{pmatrix} = 0
$$

where $r=\eta_4^2+\eta_5^2$. The special structure of $\widetilde{n}$ follows from the symmetry properties of the Bénard problem given in Section 3. Since $g_5(\mathcal{R}_0,k_0,0,0)\neq 0$, the expansion

$$g_4(\mathcal{R},k,\eta_1^2,r)=-\tfrac{\sqrt{3}}{2}(\mathcal{R}-\mathcal{R}_0)+3(k-k_0)^2-\tfrac{7\mathcal{P}^2+34\mathcal{P}-23}{200}\eta_1^2+\tfrac{3\mathcal{P}^2}{32}r+\cdots$$

yields immediately the unique solution family $\eta_{\rho,\tau}=(0,\alpha,\beta,\rho\cos\tau,\rho\sin\tau)^T$, $\alpha,\beta\in\mathbb{R}$, $\tau\in[0,2\pi)$, with $\rho=\sqrt{r}$ and $\mathcal{R}=\widetilde{\mathcal{R}}(k,\rho^2)$.

In Section 5 we analyze the stability of the roll solution $z_{\rho,k}$ associated to $\eta=(0,0,0,\rho,0)^T$. We investigate perturbations in the form

$$\widehat{z}(t,x_1,x_2,x_3)=e^{\lambda t+i(\sigma_1x_1+\sigma_2x_2)}W(kx_1,x_3)\quad\text{with } W\in Z$$

which leads to a (ρ,k,σ,λ)–depending linear problem which is treated again by the LSR. Using $W_0=\sum\omega_j\xi_j\in Z_0$ we find the reduced problem $\widetilde{b}(\rho,k,\sigma,\lambda)\omega=0$ where the matrix $\widetilde{b}\in\mathbb{C}^{5\times5}$ has the form

$$\widetilde{b}=\begin{pmatrix}\gamma_{11}&-\sigma_1\sigma_2\gamma_{12}&i\sigma_1&i\sigma_1\rho\gamma_{14}&\rho\gamma_{15}\\-\sigma_1\sigma_2\gamma_{21}&\gamma_{22}&i\sigma_2&i\sigma_2\rho\gamma_{24}&-\sigma_1\sigma_2\rho\gamma_{25}\\-i\sigma_1&-i\sigma_2&0&0&0\\i\sigma_1\rho\gamma_{41}&i\sigma_2\rho\gamma_{42}&0&\gamma_{44}&i\sigma_1\gamma_{45}\\\rho\gamma_{51}&-\sigma_1\sigma_2\rho\gamma_{52}&0&i\sigma_1\gamma_{54}&\gamma_{55}\end{pmatrix}$$

where all γ_{jl} are analytical functions of $(\rho^2,k,\sigma_1^2,\sigma_2^2,\lambda)$. This special form of $\widetilde{b}$ exploits the reflection symmetries of the Bénard rolls. Using the structure of the third column and the third row, the problem $\widetilde{b}\omega=0$ can be reduced to $b^*\omega^*=0$ for an equivalent 3×3 matrix b^* which then corresponds to the matrices which were derived in [BuB84] and [Be94] by purely formal methods. Expanding the eigenvalue equation

$$0=\Lambda(\rho,k,\sigma,\lambda)\stackrel{\text{def}}{=}\det b^*(\rho,k,\sigma,\lambda)$$

we obtain instability of $z_{\rho,k}$ whenever there exists (σ,λ) such that $\Lambda(\rho,k,\sigma,\lambda)=0$ and $\operatorname{Re}\lambda>0$. To lowest order this equation is

$$0=|\sigma|^2\Big(\frac{9}{16}(\mathcal{P}+1)^2\lambda^3+p_2(\rho^2,\sigma_1^2,\sigma_2^2)\lambda^2+p_1(\rho^2,\sigma_1^2,\sigma_2^2)\lambda+p_0(\rho^2,k,\sigma_1^2,\sigma_2^2)\Big),\qquad(1.7)$$

which can be analyzed by Hurwitz's stability criterium [Hu895]:

$$p_j\geq0\quad\text{for } j=0,1,2\quad\text{ and }\quad p_1p_2-\frac{9}{16}(\mathcal{P}+1)^2|\sigma|^2p_0\geq0.\qquad(1.8)$$

These inequalities are necessary and sufficient for all three roots to satisfy $\operatorname{Re}\lambda\leq0$, and thus any violation gives rise to an instability.

From this expansion the classical instabilities will be derived; the Eckhaus instability with $\sigma=(\sigma_1,0)$, the zigzag instability with $\sigma=(0,\sigma_2)$, the monotonic skew–varicose (MSV) instability, where $\lambda>0$ and $\sigma_1\sigma_2\neq0$, and the oscillatory skew–varicose

(OSV) instability where $|\mathrm{Im}\,\lambda|\,\mathrm{Re}\,\lambda \neq 0$ and $\sigma_1\sigma_2 \neq 0$. The main conlusion is that small convection rolls existing for the parameter values $(\mathcal{R}, k) \approx (\mathcal{R}_0, k_0)$ are unstable if $\mathcal{P} < \mathcal{P}_{\mathrm{MSV}} \approx 0.78197$. For larger $\mathcal{P}$ the rolls are unstable with respect to the MSV instability if

$$\mathcal{R} > \max\{\widetilde{\mathcal{R}}(k,0),\, \mathcal{R}_{\mathrm{MSV}}(k)\} \quad \text{where } \mathcal{R}_{\mathrm{MSV}}(k) = \mathcal{R}_0 - \frac{6\sqrt{6}}{7}(k-k_0) + o(|k-k_0|).$$

Moreover, small rolls are unstable with respect to the OSV instability for $k < k_0$ and $\widetilde{\mathcal{R}}(k,0) \leq \mathcal{R} \leq \mathcal{R}_{\mathrm{OSV}}(k)$ where

$$\mathcal{R}_{\mathrm{OSV}}(k) = \mathcal{R}_0 - \frac{6\sqrt{6}\,(\mathcal{P}+1)^2}{3\mathcal{P}^2\sqrt{(\mathcal{P}+1)(\mathcal{P}+5)} + (\mathcal{P}+3)(3\mathcal{P}^2+2\mathcal{P}+2)}(k-k_0) + o(|k-k_0|).$$

A more exact statement is given in Theorem 5.8.

The instability of all rolls in the case $\mathcal{P} < \mathcal{P}_{\mathrm{MSV}}$ was already reported in [ZS83], but this was neither recovered by the analysis in [BuB84, Be94] nor by the numerical approach in [BoB85] where only the bounds $\mathcal{R}_{\mathrm{MSV}}$ and $\mathcal{R}_{\mathrm{OSV}}$ were given. This discrepancy led to a considerable discussion in the physics literature ([Be94, DP94]) usually concluding that the result in [ZS83] are incorrect. Our work reconciles the two analyses: in both cases the Hurwitz criterion (1.8) was only evaluated in certain limits leading to two different sufficient conditions for instability, and the union of these two conditions provides the best answer. For more details see our discussion in Section 6.

We believe that our method leads to calculations which are completely equivalent to most of the formal approaches. However, our sound mathematical basis allows us to validitate that certain effects seen in lower order terms are really stable with respect to higher order corrections. As an example we mention the rôle of Galileian invariance which appears here as a crucial mechanism (see Lemma 5.3). Moreover, the systematic approach based on the LSR enables us to do many calculations automatically either by numerical codes or by computer algebra.

Finally, we warn the reader that, up to now, only sufficient conditions for instability are derived and that proofs of (linear) stability are still missing. In particular, we do not study instabilities which appear with respect to large vectors σ, like the cross–roll instability, cf. [NPL93, DP94].

Recently, in [KvW96] a first stability result was obtained for the case of the two–dimensional Rayleigh–Bénard problem. Since no dependence on x_2 is present (i.e. $\sigma_2 = 0$), the eigenvalue equation $\Lambda(\rho, k, (\sigma_1, 0), \lambda) = 0$ simplifies greatly, and the only remaining instability is the Eckhaus instability. Between the Eckhaus bounds the stability of the roll solutions is then proved. The stability of rolls in the Swift–Hohenberg problem (see (2.9)) with two unbounded direction (i.e. $\sigma \in \mathbb{R}^2$) is established in [Ku96, Mi96].

2 General theory

We consider systems of partial differential equations which are posed over unbounded domains of the form $\mathcal{Q}_\infty = \mathbb{R}^d \times \Sigma$ with variables $(x, \widetilde{x}) \in \mathbb{R}^d \times \Sigma$. We assume for simplicity the form

$$M\partial_t u = A_\mu(\partial_x)u + N(\mu, \partial_x, u) \quad \text{in } \mathcal{Q}_\infty \tag{2.1}$$

where $A_\mu(\partial_x)$ is a linear differential operator in the $(x, \widetilde{x})$ variables and suitable boundary conditions on $\partial\mathcal{Q}_\infty = \mathbb{R}^d \times \partial\Sigma$ are specified. The vector $\mu \in \mathbb{R}^p$ collectively denotes all the parameters of the problem. Note that the problem is translational invariant (no x–dependence) while dependence on the cross–sectional variable $\widetilde{x}$ and appearence of derivatives $\partial_{\widetilde{x}}$ is allowed but not explicitly displayed. For notational simplicity we have assumed that M is a constant operator, however, it is easy to see that the whole theory goes through for more general $M = M(\mu, \partial_x, u)$.

A periodic pattern $\widetilde{u}$ is characterized by a lattice group $\mathcal{L} \subset \mathbb{R}^d$ such that $\widetilde{u}(x+\ell, \cdot) = \widetilde{u}(x, \cdot)$ for all $\ell \in \mathcal{L}$ and an additional symmetry group $\Gamma \subset O(d)$ which corresponds to rotations and reflections of the pattern. For simplicity of this presentation, we neglect the latter symmetry in this general part and refer to [GSS88] (especially Case Study 4) for a discussion. However, in the application to the Bénard problem given below, we heavily use the underlying symmetries.

The dual lattice group $\mathcal{L}^* \subset \mathbb{R}^d$ is given by

$$\mathcal{L}^* = \{\, h \in \mathbb{R}^d \,:\, h \cdot \ell \in 2\pi\mathbb{Z} \text{ for all } \ell \in \mathcal{L}\,\},$$

and the periodicity domain $\Omega = \mathcal{T} \times \Sigma \subset \mathbb{R}^d \times \Sigma$ is $\mathcal{Q}_\infty/_{\mathcal{L}} = \mathcal{T} \times \Sigma$, where $\mathcal{T} = \mathbb{R}^d/_{\mathcal{L}}$ is a torus of dimension less than or equal d. Here we assume that $\mathcal{L}$ contains at least d linearly independent vectors. If the connected component of $\mathcal{L}$ which contains 0 is $\widetilde{d}$–dimensional, then $\mathcal{T}$ is a $(d - \widetilde{d})$–dimensional torus. Under this condition on $\mathcal{L}$, the dual lattice $\mathcal{L}^*$ is discrete, and a pattern $\widetilde{u}$ with lattice $\mathcal{L}$ can be written as a generalized Fourier series $\widetilde{u}(x, \widetilde{x}) = \sum_{h \in \mathcal{L}^*} e^{ih\cdot x} b_h(\widetilde{x})$ where

$$b_h(\widetilde{x}) = \frac{1}{|\mathcal{T}|} \int_{\mathcal{T}} e^{-ih\cdot x} \widetilde{u}(x, \widetilde{x})\, dx. \tag{2.2}$$

Varying the parameter μ the lattice of a pattern might slightly change under a smooth family of affine transformation. To fix the lattice group we may rescale $x = T(\mu)y + b(\mu)$ such that the y–variable lives in the constant domain $\mathcal{T}$ associated to the constant group $\mathcal{L}$. Hence, without loss of generality we assume constant $\mathcal{L}$ from now on.

We define now function spaces Z and Y over $\mathcal{Q}_\infty$ which contain $\mathcal{L}$–periodic funtions, or, equivalently we consider Z and Y as function spaces over the periodicity domain $\Omega = \mathcal{T} \times \Sigma$. Moreover, the space Z contains the boundary conditions on $\partial\Omega = \mathcal{T} \times \partial\Sigma$ associated with (2.1). Our main assumption is that $A_0(\partial_x) : Z \to Y$ is a Fredholm operator of index 0. That means that $A_0(\partial_x)$ is a bounded linear operator from Z to

Y, such that $\mathrm{kernel}\,[A_0(\partial_x)] = \{\, u \in Z \,:\, A_0(\partial_x)u = 0 \,\}$ is finite dimensional and that $\mathrm{range}\,[A_0(\partial_x)] = \{\, A_0(\partial_x)u : u \in Z \,\} \subset Y$ is a closed subspace of Y with finite codimension which is equal to $\dim(\mathrm{kernel}\,[A_0(\partial_x)])$. Moreover we assume $M : Z \to Y$ and $N(\cdot, \partial_x, \cdot) \in \mathcal{C}^m(\mathbb{R}^p \times Z, Y)$ for some $m \geq 1$.

In typical applications the spectral problem

$$\lambda M w = A_0(\partial_x) w, \quad w \in Z,$$

will have a discrete spectrum (since the domain $\Omega = \mathcal{T} \times \Sigma$ has a compact closure), but this is not necessary for the present method. Of course we are interested in instabilities of bifurcating steady solutions, hence we assume that $\lambda = 0$ is an isolated eigenvalue, i.e. $\dim(\mathrm{kernel}\,[A_0(\partial_x)]) > 0$. Moreover, the trivial solution should not be linearly unstable otherwise all bifurcating solutions would inherit this instability. Thus, our method is useful only if $\lambda M - A_0(\partial_x) : Z \to Y$ is invertible for all λ with $\mathrm{Re}\,\lambda > 0$. Eigenvalues on the imaginary axis are allowed, but instabilities arising from there will not be captured by our method.

Since $A_0(\partial_x) : Z \to Y$ has Fredholm index 0 there exist bounded projections $Q : Z \to Z$ and $P : Y \to Y$ such that $Z_0 = QZ$ and $Y_0 = PY$ have the same finite dimension. Let $Z_1 = (I - Q)Z$ and $Y_1 = (I - P)Y$ and decompose the steady problem into

$$\begin{aligned} P[A_\mu(\partial_x)(u_0 + u_1) + N(\mu, \partial_x, u_0 + u_1)] &= 0, \\ (I - P)[A_\mu(\partial_x)(u_0 + u_1) + N(\mu, \partial_x, u_0 + u_1)] &= 0, \end{aligned} \tag{2.3}$$

where $u_j \in Z_j$. By construction $A_0 : Z_1 \to Y_1$ has a bounded inverse which allows us to solve the second equation in (2.3) locally by the implicit function theorem $u_1 = \mathcal{U}(\mu, u_0)$. This solution can be inserted into the first equation to derive the *reduced problem*

$$0 = n(\mu, u_0) \stackrel{\mathrm{def}}{=} P\left[A_\mu(\partial_x)(u_0 + \mathcal{U}(\mu, u_0)) + N(\mu, \partial_x, u_0 + \mathcal{U}(\mu, u_0))\right], \tag{2.4}$$

which is also called the *bifurcation equation.* This reduction of the problem on Z to the finite–dimensional problem on Z_0 is called the *Liapunov–Schmidt reduction* (LSR) which is a well–established tool in bifurcation theory for more than 30 years, cf. [CH88, GS85].

We now want to proceed further and investigate the stability of any of the bifurcated periodic patterns u_μ. The linearization around u_μ reads

$$M \partial_t v = A_\mu(\partial_x) v + D_u N(\mu, \partial_x, u_\mu)[v]. \tag{2.5}$$

We call u_μ *linearly unstable* if there exists an initial condition v_0 which is bounded over $\mathcal{Q}_\infty$ such that the solution $v = v(t, x)$ of (2.5) with $v(0, x) = v_0(x)$ exists and is not bounded uniformly for $(t, x) \in [0, \infty) \times \mathbb{R}^d$.

A first step is to consider stability of u_μ with respect to perturbations in Z, i.e. with the same periodicity class. This could be done by using center manifold techniques or by the *principle of reduced stability* which was developed for the case $M = I$. The latter theory is intimately related to the LSR, see [GS85], Ch. I, Thm. 4.1 as well as [KL83, Va84]. However, as discussed in [Re89, Da91] the conditions for general situations are very complicated. Thus, we prefer our method which will include this theory as a special case.

As our problem is posed on all of $\mathcal{Q}_\infty = \mathbb{R}^d \times \Sigma$ we are allowed to consider perturbations of the form $v(t,x) = e^{i\sigma\cdot x} w(t,x)$ for any sideband vector $\sigma \in \mathbb{R}^d$ and $w(t,\cdot) \in Z$. Since the problem has no time dependence we can also look for special perturbations growing exponentially in time: $v(t,x) = e^{\lambda t + i\sigma\cdot x} W(x)$ with $W \in Z$. Inserting this Ansatz into (2.5) yields a solution whenever W satisfies the spectral problem

$$0 = \mathcal{B}(\mu,\sigma,\lambda)W \stackrel{\text{def}}{=} [A_\mu(i\sigma + \partial_x) + D_u N(\mu, i\sigma + \partial_x, u_\mu) - \lambda M]\, W. \tag{2.6}$$

Note that $A_\mu(i\sigma + \partial_x)W = e^{-i\sigma\cdot x}\left(A_\mu(\partial_x)[e^{i\sigma\cdot x}W]\right)$ and similarly for the term $D_u N$. (Here we see the reason for displaying ∂_x explicitly.)

The key idea of our method is to find nontrivial solutions of this linear problem by the same method as we used for the nonlinear problem, namely the LSR. Since $\mathcal{B}(0,0,0) = A_0(\partial_x)$ we can use the same projections P and Q as for (2.3) and let $W = W_0 + W_1$ with $W_0 = QW$.

$$\begin{aligned} P\mathcal{B}(\mu,\sigma,\lambda)[W_0 + W_1] &= 0, \\ (I-P)\mathcal{B}(\mu,\sigma,\lambda)[W_0 + W_1] &= 0. \end{aligned} \tag{2.7}$$

Again the second equation can be solved locally for small (μ,σ,λ) as $W_1 = \mathcal{W}(\mu,\sigma,\lambda)W_0$; and we obtain the reduced spectral problem

$$0 = b(\mu,\sigma,\lambda)W_0 \stackrel{\text{def}}{=} P\mathcal{B}(\mu,\sigma,\lambda)[W_0 + \mathcal{W}(\mu,\sigma,\lambda)W_0]. \tag{2.8}$$

Thus nontrivial solutions W_0 can be found if and only if $\det b(\mu,\sigma,\lambda) = 0$. We summarize the result as follows.

Theorem 2.1
(Principle of Reduced Instability) *Let (μ, u_μ) be a small steady solution of* (2.1) *obtained by LSR as indicated above. Let $b(\mu,\sigma,\lambda) : Z_0 \to Y_0$ in* (2.8) *describe the reduced spectral problem around (μ, u_μ). Then, u_μ is spectrally unstable if there exists (σ,λ) with $\operatorname{Re}\lambda > 0$ such that $\det b(\mu,\sigma,\lambda) = 0$.*

The result is now obvious, since the assumption of the theorem imply the existence of a nontrivial solution W_0 for (2.8) which immediately leads to the perturbation

$$v(t,x) = e^{\lambda t + i\sigma\cdot x}\left(W_0(x) + (\mathcal{W}(\mu,\sigma,\lambda)W_0)(x)\right)$$

which is bounded over $\mathcal{Q}_\infty = \mathbb{R}^d \times \Sigma$ and grows exponentially in time.

The name 'sideband instability' comes about as follows. The basic pattern u_μ consists of modes which have wave vectors $h \in \mathcal{L}^*$, while the perturbation $v(t,\cdot)$ has modes with wave vectors $h+\sigma$. Of course, the name sideband is only justified if $|\sigma|$ is small compared to the distances within $\mathcal{L}^*$. When treating $\sigma \in \mathbb{R}^d$ as a real parameter the set $S(\mu) \subset \mathbb{R}^d$ of instability points σ for u_μ is an open set

$$S(\mu) = \{\, \sigma \in \mathbb{R}^d \;:\; \text{there exists } \lambda \in \mathbb{C} \text{ with } \operatorname{Re}\lambda > 0 \text{ and } \det b(\mu,\sigma,\lambda) = 0 \,\}.$$

If $\sigma = 0$ is contained in $S(\mu)$ then we have already instability with respect to periodic perturbations in Z. More important is the case $0 \notin S(\mu)$, which has two subcases: (i) if $0 \in \overline{S(\mu)}$ (closure of $S(\mu)$ in $\mathbb{R}^d$), we have an *attached* sideband of unstable modes, and (ii) if $0 \notin \overline{S(\mu)}$, we have a *detached* sideband instability.

Example. For a simple application of this theory we refer to [Mi95]. There the Swift–Hohenberg equation

$$\partial_t u = -(1+\Delta)^2 u + \varepsilon^2 u - u^3,\;\; x \in \mathbb{R}^2, \tag{2.9}$$

is considered which has x_2–independent roll solutions $u_{\varepsilon,k}(x_1) = \rho\cos(kx_1) + \mathcal{O}(\rho^2)$ where $\rho = \sqrt{\frac{8}{3}(\varepsilon^2 - (1-k^2)^2)}$. The space $Z_0 = Y_0 = \operatorname{span}\{\cos(kx_1), \sin(kx_1)\}$ is two–dimensional and the reduced spectral problem leads to the 2×2–matrix

$$\begin{aligned}\widetilde{b}(\varepsilon,k,\sigma_1,\sigma_2,\lambda) \;=\; & \begin{pmatrix} \nu(k,\sigma) + c(\varepsilon,k) - \lambda & i\delta(k,\sigma) \\ -i\delta(k,\sigma) & \nu(k,\sigma) - \lambda \end{pmatrix} \\ & + \begin{pmatrix} \mathcal{O}(\rho^4(|\sigma|^2 + |\lambda|)) & \mathcal{O}(\rho^4(|\sigma| + |\lambda|)) \\ \mathcal{O}(\rho^4(|\sigma| + |\lambda|)) & \mathcal{O}(\rho^4(|\sigma|^2 + |\lambda|)) \end{pmatrix},\end{aligned}$$

with coefficients $c(\varepsilon,k) = -\frac{3}{4}\rho^2 + \mathcal{O}(\rho^4)$, $\nu(k,\sigma) = -4\sigma_1^2 + 2(1-k^2)|\sigma|^2 - |\sigma|^4$, and $\delta(k,\sigma) = 4\sigma_1 k(1-k^2-|\sigma|^2)$. As a result one finds two small eigenvalues having the expansion

$$\begin{aligned}\lambda_1 &= c(\varepsilon,k) + \mathcal{O}(|\sigma|^2),\\ \lambda_2 &= \tfrac{8}{c(\varepsilon,k)}(3(1-k^2)^2 - \varepsilon^2 + \mathcal{O}(\rho^2))\sigma_1^2 + \tfrac{2}{c(\varepsilon,k)}(1-k^2+\mathcal{O}(\rho^2))\sigma_2^2 + \mathcal{O}(|\sigma|^4).\end{aligned}$$

The second eigenvalue is positive for sufficiently small $\sigma \neq 0$ if one of the coefficients of σ_j^2 is positive. Hence, we find the *Eckhaus instability* (with $\sigma = (\sigma_1, 0)$) for $|1-k^2| > \varepsilon/\sqrt{3} + \mathcal{O}(\varepsilon^2)$ and the *zigzag instability* (with $\sigma = (0,\sigma_2)$) for $k < 1 + \mathcal{O}(\varepsilon^2)$. Moreover, in [Mi95] the sets $S(\varepsilon,k)$ of all unstable sideband vectors are characterized.

A similar instability theory was developed in [BM95, BM96], however there the spatial center manifold reduction (SCMR) was used instead of the LSR. The use of SCMR and the use of LSR are very similar, however there are certain restrictions and advantages for each method. The LSR applies only to periodic patterns but is technically simpler and available for problems with several unbounded directions. The SCMR is restricted to one unbounded spatial variable but allows for more general solutions; and additionally it gives more information on the spatial dynamics.

In both cases it is important to note that only sufficient conditions for instability are derived. Even if we show that for all small (μ, σ) the function $\det b(\mu, \sigma, \lambda)$ has no small solutions λ with $\operatorname{Re}\lambda > 0$ we cannot conclude stability. There might be unstable modes with either σ or λ large which are not captured by the local reduction methods. Only if more global information is at hand, which allows us to conclude that these local modes are the most unstable ones, then a stability result can be derived, see [Ku96, KvW96, Mi96].

One of the major advantages of the unified treatment of the nonlinear bifurcation problem and the linear stability problem is that all the functional analytic framework and much of the computational effort have to be done only once. As in the above example it is often sufficient to treat the linear stability problem in only two steps. First we can calculate the LSR of the linear problem only (linearization at $u \equiv 0$), which has a simple structure due to the translational invariance of the problem. Second we can find the lowest order influence of the nonlinearity (i.e. $D_u N(\mu, \partial_x, u_\mu)$) by the following theorem which was already used in [Mi95], Thm. 2.1 (see also [BM95] for the analogue in the case of the SCMR).

Theorem 2.2
Let $u_1 = \mathcal{U}(\mu, u_0)$ be obtained from the LSR of (2.3) such that $n(\mu, u_0) = 0$ (i.e. (2.4)) is the reduced problem. Then the reduction $W_1 = \mathcal{W}(\mu, \sigma, \lambda)W_0$ of the spectral problem (2.6) at $(\mu, u_\mu) = (\mu, u_{\mu 0} + \mathcal{U}(\mu, u_{\mu 0}))$ satisfies

$$\begin{aligned} \mathcal{W}(\mu, 0, 0) &= D_{u_0}\mathcal{U}(\mu, u_{\mu 0}) \quad \text{in } \mathcal{L}(Z_0, Y_1), \\ \text{and} \quad b(\mu, 0, 0) &= D_{u_0} n(\mu, u_{\mu 0}) \quad \text{in } \mathcal{L}(Z_0, Y_0). \end{aligned}$$

Proof: We have $n(\mu, u_0) = P\mathcal{N}(\mu, u_0 + \mathcal{U}(\mu, u_0))$ and $(I - P)\mathcal{N}(\mu, u_0 + \mathcal{U}(\mu, u_0)) = 0$. Taking the u_0–derivative at $(\mu, u_{\mu 0})$ in both equations gives

$$\begin{gathered} D_{u_0} n(\mu, u_{\mu 0})[W_0] = P D_u \mathcal{N}(\mu, u_\mu)(W_0 + D_{u_0}\mathcal{U}(\mu, u_{\mu 0})[W_0]), \\ (I - P) D_u \mathcal{N}(\mu, u_\mu)(W_0 + D_{u_0}\mathcal{U}(\mu, u_{\mu 0})[W_0]) = 0. \end{gathered}$$

However, by the uniqueness of the reduction functions $\mathcal{U}$ and $\mathcal{W}$ we find $\mathcal{W}(\mu, 0, 0) = D_{u_0}\mathcal{U}(\mu, u_{\mu 0})$, and thus $b(\mu, 0, 0) = D_{u_0} n(\mu, u_{\mu 0})$ follows. $\square$

Recalling that we started out with a translationally invariant problem and noting that u_μ really depends nontrivially on $x \in \mathbb{R}^d$, we see that some of the derivatives $\partial_{x_j} u_\mu$ are nontrivial and satisfy $D_u \mathcal{N}(\mu, u_\mu)W = 0$. These vectors as well as associated Jordan blocks lead to nontrivial null vectors and Jordan blocks in the reduced problem. This will be important in the Bénard problem where the Galileian invariance enforces a Jordan block of length 2.

Theorem 2.3
(a) If $D_u\mathcal{N}(\mu,u_\mu)\widetilde{w}=0$, then $b(\mu,0,0)Q\widetilde{w}=0$.

b) If, in addition, $M\widetilde{w}=D_u\mathcal{N}(\mu,u_\mu)\widetilde{v}$, then $b(\mu,0,0)Q\widetilde{v}+\frac{\partial}{\partial\lambda}b(\mu,0,0)Q\widetilde{w}=0$.

Proof: (a). We obviously have $\widetilde{w}_1=(I-Q)\widetilde{w}=\mathcal{W}(\mu,0,0)\widetilde{w}_0$ where $\widetilde{w}_0=Q\widetilde{w}$. Hence, the result is clear.

(b). The relation of $\widetilde{v}$ and $\widetilde{w}$ can also be written as $\mathcal{B}(\mu,0,0)\widetilde{v}+\frac{\partial}{\partial\lambda}\mathcal{B}(\mu,0,0)\widetilde{w}=0$, which is analogous to the result for the reduced problem. For all λ we have

$$(D_u\mathcal{N}(\mu,u_\mu)-\lambda M)(\lambda\widetilde{v}+\widetilde{w})+\lambda^2M\widetilde{v}=0.$$

Now choose any bounded linear operator K with $K\widetilde{v}=0$ and $K\widetilde{w}=M\widetilde{v}$ and consider the LSR for the linear problem

$$(D_u\mathcal{N}(\mu,u_\mu)-\lambda M+\delta K)W=0$$

with the same splitting $W=W_0+W_1$ with $W_0=QW$ as above. For small $(\lambda,\delta)\in\mathbb{C}^2$ we obtain a reduced problem $m(\mu,\lambda,\delta)W_0=0$. By Theorem 2.2, we know that $m(\mu,0,0)$ as well as $b(\mu,0,0)$ coincide with $D_{u_0}n(\mu,u_{\mu 0})$.

By construction $(D_u\mathcal{N}-\lambda M+\lambda^2K)(\lambda\widetilde{v}+\widetilde{w})\equiv 0$, and hence after reduction we have $m(\mu,\lambda,\lambda^2)Q(\lambda\widetilde{v}+\widetilde{w})\equiv 0$. Obviously the relation $b(\mu,0,\lambda)=m(\mu,\lambda,\lambda^2)+\mathcal{O}(|\lambda|^2)$ holds. Hence, $b(\mu,0,0)Q\widetilde{w}=0$ and $b(\mu,0,0)Q\widetilde{v}+\partial_\lambda b(\mu,0,0)Q\widetilde{w}=0$, which is the desired result. □

We mention another important consequence of the translational invariance of the problem. The linear problem

$$(A_0(i\sigma+\partial_x)-\lambda M)w=0$$

has x–independent coefficients. Define Z^Σ and Y^Σ to be the Banach spaces of those functions on Σ which can be obtained as Fourier coefficients in (2.2). Then, $\mathcal{B}(0,\sigma,\lambda)=A_0(i\sigma+\partial_x)-\lambda M$ maps the function $u=e^{ih\cdot x}\phi$ with $\phi\in Z^\Sigma$ onto $e^{ih\cdot x}\psi$, where

$$\psi=[A_0(i\sigma+ih)-\lambda M]\phi\in Y^\Sigma.$$

Thus, we are left with a differential operator on the domain Σ only, which degenerates to an algebraic problem if Σ is zero–dimensional as in the example of the Swift–Hohenberg equation (2.9). Now the kernel Z_0 of $\mathcal{B}(0,0,0)$ has the representation

$$Z_0=\{\textstyle\sum_{j=0}^p e^{ih^{(j)}\cdot x}\left(\sum_{l=1}^{m_j}a_{jl}\phi_{jl}\right)\;:\;a_{jl}\in\mathbb{C}\text{ with }a_{p-j,l}=\overline{a_{jl}}\}$$

where $h^{(j)}=-h^{(p-j)}\in\mathcal{L}^*$, $\phi_{jl}=\overline{\phi_{p-j,l}}\in Z^\Sigma$, and $\dim Z_0=m_0+\ldots+m_p$.

This shows that the reduced problem $b(0,\sigma,\lambda)$ decouples into $(p+1)$ blocks of lengths $m_0, m_1, \dots, m_p$, respectively. Write the projections $P: Y \to Y_0$ and $Q: Z \to Z_0$ in the form

$$\begin{aligned} Pv &= \sum_{j=0}^{p} \sum_{l=1}^{m_j} \langle \textstyle\int_{\mathcal{T}} e^{-ih^{(j)}\cdot y} v(y)\, dy, \widetilde{\psi}_{jl} \rangle_\Sigma \, e^{ih^{(j)}\cdot x}\, \psi_{jl}, \\ Qw &= \sum_{j=0}^{p} \sum_{l=1}^{m_j} \langle \textstyle\int_{\mathcal{T}} e^{-ih^{(j)}\cdot y} w(y)\, dy, \widetilde{\phi}_{jl} \rangle_\Sigma \, e^{ih^{(j)}\cdot x}\, \phi_{jl}, \end{aligned}$$

where $\langle\cdot,\cdot\rangle_\Sigma$ is the L^2–scalar product on Σ. The block of $b(0,\sigma,\lambda)$ which is associated to $h^{(j)}$ is constructed as follows. For $l = 1, \dots, m_j$, solve the following linear problem for $(W_{jl}, \alpha_{jl1}, \dots, \alpha_{jlm_j}) \in Z^\Sigma \times \mathbb{C}^{m_j}$

$$\begin{aligned} (A_0(i\sigma + ih^{(j)}) - \lambda M)[\phi_{jl} + W_{jl}] &= \textstyle\sum_{q=1}^{m_j} \alpha_{jlq}\psi_{jq}, \\ \langle W_{jl}, \widetilde{\psi}_{jq} \rangle &= 0 \text{ for } q = 1, \dots, m_j. \end{aligned}$$

For sufficiently small (σ,λ) there is a unique solution, since the bordered operator

$$\begin{pmatrix} A_0(ih^{(j)}) & \psi_{j1} & \cdots & \psi_{jm_j} \\ \langle\,\cdot\,, \widetilde{\psi}_{j1}\rangle & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ \langle\,\cdot\,, \widetilde{\psi}_{jm_j}\rangle & 0 & \cdots & 0 \end{pmatrix} \text{ from } Z^\Sigma \times \mathbb{C}^{m_j} \text{ into } Y^\Sigma \times \mathbb{C}^{m_j}$$

has a bounded inverse due to the construction according to Fredholm's alternative. With the solution $W_{jl}(\sigma,\lambda)$ and $\alpha_{jlq}(\sigma,\lambda)$ the reduced linear problem satisfies the relations

$$\mathcal{W}(0,\sigma,\lambda)W_0 = e^{ih^{(j)}\cdot x} \sum_{l=1}^{m_j} a_l W_{jl}(\sigma,\lambda), \quad b(0,\sigma,\lambda)W_0 = e^{ih^{(j)}\cdot x} \sum_{q=1}^{m_j} \left[\sum_{l=1}^{m_j} a_l \alpha_{jlq}(\sigma,\lambda) \right] \psi_{jq},$$

for $W_0 = e^{ih^{(j)}\cdot x} \left(\sum_{l=1}^{m_j} a_l \phi_{jl} \right)$. Hence, the functions $\alpha_{jlq}(\sigma,\lambda)$ appear as matrix entries when $b(0,\sigma,\lambda)$ is represented in the given bases of Z_0 and Y_0.

It may be also convenient to use a real basis for Z_0 and Y_0. Consider the case, to be used in the Bénard problem below, that $h^{(j)} \neq 0$ and $m_j = 1$. The associated complex basis elements are $\zeta_j = e^{ih^{(j)}\cdot x}\phi_{j1}$ and $\zeta_{p-j} = \overline{\zeta}_j$ in Z_0 and $\kappa_j = e^{ih^{(j)}\cdot x}\psi_{j1}$ and $\kappa_{p-j} = \overline{\kappa}_j$ in Y_0, such that $b(0,\sigma,\lambda)\zeta_j = \alpha_{j,1,1}(\sigma,\lambda)\kappa_j$ and $b(0,\sigma,\lambda)\zeta_{p-j} = \alpha_{p-j,1,1}(\sigma,\lambda)\kappa_{p-j}$.

Lemma 2.4
For any $(c_1,c_2) \in \mathbb{R}^2$ we have

$$b(0,\sigma,\lambda)\,(c_1 \mathrm{Re}\,\zeta_j + c_2 \mathrm{Im}\,\zeta_j) = d_1 \mathrm{Re}\,\kappa_j + d_2 \mathrm{Im}\,\kappa_j$$

with

$$\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} \nu(\sigma,\lambda) & -i\delta(\sigma,\lambda) \\ i\delta(\sigma,\lambda) & \nu(\sigma,\lambda) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

where $2\nu(\sigma,\lambda) = \alpha_{j11}(\sigma,\lambda) + \overline{\alpha_{j11}(-\sigma,\overline{\lambda})}$ and $2\delta(\sigma,\lambda) = \alpha_{j11}(\sigma,\lambda) - \overline{\alpha_{j11}(-\sigma,\overline{\lambda})}$.

Proof: As we started with a real problem we have $\overline{\mathcal{B}(0,\sigma,\lambda)} = \mathcal{B}(0,-\sigma,\overline{\lambda})$ which leads by the uniqueness of the LSR to $\overline{\mathcal{W}(0,\sigma,\lambda)} = \mathcal{W}(0,-\sigma,\overline{\lambda})$ and hence to $\overline{b(0,\sigma,\lambda)} = b(0,-\sigma,\overline{\lambda})$. In particular, this implies $\alpha_{p-j,11}(-\sigma,\overline{\lambda}) = \overline{\alpha_{j11}(\sigma,\lambda)}$ which leads to the desired result. □

The above considerations show that the reduced spectral problem $b(\mu,\sigma,\lambda)W_0 = 0$ can be calculated to lowest order with little additional effort compared to the effort for obtaining the reduced bifurcation problem. Assume that $b(0,\sigma,\mu)$ and $n(\mu,u_0)$ are known (to a sufficiently high order), then $b(0,0,0)=0$ and Theorem 2.2 implies

$$b(\mu,\sigma,\lambda) = b(0,\sigma,\lambda) + D_{u_0}n(\mu,u_{\mu 0}) + \mathcal{O}(|\mu|^{\varepsilon_1}(|\sigma|^{\varepsilon_2}+|\lambda|^{\varepsilon_3})) \tag{2.10}$$

with $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 1$. Often additional structure of the problem (e.g. symmetries) provides even larger ε_j. The order of approximation in (2.10) turns out to be sufficient in the Swift–Hohenberg problem, yet for the Bénard problem treated below we have to calculate a few more coefficients in order to study the relevant instabilities.

3 The Rayleigh–Bénard problem

The Rayleigh–Bénard convection is described by the Navier–Stokes equations for the divergence free velocity field u of the fluid and the temperature distribution θ, which generates a buoyancy force:

$$\left.\begin{array}{rcl} \partial_t u + (u\cdot\nabla)u + \nabla p - \mathcal{R}\,\theta e_3 - \Delta u &=& 0,\\ \mathcal{P}(\partial_t\theta + u\cdot\nabla\theta) - \mathcal{R}\,u_3 - \Delta\theta &=& 0,\\ \nabla\cdot u &=& 0, \end{array}\right\} \text{ for } x\in\mathbb{R}^2\times(0,\pi); \tag{3.1}$$

with stress–free boundary conditions

$$\partial_3 u_1 = \partial_3 u_2 = u_3 = \theta = 0 \text{ for } x_3 = 0 \text{ or } x_3 = \pi.$$

The parameter $\mathcal{R}$ is related to the Rayleigh number via $\mathrm{Ra} = \mathcal{R}^2$ and used as bifurcation parameter. The Prandtl number $\mathcal{P} > 0$ is treated as fixed but arbitrary and therefore normally the dependence on $\mathcal{P}$ will be omitted. For additional physical background we refer to [BuB84, Be94] and the references therein. We have used the special scaling of [SvW92] which makes the linearized operator symmetric in the standard L^2 scalar product. We will use the abbreviation $z = (u,\theta,p)^T \in \mathbb{R}^5$.

The system has a rich symmetry structure playing an important rôle in the subsequent analysis. The Euclidean group of $\mathbb{R}^2$ is generated by translations by $\tau = (\tau_1,\tau_2)\in\mathbb{R}^2$ and rotations through the angle $\alpha \in S^1 = \mathbb{R}/_{2\pi\mathbb{Z}}$:

$$\begin{pmatrix} u\\ \theta\\ p\end{pmatrix}(x) \mapsto \begin{pmatrix} R(-\alpha)u\\ \theta\\ p\end{pmatrix}(R(\alpha)x + (\tau_1,\tau_2,0)), \text{ where } R(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0\\ \sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{pmatrix}.$$

We have reflection symmetries $K^{(j)}$, $j = 1, 2$, corresponding to reflections of the x_j–coordinate

$$\begin{aligned} K^{(1)} :\ \ & z(x) \mapsto (-u_1, u_2, u_3, \theta, p)(-x_1, x_2, x_3), \\ K^{(2)} :\ \ & z(x) \mapsto (u_1, -u_2, u_3, \theta, p)(x_1, -x_2, x_3). \end{aligned}$$

There is a further reflection symmetry about the midplane which arises from the Boussinesq approximation:

$$K^{(3)} :\ z(x) \mapsto (u_1, u_2, -u_3, -\theta, p)(x_1, x_2, \pi - x_3).$$

See also [GSS88], Case Study 4, for a discussion of these symmetries. However, there is an additional (generalized) symmetry, which comes from the Galileian invariance associated with a moving frame with speed $v = (v_1, v_2, 0)^T$. This transformation is given explicitly below in (3.2), and it will play an impotant rôle in the discussion of the sideband instabilities (see Lemma 5.3).

When trying to set up an instability theory for the Navier–Stokes problem in unbounded domains one has to be aware of the problems at infinity when no decay assumptions on the flow are imposed. Since it is our aim to allow for periodic solutions (which are in $W^{k,\infty}(\mathbb{R}^2 \times (0, \pi))$) we have to impose some conditions at infinity in order to control the mass flux there. If $z = (u, \theta, p)$ solves the Bénard problem and $\widetilde{x} = (\widetilde{x}_1(t), \widetilde{x}_2(t), 0)$ and $q = q(t) \in \mathbb{R}$ are any smooth functions, then

$$\begin{aligned} \widetilde{u}(t, x) &= u(t, x - \widetilde{x}(t)) + \dot{\widetilde{x}}(t), \\ \widetilde{\theta}(t, x) &= \theta(t, x - \widetilde{x}(t)), \\ \widetilde{p}(t, x) &= p(t, x - \widetilde{x}(t)) + q(t) - \ddot{\widetilde{x}}(t) \cdot x, \end{aligned} \tag{3.2}$$

solves the Bénard problem as well. For $\widetilde{x}(t) = (v_1 t, v_2 t, 0)$ this is exactly the Galileian transformation into a moving frame with constant speed v.

A rigorous theory can be developed when periodicity in (x_1, x_2) on some periodicity domain is assumed. For simplicity we restrict our analysis to the case of a large rectangle $\mathcal{T}_\ell = (0, \ell_1) \times (0, \ell_2)$. The periodicity of velocity and temperature is reasonable, however there is a choice for the pressure, namely to impose (i) periodicity on p or (ii) periodicity in ∇p only. According to the above non–uniqueness case (ii) needs an extra condition on the mean value of the first two components of u. Since there is no horizontal forcing in the problem, it is easy to see that the mean values of the horizontal components of u satisfy

$$\frac{d}{dt} \int_{\mathcal{T}_\ell \times (0,\pi)} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} dx + \int_{\mathcal{T}_\ell \times (0,\pi)} \begin{pmatrix} \partial_{x_1} p \\ \partial_{x_2} p \end{pmatrix} dx = 0$$

in case (i) and (ii). Hence, while the mean flow is constant in case (i), the pressure gradient has zero mean in case (ii). From this we conclude that case (ii) is a special case of (i), namely that with mean flow zero.

Thus, we can use the following spaces. Let $\mathcal{Q}_\ell = \mathcal{T}_\ell \times (0,\pi)$ and

$$\mathcal{H}_\ell = \{\, (u,\theta) \in H^1(\mathcal{Q}_\ell)^4 \;:\; \operatorname{div} u \equiv 0, u_3 = 0 \text{ on } \mathcal{T}_\ell \times \{0,\pi\} \,\}.$$

The problem (3.1) can be written as an evolution problem

$$\frac{d}{dt}\begin{pmatrix} u \\ \theta \end{pmatrix} = L_{\ell,\mathcal{R}}\begin{pmatrix} u \\ \theta \end{pmatrix} + F(\mathcal{R},u,\theta) \tag{3.3}$$

in $\mathcal{H}_\ell$ after applying the suitable Stokes projection. In [SvW92] it is shown that this initial value problem is locally well posed for any ℓ.

Now assume we have found a periodic steady state $\widetilde{U} = (\widetilde{u},\widetilde{\theta}) \in \mathcal{H}_\ell$. We may investigate its spectral properties in the space $\mathcal{H}_\ell$ by classical methods from functional analysis. Here we want to take into account more general perturbations, but keeping the restriction to periodicity. We solve this apparent dilemma by considering the solution $\widetilde{z}$ as a solution on all larger domains which are compatible with the period of the pattern $\widetilde{z}$. Thus, our instablity definition reads as follows.

Definition 3.1
Assume that $\widetilde{U}$ is a steady solution of the Bénard problem satisfying $\widetilde{U}(x+(n_1\ell_1, n_2\ell_2, 0)) = \widetilde{U}(x)$ for all $x \in \mathcal{Q}_\infty = \mathbb{R}^2 \times (0,\pi)$ and all $n_1, n_2 \in \mathbb{Z}$, where $\ell_1, \ell_2 > 0$ are fixed.

Then, the solution $\widetilde{U}$ is called (linearly) **unstable** *on $\mathcal{Q}_\infty$ if there exists any $n_1, n_2 > 0$ such that $\widetilde{U}$ is a (linearly) unstable equilibrium point of* (3.3) *when considered in the function space $\mathcal{H}_{(n_1\ell_1, n_2\ell)}$.*

In this definition the case in which (n_1, n_2) is very large is of particular interest. A sideband instability corresponds to the case where (neutral) stability holds for small (n_1, n_2) but instability appears for large n. As developed above we will study the instability problem by introducing the sideband vector $\sigma = (\sigma_1, \sigma_2)$ as a small continuous parameter. However, according to the definition we have to consider perturbations in $\mathcal{H}_{(n_1\ell_1, n_2\ell_2)}$, namely in the form $v(t,x) = e^{\lambda t + i\sigma\cdot x} W(x)$ with $W \in \mathcal{H}_\ell$. Thus, σ should satisfy $e^{i\sigma\cdot(n_1\ell_1, n_2\ell_2)} = 1$, or, which is equivalent, lie in the discrete set

$$\Sigma_{(n_1\ell_1, n_2\ell_2)} = \{\, (\tfrac{2\pi m_1}{n_1\ell_1}, \tfrac{2\pi m_2}{n_2\ell_2}) \in \mathbb{R}^2 \;:\; m_1, m_2 \in \mathbb{Z} \,\}.$$

Hence, if our analysis with continuous $\sigma \in \mathbb{R}^2$ provides any open set $S(\mathcal{R},\ell) \subset \mathbb{R}^2$ in which nontrivial solutions with $\operatorname{Re}\lambda > 0$ exists, then by making n sufficiently large we can find points of $\Sigma_{(n_1\ell_1, n_2\ell_2)}$ lying inside of $S(\mathcal{R},\ell)$. Thus instability in the sense of the above definition can be concluded. Using this philosophy, we are able to establish rigorous instability results for the Bénard rolls on suitably large periodicity domains.

4 The linear problem and bifurcation of steady rolls

We look for the solutions of the linear problem (linearized at the trivial flow $u = 0$). We use a Fourier Ansatz in all three spatial direction, where the x_3–direction has boundary conditions which make the Fourier spectrum discrete:

$$z = e^{i\sigma\cdot x+\lambda t}\varphi_n(\alpha)(x_3) \quad \text{with } \varphi_n(\alpha)(x_3) = \begin{pmatrix} \alpha_1\cos(nx_3) \\ \alpha_2\cos(nx_3) \\ \alpha_3\sin(nx_3) \\ \alpha_4\sin(nx_3) \\ \alpha_5\cos(nx_3) \end{pmatrix}, \tag{4.1}$$

$\sigma = (\sigma_1, \sigma_2, 0)^T$, $\alpha \in \mathbb{C}^5$, and $\alpha_3 = \alpha_4 = 0$ for $n = 0$. There are nontrivial solutions of the linear problem if α satisfies $F_n(\mathcal{R}, \sigma_1, \sigma_2, \lambda)\alpha = 0$, where $F_n(\mathcal{R}, \sigma_1, \sigma_2, \lambda) =$

$$\begin{pmatrix} \lambda+n^2+|\sigma|^2 & 0 & 0 & 0 & i\sigma_1 \\ 0 & \lambda+n^2+|\sigma|^2 & 0 & 0 & i\sigma_2 \\ 0 & 0 & \lambda+n^2+|\sigma|^2 & -\mathcal{R} & -n \\ 0 & 0 & -\mathcal{R} & \mathcal{P}\lambda+n^2+|\sigma|^2 & 0 \\ -i\sigma_1 & -i\sigma_2 & -n & 0 & 0 \end{pmatrix}, \tag{4.2}$$

and $n = 1, 2, \ldots$. Calculating the determinant we find

$$\det F_n = (\lambda+n^2+|\sigma|^2)\Big[\mathcal{R}^2|\sigma|^2 - (\lambda+n^2+|\sigma|^2)(\mathcal{P}\lambda+n^2+|\sigma|^2)(n^2+|\sigma|^2)\Big],$$

and, from $\det F_n = 0$, the eigenvalues are

$$\lambda_1 = -n^2-|\sigma|^2, \quad \lambda_{2,3} = -\frac{1}{2\mathcal{P}}(\mathcal{P}+1)(n^2+|\sigma|^2)\Big(1\pm\sqrt{1-4\mathcal{P}\delta/(\mathcal{P}+1)^2}\Big)$$

where $\delta = 1-\mathcal{R}^2|\sigma|^2/(n^2+|\sigma|^2)^3$. In particular, for small δ, we obtain

$$\lambda_3 = -\frac{n^2+|\sigma|^2}{\mathcal{P}+1}\delta + \mathcal{O}(\delta^2) \quad \text{for } \delta \to 0.$$

Thus, for $n = 0$ and $\sigma = 0$ we have a three–dimensional neutral direction (recall $\alpha_3 = \alpha_4 = 0$) which correspond to translations in the (x_1, x_2)–directions and to adding a constant to the pressure. Eigenvalues with positive real part exist only when δ becomes negative, which is the case for $\mathcal{R}^2 > 27n^2/4$. Hence unstable directions occur first for $n = 1$ and are associated to $\mathcal{R} > \mathcal{R}_0 = 3\sqrt{3}/2$. The relevant wave vector $\sigma = (\sigma_1, \sigma_2, 0)^T$ then satisfies $|\sigma| \approx 1/\sqrt{2}$. Our further study will be restricted to Rayleigh numbers with $\mathcal{R} \in (\mathcal{R}_0, \mathcal{R}_0+\varepsilon)$, then the only unstable modes can appear for $n = 1$, but $n = 0$ also contains neutral modes.

Our main interest lies in roll patterns which are defined by periodicity in one unbounded direction while they are independent of the other direction. Without loss of generality (using the rotational invariance of the problem) we may assume that the rolls are independent of x_2 and periodic in x_1, namely $\partial_2 z = 0$ and $z(x_1 + 2\pi/k, x_3) = z(x_1, x_3)$, where k is the wave number of the periodic pattern.

Thus, we introduce the scaled variable $y = kx_1 \in S^1 = \mathbb{R}/_{2\pi\mathbb{Z}}$ and the basic domain $\Omega = S^1 \times (0, \pi)$. Moreover, we define the spaces

$$\begin{aligned} &Y = \left(L^2(\Omega)\right)^4 \times H^1(\Omega), \\ &Z = \{(u, \theta, p) \in (H^2(\Omega))^4 \times H^1(\Omega) : \partial_3 u_1 = \partial_3 u_2 = u_3 = \theta = 0 \text{ on } \partial\Omega\,\}, \\ &\langle z, \widetilde{z} \rangle = \tfrac{1}{2\pi^2} \textstyle\int_\Omega z(y, x_3) \cdot \widetilde{z}(y, x_3)\, dy\, dx_3. \end{aligned} \tag{4.3}$$

Note that the spaces $H^j(\Omega)$ automatically include the assumption of periodicity in y. For all integrals over Ω we will drop the integration variables $dy\, dx_3$ whenever there is no danger of confusion. We further use the scaled versions of the gradient and Laplace operator on Ω:

$$\nabla_k = (k\partial_y, 0, \partial_3)^T, \quad \Delta_k = k^2\partial_y^2 + \partial_3^2.$$

The steady roll patterns satisfy the nonlinear problem

$$\mathcal{N}(\mathcal{R}, k, z) \stackrel{\text{def}}{=} \begin{pmatrix} -\Delta_k u - \mathcal{R}\theta e_3 + \nabla_k p + (u \cdot \nabla_k) u \\ -\Delta_k \theta - \mathcal{R} u_3 + \mathcal{P} u \cdot \nabla_k \theta \\ -\nabla_k \cdot u \end{pmatrix} = 0. \tag{4.4}$$

The map $\mathcal{N}$ is analytic from Z into Y, since the nonlinear terms are quadratic with both factors in $H^1(\Omega) \subset L^4(\Omega)$ by Sobolev's embedding theorem (recall that Ω is two–dimensional).

Note that the second component of $\mathcal{N} = 0$ reads $-\Delta_k u_2 + u_1 k\partial_y u_2 + u_3\partial_3 u_2 = 0$, which is a linear problem having the trivial solutions $u_2 = a_2 =$const. Multiplying this equation by u_2 and integrating by parts shows that these are the only solutions:

$$\textstyle\int_\Omega |\nabla_k u_2|^2 = -\int_\Omega u_2 \Delta_k u_2 = -\int_\Omega u_2 (u \cdot \nabla_k) u_2 = -\int_\Omega (u \cdot \nabla_k)[\frac{1}{2}u_2^2] = \int_\Omega (\nabla_k \cdot u)\frac{1}{2}u_2^2 = 0,$$

where all boundary terms cancel due to $z \in Z$. Moreover, note that u_2 does not enter in any of the other four equations. Thus, the u_2 component decouples; nevertheless we keep it in the following calculations since this decoupling will not be valid when the instability is investigated.

We are interested in the bifurcations of nontrivial patterns from the trivial solution $z = 0$. The linear theory tells us that $z = 0$ remains stable as long as the Rayleigh number stays less than 27/4, or in our notation $\mathcal{R} \le \mathcal{R}_0 = 3\sqrt{3}/2$. When $\mathcal{R}$ passes through $\mathcal{R}_0$ the waves with wave number k close to $k_0 = 1/\sqrt{2}$ become unstable. Hence, the basic linear operator we have to study is

$$A_0 \stackrel{\text{def}}{=} D_z \mathcal{N}(\mathcal{R}_0, k_0, 0) : \begin{pmatrix} u \\ \theta \\ p \end{pmatrix} \mapsto \begin{pmatrix} -\Delta_{k_0} u - \mathcal{R}_0 \theta e_3 + \nabla_{k_0} p \\ -\Delta_{k_0} \theta - \mathcal{R}_0 u_3 \\ -\nabla_{k_0} \cdot u \end{pmatrix} \tag{4.5}$$

and it is easy to see that A_0 is symmetric with respect to the scalar product given above, i.e., $\langle A_0 z, \widetilde{z}\rangle = \langle z, A_0\widetilde{z}\rangle$ for all $z, \widetilde{z} \in Z$.

This operator can be analyzed by expanding the elements in Z and Y into Fourier series as follows:

$$z(y,x_3) = \sum_{n\in\mathbb{N}_0}\sum_{m\in\mathbb{Z}} e^{imy}\varphi_n(\alpha^{(n,m)})(x_3)$$

where φ_n was defined in (4.1) and $\alpha^{(n,-m)} = \overline{\alpha}^{(n,m)} \in \mathbb{C}^5$.

The five–dimensional spaces $Z_{(n,m)} = \{\, e^{imy}\varphi_n(\alpha) \in Z \,:\, \alpha \in \mathbb{C}^5 \,\}$ are left invariant by the application of A_0 since

$$A_0 \sum_{n,m} e^{imy}\varphi_n(\alpha^{(n,m)}) = \sum_{n,m} e^{imy}\varphi_n(F^0(n,m)\alpha^{(n,m)}) \tag{4.6}$$

where $F^0(n,m) = F_n(\mathcal{R}_0, k_0 m, 0, 0) \in \mathbb{C}^{5\times5}$ as defined in (4.2). Obviously the kernel of A_0 is found as a direct sum from the kernels of the $F^0(n,m)$.

Lemma 4.1
The operator $A_0 : Z \subset Y \to Y$ is symmetric in Y with respect to the scalar product $\langle\cdot,\cdot\rangle$. Moreover, $A_0 : Z \to Y$ is a Fredholm operator of index zero with a five–dimensional kernel: span $\{\xi_1,\dots,\xi_5\}$ *where*

$$\xi_1 = \begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix},\ \xi_2 = \begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix},\ \xi_3 = \begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix},\quad \begin{matrix}\xi_4 = \operatorname{Re}\zeta,\\ \xi_5 = \operatorname{Im}\zeta,\end{matrix} \quad \textit{with } \zeta = e^{iy}\begin{pmatrix}\sqrt{2}\cos x_3\\ 0\\ -i\sin x_3\\ -i\sqrt{3}\sin x_3\\ 3i\cos x_3\end{pmatrix}.$$

Note that $\xi_4 = K^{(2)}\xi_4 = -K^{(1)}K^{(3)}\xi_4$ and $\xi_5(y) = \xi_4(y-\pi/2)$

Proof: The result on the kernel follows easily by studying the kernels of each $F^0(n,m)$ separately and then using (4.6).

The symmetry of A_0 follows by an explicit calculation. The Fredholm property can be derived from classical elliptic regularity theory (see e.g. [RR92] Ch. 8) or by writing $A_0 z = g \in Y$ in its Fourier representation $F^0(n,m)\alpha^{(n,m)} = \gamma^{(n,m)}$ and estimating $\alpha^{(n,m)}$ directly. $\square$

The question of existence of nontrivial roll patterns is now investigated by applying the LSR to the nonlinear equation $\mathcal{N}(\mathcal{R},k,z) = 0$. Therefore, we rewrite the mapping $\mathcal{N}$ as $\mathcal{N}(\mathcal{R},k,z) = A_0 z + N(\mathcal{R},k,z)$ with $N(\mathcal{R},k,z) = \mathcal{O}(\varepsilon\|z\|_Z + \|z\|_Z^2)$ where $\varepsilon = |\mathcal{R}-\mathcal{R}_0| + |k-k_0|$. Moreover, we let $Z_0 = Y_0 = \text{kernel}\,[A_0]$ and define the orthogonal projection P from Y onto Y_0 which also projects Z onto Z_0. The kernels of P are denoted by Y_1 and Z_1, respectively. Thus, we decompose $z = z_0 + z_1$ with $z_j \in Z_j$ and, since $PA_0 z = 0$ and $(I - P)A_0(z_0 + z_1) = A_0 z_1$, the nonlinear problem (4.4) can be written as

$$\begin{aligned} PN(\mathcal{R}, k, z_0 + z_1) &= 0, \\ A_0 z_1 + (I - P)N(\mathcal{R}, k, z_0 + z_1) &= 0. \end{aligned} \tag{4.7}$$

The second equation in (4.7) has a locally unique solution $z_1 = \mathcal{Z}(\mathcal{R}, k, z_0) \in Z_1$ which depends smoothly (in fact, analytically) on $(\mathcal{R}, k, z_0)$ and is such that

$$\mathcal{Z}(\mathcal{R}, k, z_0) = \mathcal{O}(\varepsilon\|z_0\| + \|z_0\|^2).$$

Substituting this relation into the first equation of (4.7) we obtain the bifurcation equation

$$0 = n(\mathcal{R}, k, z_0) \stackrel{\text{def}}{=} PN(\mathcal{R}, k, z_0 + \mathcal{Z}(\mathcal{R}, k, z_0)), \tag{4.8}$$

which is equivalent to

$$0 = \widetilde{n}(\mathcal{R}, k, \eta) \stackrel{\text{def}}{=} \Big(\langle \xi_j, n(\mathcal{R}, k, \sum_{l=1}^{5} \eta_l \xi_l)\rangle\Big)_{j=1,\ldots,5} \in \mathbb{R}^5. \tag{4.9}$$

The function $\widetilde{n}$ is the coordinate representation of the function n, which has values in Y_0, with respect to the basis $\{\xi_1, \ldots, \xi_5\}$. Hence, finding nontrivial roll patterns is equivalent to finding $\eta \in \mathbb{R}^5$ with $\widetilde{n}(\mathcal{R}, k, \eta) = 0$.

Before giving the calculations of the relevant lowest order terms in an expansion of $\widetilde{n}$ we discuss the general structure of $\widetilde{n}$ which is enforced by the symmetries of the problem. We have invariance of the problem under adding multiples of ξ_2 and ξ_3 to z. Hence, $\mathcal{Z}$ and thus $\widetilde{n}$ do not depend on (η_2, η_3). Moreover, we have equivariance of the problem under translation in y–direction

$$T_\tau : z(\cdot) \mapsto z(\cdot - \tau)$$

and under the reflections $K^{(j)}$, $j = 1, 2, 3$. These symmetries induce symmetries on the bifurcation equation relative to the following action on $\mathbb{R}^5$:

$$\begin{aligned} \widehat{K}^{(1)} &: \ \eta \mapsto \mathrm{diag}(-1, 1, 1, -1, 1)\eta, \\ \widehat{K}^{(2)} &: \ \eta \mapsto \mathrm{diag}(1, -1, 1, 1, 1)\eta, \\ \widehat{K}^{(3)} &: \ \eta \mapsto \mathrm{diag}(1, 1, 1, -1, -1)\eta, \end{aligned} \qquad \widehat{T}_\tau : \ \eta \mapsto \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \cos(\tau)\eta_4 - \sin(\tau)\eta_5 \\ \sin(\tau)\eta_4 + \cos(\tau)\eta_5 \end{pmatrix}. \tag{4.10}$$

Notice that $\widehat{K}^{(3)} = \widehat{T}_\pi$.

Lemma 4.2
The function $\widetilde{n}$ in (4.9) takes the form

$$\widetilde{n}(\mathcal{R},k,\eta)=\begin{pmatrix} g_1(\mathcal{R},k,\eta_1^2,r)\eta_1 \\ 0 \\ 0 \\ g_4(\mathcal{R},k,\eta_1^2,r)\eta_4-g_5(\mathcal{R},k,\eta_1^2,r)\eta_1\eta_5 \\ g_5(\mathcal{R},k,\eta_1^2,r)\eta_1\eta_4+g_4(\mathcal{R},k,\eta_1^2,r)\eta_5 \end{pmatrix} \quad \text{where } r=\eta_4^2+\eta_5^2,$$

Proof: As mentioned above, $\widetilde{n}$ is independent of η_2 and η_3. Moreover, the equivariance with respect to $\widehat{K}^{(2)}$ implies that $\widetilde{n}_2$ is odd in η_2. This is only possible if $\widetilde{n}_2\equiv 0$. The third component is zero since $\widetilde{n}_3=\langle \xi_3,\mathcal{N}(\mathcal{R},k,z)\rangle=-\int_\Omega \nabla_k\cdot u=0$ due to the boundary conditions for elements in Z.

The equivariance with respect to $\widehat{T}_\tau$ implies

$$\begin{pmatrix}\widetilde{n}_1\\ \widetilde{n}_4\\ \widetilde{n}_5\end{pmatrix}=\begin{pmatrix}\widetilde{g}_1\\ 0\\ 0\end{pmatrix}+\widetilde{g}_4\begin{pmatrix}0\\ \eta_4\\ \eta_5\end{pmatrix}+\widetilde{g}_5\begin{pmatrix}0\\ -\eta_5\\ \eta_4\end{pmatrix},$$

where each $\widetilde{g}_j$ depends on $(\mathcal{R},k,\eta_1,r)$ (see Proposition VIII,2.3 in [GS85]). Using $\widehat{K}^{(1)}$ we find that $\widetilde{g}_1$ and $\widetilde{g}_5$ must be odd in η_1 whereas $\widetilde{g}_4$ is even. $\square$

A nice consequence of the special form of $\widetilde{n}$ is that all the functions g_j can be calculated by setting $\eta_2=\eta_3=\eta_5=0$. In particular, we find

$$\mathcal{Z}(\mathcal{R},k,\eta_1\xi_1+\eta_4\xi_4)=\eta_1\eta_4 Z_{14}+\eta_4^2 Z_{44}+\eta_4(k-k_0)Z_{4k}+\eta_4(\mathcal{R}-\mathcal{R}_0)Z_{4\mathcal{R}}+\ldots$$

For explicit expressions for Z_{14}, Z_{44}, Z_{4k}, and $Z_{4\mathcal{R}}$ see Appendix A. For the further analysis it will only be important to calculate a finite number of coefficients, namely c_j for $j=1,..,5$ in

$$\begin{aligned} g_1(\mathcal{R}_0,k_0,0,r) &= c_1 r+\mathcal{O}(r^2),\\ g_4(\mathcal{R},k,0,r) &= c_2(\mathcal{R}-\mathcal{R}_0)+c_3(k-k_0)^2+c_4 r+\ldots, \\ g_5(\mathcal{R}_0,k_0,0,0) &= c_5. \end{aligned} \tag{4.11}$$

The calculation of these coefficients is given in Appendix A, and the result is

$$c_1=-3(\mathcal{P}+1)/20,\ c_2=-\sqrt{3}/2,\ c_3=3,\ c_4=3\mathcal{P}^2/32,\ c_5=-3\sqrt{2}(\mathcal{P}+1)/8.$$

We conclude from $c_5\neq 0$ that small solutions can only exist for $\eta_1=0$, which means that the mean flow in x_1–direction is zero. In this case, the solutions are given by $\eta=(0,\eta_2,\eta_3,\rho\cos(\tau),\rho\sin(\tau))^T$ where $\rho=\sqrt{r}$ solves the scalar equation $g_4(\mathcal{R},k,0,\rho^2)=0$. As $\mathcal{R}=(1+k^2)^{3/2}/k$ is the line of neutral stability we know that $g_4((1+k^2)^{3/2}/k,k,0,0)=0$. Using $c_4>0$ we conclude that nontrivial solutions exist exactly for those $(\mathcal{R},k)\approx(\mathcal{R}_0,k_0)$ with $g_4(\mathcal{R},k,0,0)<0$. Thus, we have proved the following (well–known) result (cf. [GSS88], Case Study 4 and the references therein).

Theorem 4.3
For all $(\mathcal{R}, k)$ close to $(\mathcal{R}_0, k_0)$ and satisfying $\mathcal{R} > (1+k^2)^{3/2}/k$ there exists a roll solution which is unique up to a shift in x_1–direction. Let $\mathcal{R} = \widetilde{\mathcal{R}}(k, r)$ be the unique solution of $g_4(\mathcal{R}, k, 0, r) = 0$ and $\rho = \sqrt{r}$, then,

$$z_{\rho,k} = \rho\xi_4 + \mathcal{Z}(\widetilde{\mathcal{R}}(k,\rho^2), k, \rho\xi_4) \in Z$$

is the unique roll solution of (4.4) *with u_1 an even function in y and $u_1 > 0$ at $(y, x_3) = (0,0)$.*

We will continue to use (ρ, k) as parameters for our problem and express $\mathcal{R}$ as

$$\begin{aligned}\mathcal{R} = \widetilde{\mathcal{R}}(k,\rho^2) &= \widetilde{\mathcal{R}}(k,0) - \tfrac{c_4}{c_2}\rho^2 + \mathcal{O}(|k-k_0|\rho^2 + \rho^4),\\ &= \mathcal{R}_0 + 2\sqrt{3}(k-k_0)^2 + \tfrac{\sqrt{3}\mathcal{P}^2}{16}\rho^2 + \mathcal{O}(|k-k_0|^3 + |k-k_0|\rho^2 + \rho^4).\end{aligned}$$

As a basis of our following stability analysis we choose the roll solution $z_{\rho,k} \in Z$ of Theorem 4.3. Note that $z_{\rho,k}$ is associated to the solution $\eta = \eta_\rho = (0,0,0,\rho,0)^T$ of (4.9) resp. the solution $z_0 = \rho\xi_4$ of the reduced problem (4.8). Moreover, we have the expansion

$$z_{\rho,k} = \rho\xi_4 + \rho^2 Z_{44} + \rho^3\Big(Z_{444} - \frac{c_4}{c_2}Z_{4\mathcal{R}}\Big) + \rho(k-k_0)Z_{4k} + \mathcal{O}(\rho(\rho^3 + |k-k_0|^2)). \quad (4.12)$$

The additional term $Z_{4\mathcal{R}}$ is present since $\mathcal{R} = \widetilde{\mathcal{R}}$ is considered as a function of ρ.

5 Sideband instabilities

We assume that $\widehat{z}$ is a roll solution as constructed above, i.e. $\widehat{z} = z_{\rho,k}$. We want to study the instability of this solution as a steady state of the full Rayleigh–Bénard problem (3.1) in the sense of Definition 3.1. The linearization is obtained by letting $z = \widehat{z} + \varepsilon w$, with $w = (v, \vartheta, q)$, and considering only terms which are linear in ε:

$$\left.\begin{aligned}\partial_t v + (\widehat{u}\cdot\nabla)v + (v\cdot\nabla)\widehat{u} + \nabla q - \widetilde{\mathcal{R}}\vartheta e_3 - \Delta v &= 0,\\ \mathcal{P}(\partial_t\vartheta + \widehat{u}\cdot\nabla\vartheta + v\cdot\nabla\widehat{\theta}) - \widetilde{\mathcal{R}}v_3 - \Delta\vartheta &= 0,\\ \nabla\cdot v &= 0,\end{aligned}\right\}\ \text{for } x \in \mathbb{R}^2\times(0,\pi);$$
$$\partial_3 v_1 = \partial_3 v_2 = v_3 = \vartheta = 0 \text{ for } x_3 = 0 \text{ or } x_3 = \pi. \quad (5.1)$$

As discussed above we study perturbations of the form

$$w(t,x) = e^{\lambda t + i(\sigma_1 x_1 + \sigma_2 x_2)}W(y, x_3), \text{ where } y = kx_1 \text{ and } W \in Z.$$

Inserting this Ansatz into (5.1) we obtain a linear problem for $W \in Z$ which depends on the additional parameters λ and σ. In fact, we have complexified the linear problem such that $W \in Z^{\mathbb{C}} = Z \oplus iZ$, however, for notational convenience we omit the superscript $\mathbb{C}$ in the following.

The problem for $W=(v,\vartheta,q)\in Z$ reads as follows:

$$\left.\begin{array}{rcl} \lambda v+(\widehat{u}\cdot\nabla_{k,\sigma})v+(v\cdot\nabla_k)\widehat{u}+\nabla_{k,\sigma}q-\widetilde{\mathcal{R}}\vartheta e_3-\Delta_{k,\sigma}v &=& 0,\\ \mathcal{P}(\lambda\vartheta+\widehat{u}\cdot\nabla_{k,\sigma}\vartheta+v\cdot\nabla_k\widehat{\theta})-\widetilde{\mathcal{R}}v_3-\Delta_{k,\sigma}\vartheta &=& 0,\\ -\nabla_{k,\sigma}\cdot v &=& 0, \end{array}\right\}\ \text{in } \mathbb{R}^2\times(0,\pi); \tag{5.2}$$

with boundary conditions

$$\partial_3 v_1=\partial_3 v_2=v_3=\vartheta=0 \text{ for } x_3=0 \text{ or } x_3=\pi.$$

We continue to use the abbreviations

$$\nabla_{k,\sigma}=\begin{pmatrix} i\sigma_1+k\partial_y\\ i\sigma_2\\ \partial_3\end{pmatrix},\quad \Delta_{k,\sigma}=k^2\partial_y^2+\partial_3^2+2i\sigma_1 k\partial_y-|\sigma|^2=\Delta_k+2i\sigma_1 k\partial_y-|\sigma|^2.$$

It is important to note that the gradient operator $\nabla_{k,\sigma}$ is applied to the linear perturbation W while the steady solution $\widehat{z}$ is acted upon by $\nabla_k=\nabla_{k,0}$.

We can write this systems as a general linear problem

$$\mathcal{B}(\rho,k,\sigma,\lambda)W=0, \tag{5.3}$$

where $\mathcal{B}(\ldots)$ is a bounded operator from Z into Y depending smoothly on the variables (ρ,k,σ,λ). It can be written as

$$\mathcal{B}(\rho,k,\sigma,\lambda)=D_z\mathcal{N}(\widetilde{\mathcal{R}}(k,\rho^2),k,\nabla_{k,\sigma},z_{\rho,k})-\lambda M,$$

where $M=\operatorname{diag}(-1,-1,-1,-\mathcal{P},0)$. Thus, we have $\mathcal{B}(\rho,k,0,0)=D_z\mathcal{N}(\rho,k,z_{\rho,k})$ and so $\mathcal{B}(0,k_0,0,0)=A_0$.

The existence of nontrivial solutions W for (5.3) can again be studied by the help of the LSR. Again we use the splitting $W=W_0+W_1$ with $W_j\in Z_j$ and (5.3) is thus equivalent to

$$\begin{array}{rcl} P\mathcal{B}(\rho,k,\sigma,\lambda)(W_0+W_1) &=& 0,\\ (I-P)\mathcal{B}(\rho,k,\sigma,\lambda)(W_0+W_1) &=& 0. \end{array} \tag{5.4}$$

The second equation is solvable in the form $W_1=\mathcal{W}(\rho,k,\sigma,\lambda)W_0$ by the implicit function theorem for (ρ,k,σ,λ) sufficiently close to $(0,k_0,0,0)$.

Using the coordinates $\omega=(\omega_1,\ldots,\omega_5)^T\in\mathbb{C}^5$ in Z_0 via $W_0=\sum_{j=1}^5\omega_j\xi_j$ we obtain $W_1=\sum_{j=1}^5\omega_j\mathcal{W}_j(\rho,k,\sigma,\lambda)$. Introducing this into the first equation of (5.4) the reduced linear problem takes the form

$$\widetilde{b}(\rho,k,\sigma,\lambda)\omega=0,\ \text{where } \widetilde{b}(\ldots)\in\mathbb{C}^{5\times 5} \tag{5.5}$$

and $\widetilde{b}(\ldots)_{jl}=\langle\mathcal{B}(\ldots)[\xi_l+\mathcal{W}_l(\ldots)],\xi_j\rangle$.

Thus, the problem of linearized instability is reduced to the study of the matrix $\widetilde{b}$. For a given (ρ, k) we can now prove spectral instability of the roll pattern $z_{\rho,k}$ by finding a solution $(\sigma, \lambda, \omega) \in \mathbb{R}^2 \times \mathbb{C} \times \mathbb{C}^5$ with $\operatorname{Re} \lambda > 0$ and $\omega \notin \mathbb{C} e_3$. (This last condition on ω is necessary to exclude solutions which only grow exponentially in the mean value of the pressure.)

As in the case of the nonlinear bifurcation problem, we can give more structure to the matrix $\widetilde{b}$ by using the special form and the symmetries of the problem. This will minimize the amount of calculations for the relevant coefficients in the lowest order expansion.

Lemma 5.1
The matrix $\widetilde{b}$ takes the form

$$\widetilde{b} = \begin{pmatrix} \gamma_{11} & -\sigma_1\sigma_2\gamma_{12} & i\sigma_1 & i\sigma_1\rho\gamma_{14} & \rho\gamma_{15} \\ -\sigma_1\sigma_2\gamma_{21} & \gamma_{22} & i\sigma_2 & i\sigma_2\rho\gamma_{24} & -\sigma_1\sigma_2\rho\gamma_{25} \\ -i\sigma_1 & -i\sigma_2 & 0 & 0 & 0 \\ i\sigma_1\rho\gamma_{41} & i\sigma_2\rho\gamma_{42} & 0 & \gamma_{44} & i\sigma_1\gamma_{45} \\ \rho\gamma_{51} & -\sigma_1\sigma_2\rho\gamma_{52} & 0 & i\sigma_1\gamma_{54} & \gamma_{55} \end{pmatrix}, \tag{5.6}$$

where all γ_{jl} are analytic functions of $(\rho^2, k, \sigma_1^2, \sigma_2^2, \lambda)$.

Proof: First we note that the third row and the third column of $\widetilde{b}$ can be given exactly. The reason for this is two–fold. On the one hand the third component of $\widetilde{b}\omega \in \mathbb{C}^5$ reads

$$\langle \xi_3, \mathcal{B}(\ldots)W\rangle = -\int_\Omega \nabla_{k,\sigma} v = -\int_\Omega (i\sigma\cdot v + k\partial_y v_1 + \partial_3 v_3) = -i\sigma\cdot(\omega_1,\omega_2)^T,$$

since the intergal over the derivatives cancel due to the boundary conditions. On the other hand $\omega_3 = \frac{1}{2\pi^2}\int_\Omega q$ enters $\mathcal{B}(\ldots)W$ only in the first and second equation, namely as $(i\sigma_1\omega, i\sigma_2\omega, 0, 0, 0)^T$. Hence, $(I-P)\mathcal{B}(\ldots)(W_0+W_1)$ is independent of ω_3 and so is $W_1 = \mathcal{W}(\ldots)W_0$. This proves the form of $\widetilde{b}_{jl}$ when either j or l equals 3.

Next we establish the symmetry properties of $\widetilde{b}$. Since imaginary coefficients only appear through $i\sigma$ in the full problem $\mathcal{B}W = 0$, we find that

$$\overline{\widetilde{b}(\rho,k,\sigma,\lambda)} = \widetilde{b}(\rho,k,-\sigma,\overline{\lambda}).$$

The symmetry $K^{(2)} : x_2 \mapsto -x_2$ leads to the equivariance

$$\widehat{K}^{(2)}\widetilde{b}(\rho,k,\sigma_1,-\sigma_2,\lambda)\widehat{K}^{(2)} = \widetilde{b}(\rho,k,\sigma,\lambda),$$

where the matrices $\widehat{K}^{(j)} \in \mathbb{R}^{5\times 5}$ are defined in (4.10). The basic roll pattern $z_{\rho,k}$ is still invariant under the transformation $K^{(1)}K^{(3)} : (y, x_3) \mapsto (-y, \pi - x_3)$ which amounts to

$$\widehat{K}^*\widetilde{b}(\rho,k,-\sigma_1,\sigma_2,\lambda)\widehat{K}^* = \widetilde{b}(\rho,k,\sigma,\lambda) \text{ with } \widehat{K}^* = \widehat{K}^{(1)}\widehat{K}^{(3)} = \operatorname{diag}(-1,1,1,1,-1).$$

In addition, we may change the sign of ρ, since $\mathcal{R} = \widetilde{\mathcal{R}}(k,\rho^2)$ and $z_{-\rho,k}$ is obtained from $z_{\rho,k}$ by the reflection $K^{(3)} : x_3 \mapsto \pi - x_3$. Hence,

$$\widehat{K}^{(3)}\widetilde{b}(-\rho,k,\sigma,\lambda)\widehat{K}^{(3)} = \widehat{b}(\rho,k,\sigma,\lambda).$$

These four symmetries imply the existence of the functions $\gamma_{jl}(\rho^2,k,\sigma_1^2,\sigma_2^2,\lambda)$. $\square$

The structure of the third row and column of $\widetilde{b}$ allows us to reduce the problem to a three–dimensional one for the variables $(\omega_0,\omega_4,\omega_5) \in \mathbb{C}^3$, where $\omega_0 = \sigma_2\omega_1 - \sigma_1\omega_2$. In fact, this reduction is analogous to the classical reductions of Navier–Stokes problems by enforcing the incompressibility $\nabla \cdot u = 0$ and projecting out the pressure. Moreover, the 3×3–matrix b^* defined below corresponds to the matrices in the previous studies [BuB84], equations (2.9) and (2.11), and [Be94], equation (3.11).

Lemma 5.2
Let $\widetilde{b}$ be given as in (5.6) and define $b^* =$

$$\begin{pmatrix} \sigma_2^2\gamma_{11}+\sigma_1^2\gamma_{22}+\sigma_1^2\sigma_2^2(\gamma_{12}+\gamma_{21}) & \sigma_1\sigma_2\rho(\gamma_{14}-\gamma_{24}) & -i\sigma_2\rho(\gamma_{15}+\sigma_1^2\gamma_{25}) \\ \sigma_1\sigma_2\rho(\gamma_{42}-\gamma_{41}) & \gamma_{44} & i\sigma_1\gamma_{45} \\ i\sigma_2\rho(\gamma_{51}+\sigma_1^2\gamma_{52}) & i\sigma_1\gamma_{54} & \gamma_{55} \end{pmatrix}. \tag{5.7}$$

Then we have the relation $\det\widetilde{b}(\rho,k,\sigma,\lambda) = -\det b^*(\rho,k,\sigma,\lambda)$.

Proof: Obviously, $\det\widetilde{b} = \det b^* = 0$ for $\sigma = 0$ due to (5.6) and the definition of b^*. For $\sigma \neq 0$ define

$$T = \begin{pmatrix} 0 & i\sigma_1 & i\sigma_2 & 0 & 0 \\ 0 & i\sigma_2 & -i\sigma_1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

which has $\det T = |\sigma|^2$. We find that

$$C = \overline{T}^T\widetilde{b}T = \begin{pmatrix} 0 & |\sigma|^2 & 0 & 0 & 0 \\ |\sigma|^2 & * & * & * & * \\ 0 & * & & & \\ 0 & * & & b^* & \\ 0 & * & & & \end{pmatrix}$$

which has $\det C = -|\sigma|^4 \det b^*$. This proves the result for $\sigma \neq 0$. $\square$

We apply our abstract result from Theorem 2.2 to control the terms obtained from the nonlinearity in the case $(\sigma,\lambda)=0$:

$$\widetilde{b}(\rho,k,0,0)=D_\eta\widetilde{n}(\widetilde{\mathcal{R}}(k,\rho^2),k,\eta_\rho)=\begin{pmatrix} g_1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 2\rho^2\partial g_4/\partial r & 0\\ \rho g_5 & 0 & 0 & 0 & 0\end{pmatrix}. \tag{5.8}$$

Here we have used $\eta_\rho=(0,0,0,\rho,0)^T$ and $g_4(\widetilde{\mathcal{R}},k,\eta_\rho)=0$ by definition of ρ. In this formula all g_j are evaluated at $(\widetilde{\mathcal{R}}(k,\rho^2),k,\eta_\rho)$. This yields additional information on the following functions γ_{jl} (recall that $r=\rho^2$):

$$\begin{aligned}
&\gamma_{15}(r,k,0,0,0)=\gamma_{22}(r,k,0,0,0)=\gamma_{55}(r,k,0,0,0)=0,\\
&\gamma_{11}(r,k,0,0,0)=g_1=c_1r+\mathcal{O}(r^2),\ \ \gamma_{44}(r,k,0,0,0)=2r\tfrac{\partial g_4}{\partial r}=2c_4r+\mathcal{O}(r^2),\\
&\gamma_{51}(r,k,0,0,0)=g_5=c_5+\mathcal{O}(r).
\end{aligned}$$

Moreover, employing Lemma 2.4 we can construct the part arising from the linearization at the trivial flow $\rho=0$. In particular, the terms from the mean flows decouple from those arising from $e^{\pm iy}$. Thus, we obtain the block structure

$$\widetilde{b}(0,k,\sigma,\lambda)=\begin{pmatrix}\widetilde{b}^{(123)}(k,\sigma,\lambda) & 0\\ 0 & \widetilde{b}^{(45)}(k,\sigma,\lambda)\end{pmatrix},$$

where the individual blocks have the form

$$\widetilde{b}^{(123)}(k,\sigma,\lambda)=\begin{pmatrix}\lambda+|\sigma|^2 & 0 & i\sigma_1\\ 0 & \lambda+|\sigma|^2 & i\sigma_2\\ -i\sigma_1 & -i\sigma_2 & 0\end{pmatrix} \tag{5.9}$$

and

$$\widetilde{b}^{(45)}(k,\sigma,\lambda)=\begin{pmatrix}\nu(k,\sigma_1^2,\sigma_2^2,\lambda) & -i\sigma_1\widetilde{\delta}(k,\sigma_1^2,\sigma_2^2,\lambda)\\ i\sigma_1\widetilde{\delta}(k,\sigma_1^2,\sigma_2^2,\lambda) & \nu(k,\sigma_1^2,\sigma_2^2,\lambda)\end{pmatrix}. \tag{5.10}$$

The following expansion for $\widetilde{b}^{(45)}$ is derived in Appendix B:

$$\begin{aligned}
\nu(k,\sigma_1^2,\sigma_2^2,\lambda) &= \tfrac{3}{4}(\mathcal{P}+1)\lambda+3\sigma_1^2+3\sqrt{2}(k-k_0)\sigma_2^2+\tfrac{3}{2}\sigma_2^4+\tfrac{7\mathcal{P}^2+34\mathcal{P}-23}{100}\lambda^2+\ldots\\
\widetilde{\delta}(k,\sigma_1^2,\sigma_2^2,\lambda) &= \tfrac{\sqrt{2}}{15}(45\sqrt{2}(k-k_0)+31\sigma_1^2+45\sigma_2^2+9(\mathcal{P}+1)\lambda+\ldots).
\end{aligned}$$

We can draw a few helpful conclusions about some of the γ_{jl} at $\rho=0$:

$$\gamma_{jl}(0,k,\sigma_1^2,\sigma_2^2,\lambda)=0 \text{ except for } (j,l)\in\{(1,1),(2,2),(4,4),(4,5),(5,4),(5,5)\}.$$

The two pieces of information can be put together by the general relation

$$\gamma_{jl}(\rho^2, k, \sigma_1^2, \sigma_2^2, \lambda) = \gamma_{jl}(0, k, \sigma_1^2, \sigma_2^2, \lambda) + \gamma_{jl}(\rho^2, k, 0, 0, 0) + \mathcal{O}(\rho^2(|\sigma|^2 + |\lambda|)),$$

which follows from the general fact that $f(a,b) = f(a,0) + f(0,b) + \mathcal{O}(|ab|)$ for a C^2 function satisfying $f(0,0) = 0$, cf. also (2.10).

According to the Principle of Reduced Instability (Theorem 2.1) it remains to analyze the solution set of the eigenvalue equation

$$0 = \Lambda(\rho, k, \sigma, \lambda) \stackrel{\text{def}}{=} \det b^*(\rho, k, \sigma, \lambda),$$

where $(\rho, k, \sigma, \lambda) \approx (0, k_0, 0, 0)$. If for a given (ρ, k) we can find a solution (σ, λ) of $\Lambda(\rho, k, \sigma, \lambda) = 0$ with $\operatorname{Re}\lambda > 0$, then, in view of the discussion in Section 3, instability of the roll solution $z_{\rho,k}$ in the sense of Definition 3.1 is proved.

A complete discussion of $\Lambda = 0$ turns out to be difficult and we will not be given here. We treat the cases $\sigma = (\sigma_1, 0)$ and $\sigma = (0, \sigma_2)$ exactly, whereas for $\sigma_1\sigma_2 \neq 0$ only certain scaling limits are considered.

5.1 The case $\sigma = 0$

Before going into detail we derive a useful result for the eigenvalue problem in the case of $\sigma = 0$. Hence, stability is only calculated with respect to perturbations with the same period. In that case we have to treat the matrix $\widetilde{b} \in \mathbb{R}^{5\times5}$ directly without using the reduction associated with Lemma 5.2:

$$\widetilde{b} = \begin{pmatrix} \gamma_{11} & 0 & 0 & 0 & \rho\gamma_{15} \\ 0 & \gamma_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma_{44} & 0 \\ \rho\gamma_{51} & 0 & 0 & 0 & \gamma_{55} \end{pmatrix}$$

where

$$\begin{aligned} \gamma_{11} &= \lambda + g_1(\widetilde{\mathcal{R}}, k, 0, r) + \mathcal{O}(|\lambda| r), & \gamma_{22} &= \lambda(1 + \ldots), \\ \gamma_{44} &= \tfrac{3}{4}(\mathcal{P}+1)\lambda + 2c_4\rho^2 + \ldots, & \gamma_{51} &= c_5 + \ldots \\ \gamma_{15} &= c_{15}\lambda + \ldots, & \gamma_{55} &= \tfrac{3}{4}(\mathcal{P}+1)\lambda + \ldots. \end{aligned}$$

Eliminating the third column and row which is due to the pressure indeterminacy, we obtain a four dimensional problem, where the diagonal entries γ_{22} and γ_{44} decouple. We immediately obtain the eigenvalue $\lambda_2 = 0$ from $\gamma_{22} = 0$ which is due to the mean flow in the x_2–direction. From $\gamma_{44} = 0$, we obtain the eigenvalue λ_4 which is strictly negative.

It remains to investigate the relation $0 = \gamma_{11}\gamma_{55} - \rho^2\gamma_{15}\gamma_{51}$. Of course one eigenvalue has to be $\lambda = 0$, since we have the translational mode in the x_1–direction. In fact, by Galileian invariance the zero $\lambda = 0$ has to be double. This is the content of the following lemma which is based on Theorem 2.3.

Lemma 5.3
(a) For $(\sigma, \lambda) = (0,0)$ *we have* $\widetilde{b}(\rho, k, 0, 0)e_5 = 0$*, or equivalently*

$$\gamma_{15}(\rho^2, k, 0, 0) = \gamma_{55}(\rho^2, k, 0, 0) = 0 \quad \textit{for all small } (\rho, k-k_0).$$

(b) There is a smooth function $\alpha : \mathbb{R}^2 \to \mathbb{R}$ *with* $\alpha(\rho, k) = \rho(1 + \mathcal{O}(\rho^2 + (k-k_0)^2))$ *such that the relation* $\widetilde{b}(\rho, k, 0, 0)e_1 + k\alpha(\rho, k)\partial_\lambda \widetilde{b}(\rho, k, 0, 0)e_5 = 0$ *holds, or equivalently*

$$\gamma_{11} + k\alpha(\rho, k)\rho\partial_\lambda\gamma_{15} = 0, \qquad \gamma_{51} + k\alpha(\rho, k)\partial_\lambda\gamma_{55} = 0,$$

for all small $(\rho, k-k_0)$ *and* $\sigma = \lambda = 0$*.*

Proof: We apply part (a) and (b) of Theorem 2.3 using the fact that $\widetilde{w} = -\partial_y z_{\rho,k}$ satisfies $\mathcal{B}(\rho, k, 0, 0)\widetilde{w} = D_z\mathcal{N}(\widetilde{\mathcal{R}}, k, z_{\rho,k})\widetilde{w} = 0$ and $Q\widetilde{w} = \alpha(\rho, k,)\xi_5$. The last result is easy to see since $\widetilde{w} = \partial_y z_{\rho,k}$ has average zero, i.e. $\langle \xi_j, \widetilde{w}\rangle = 0$ for $j = 1, 2, 3$. The case $j = 4$ follows from symmetry considerations, namely $K^{(1)}\xi_4 = \xi_4$ and $K^{(1)}\widetilde{w} = -\widetilde{w}$. Moreover, we have $\widetilde{w} = \rho\xi_5 + \mathcal{O}(\rho^2)$ which proves part (a).

To establish (b), we use the Galileian invariance (3.2). Consider the time dependent solution $z(t, \cdot) = \tau\xi_1 + T_{k\tau t}z_{\rho,k}$ (recall that $T_{k\tau}z = z(\cdot - k\tau)$) of Bénard's problem $M\partial_t z = \mathcal{N}(\widetilde{\mathcal{R}}, k, z)$, which is just the same roll as before but considered in a frame moving with constant velocity τ in x_1–direction. As a consequence we find

$$\mathcal{N}(\tau\xi_1 + T_{k\tau t}z_{\rho,k}) = M\partial_t(\tau\xi_1 + T_{k\tau t}z_{\rho,k}) = k\tau M T_{k\tau t}\widetilde{w},$$

for all $\tau \in \mathbb{R}$. Evaluating the derivative with respect to τ at $\tau = 0$ we arrive at

$$D_z\mathcal{N}(\widetilde{\mathcal{R}}, k, z_{\rho,k})[\xi_1 + kt\widetilde{w}] - kM\widetilde{w} = 0.$$

However, this is just the assumption of part (b) in Theorem 2.3, and the result follows with $\widetilde{v} = \xi_1 = Q\xi_1$. □

5.2 Eckhaus instability: $\sigma = (\sigma_1, 0)$

The first case we study is the two–dimensional problem (that is, with no x_2–dependence) which will lead to the Eckhaus instability. We let $\sigma_2 = 0$ and obtain

$$b^*(\rho, k, \sigma_1, 0, \lambda) = \begin{pmatrix} \sigma_1^2\gamma_{22} & 0 & 0 \\ 0 & \gamma_{44} & i\sigma_1\gamma_{45} \\ 0 & i\sigma_1\gamma_{54} & \gamma_{55} \end{pmatrix}.$$

The decoupling yields a first eigenvalue $\lambda_1 = -\sigma_1^2 + \ldots$ which is always stable.

The 2×2 block admits the expansion

$$\begin{pmatrix} \frac{3}{4}(\mathcal{P}+1)\lambda + 3\sigma_1^2 + 2c_4\rho^2 + \dots & -i\sigma_1(6(k-k_0)+\dots) \\ i\sigma_1(6(k-k_0)+\dots) & \frac{3}{4}(\mathcal{P}+1)\lambda + 3\sigma_1^2 + \dots \end{pmatrix}.$$

We immediately find that the second eigenvalue is always less than $-\frac{8}{3}c_4\rho^2/(\mathcal{P}+1)$ while the third eigenvalue is positive if and only if $\widehat{E}(\rho,k,\sigma_1) = [\gamma_{44}\gamma_{55} - \sigma_1^2\gamma_{45}\gamma_{54}]_{\lambda=0}$ is negative. Since $\widehat{E}$ has the form $\widehat{E}(\rho,k,\sigma_1) = \sigma_1^2 E(\rho^2,k,\sigma_1^2)$ with

$$E(r,k,s) = 3(3s + 2c_4 r) - 36(k-k_0)^2 + \mathcal{O}(|k-k_0|(|k-k_0|^2 + r + s) + r^2 + s^2),$$

negative values of $\widehat{E}$ (for $s = \sigma_1^2 > 0$) can occur if and only if $\rho^2 \leq r_{\mathrm{E}}(k)$. The function r_{E} is obtained from solving $E(r,k,0) = 0$ for $r = r_{\mathrm{E}}(k)$ and has the expansion $r_{\mathrm{E}}(k) = 6(k-k_0)^2/c_4 + \mathcal{O}(|k-k_0|^3)$.

Transforming back into the $(\mathcal{R},k)$ coordinates, we let $\mathcal{R}_{\mathrm{Eckhaus}}(k) \stackrel{\mathrm{def}}{=} \widetilde{\mathcal{R}}(k, r_{\mathrm{E}}(k))$ and find existence of rolls which are *Eckhaus unstable* for

$$\begin{aligned} \widetilde{\mathcal{R}}(k,0) < \mathcal{R} < \mathcal{R}_{\mathrm{Eckhaus}}(k) &= \mathcal{R}_0 + 3\cdot 2\sqrt{3}(k-k_0)^2 + \mathcal{O}(|k-k_0|^3) \\ &= \mathcal{R}_0 + 3(\widetilde{\mathcal{R}}(k,0) - \mathcal{R}_0) + \mathcal{O}(|k-k_0|^3). \end{aligned}$$

We summarize the result as follows.

Theorem 5.4
(Eckhaus instability) *The roll solution $z_{\rho,k}$ is unstable with respect to two–dimensional sideband perturbations (that is, $v(x) = e^{i\sigma_1 x_1} W(kx_1, x_3)$ with $W \in Z$) if $\widetilde{\mathcal{R}}(k,\rho^2) < \mathcal{R}_{\mathrm{Eckhaus}}(k)$.*

In [KvW96] the two–dimensional problem (no dependence on x_2) is analyzed in more detail. There stability of $z_{\rho,k}$ in the sense of Definition 3.1 is established for the case $\widetilde{\mathcal{R}}(k,\rho^2) \geq \mathcal{R}_{\mathrm{Eckhaus}}(k)$.

5.3 Zigzag instabilities: $\sigma = (0, \sigma_2)$

The second case relates to three–dimensional perturbations with $\sigma_1 = 0$. Again, the matrix b^* decouples into

$$b^* = \begin{pmatrix} \sigma_2^2\gamma_{11} & 0 & -i\sigma_2\rho\gamma_{15} \\ 0 & \gamma_{44} & 0 \\ i\sigma_2\rho\gamma_{51} & 0 & \gamma_{55} \end{pmatrix}.$$

From $\gamma_{44} = 0$ we obtain the second eigenvalue

$$\lambda_2(\rho^2, k, \sigma_2^2) = -\frac{4}{3(\mathcal{P}+1)}\Big(2c_4\rho^2 + 3\sqrt{2}(k-k_0)\sigma_2^2 + \frac{3}{2}\sigma_2^4 + \dots\Big).$$

For $k > k_0$ this eigenvalue is always negative, while for $k < k_0$ the maximum eigenvalue is achieved for finite $\sigma_2^2 = S_1(\rho^2, k)$ with $S(\rho^2, k) = -\sqrt{2}(k - k_0) + \ldots$. This maximum is positive if and only if

$$\rho^2 < r_{\mathrm{Z}}^{(1)}(k) = \frac{3}{2c_4}(k - k_0)^2 + \mathcal{O}(|k - k_0|^3) \quad \text{and } k < k_0.$$

Here $r_{\mathrm{Z}}^{(1)}$ is defined by solving $\lambda_2(r, k, S_1(r, k)) = 0$ for r.

This sideband instability is detached: for very small $|\sigma_2|$ the modes are always stable as well as for large values. Moreover, this instability only occurs where the Eckhaus instablitiy is already present, since it appears in the regime

$$\widetilde{\mathcal{R}}(k, 0) < \mathcal{R} < \mathcal{R}^{(1)}_{\text{zigzag}}(k) = \mathcal{R}_0 + \frac{3}{2}(\widetilde{\mathcal{R}}(k, 0) - \mathcal{R}_0) + \mathcal{O}(|k - k_0|^3) \quad \text{and } k < k_0,$$

where $\mathcal{R}^{(1)}_{\text{zigzag}}(k) \stackrel{\text{def}}{=} \widetilde{\mathcal{R}}(k, r_{\mathrm{Z}}^{(1)}(k)) = \mathcal{R}_0 + 3\sqrt{3}\,(k - k_0)^2 + \mathcal{O}(|k - k_0|^3)$.

The remaining 2×2 block can now be studied separately. Since we consider only $\sigma_2 \neq 0$ we may divide out σ_2 and are left with

$$\begin{pmatrix} \gamma_{11} & \rho\gamma_{15} \\ \rho\gamma_{51} & \gamma_{55} \end{pmatrix} = \begin{pmatrix} \lambda + \sigma_2^2 + c_1\rho^2 + \ldots & \rho(-\lambda\sqrt{2}\,c_1 + d_2\sigma_2^2 + \ldots) \\ \rho(c_5 + \ldots) & \frac{3(\mathcal{P}+1)}{4}\lambda + 3\sqrt{2}\,(k - k_0)\sigma_2^2 + \frac{3}{2}\sigma_2^4 + \ldots \end{pmatrix},$$

where $c_1 = -\frac{3}{20}(\mathcal{P} + 1)$, $c_5 = -\frac{3}{4\sqrt{2}}(\mathcal{P} + 1)$, and $d_2 = 1/\sqrt{2}$ (cf. Appendix C). This leads to the eigenvalue equation

$$\begin{aligned} &\tfrac{3(\mathcal{P}+1)}{4}\lambda^2 + \left(\tfrac{3(\mathcal{P}+1)}{4}\sigma_2^2 + 3\sqrt{2}\,(k - k_0)\sigma_2^2 + \tfrac{3}{2}\sigma_2^4\right)\lambda + \\ &\quad +(\sigma_2^2 + c_1\rho^2)(3\sqrt{2}\,(k - k_0)\sigma_2^2 + \tfrac{3}{2}\sigma_2^4) - \rho^2 c_5 d_2 \sigma_2^2 + \ldots = 0. \end{aligned}$$

Since $d_2 > 0$, for small σ_2 the eigenvalues are not real with $\operatorname{Im}\lambda = \mathcal{O}(\rho\sigma_2)$ and $\operatorname{Re}\lambda \leq -\frac{1}{2}\sigma_2^2 + \ldots$. However, for larger σ_2 (viz. $\sigma_2^2 > 16\sqrt{2}\rho^2/(3(\mathcal{P}+1))$) the eigenvalues become real, and a positive eigenvalue appears if the coefficient of λ^0 is negative. This coefficient is $\widetilde{F}(\rho, k, \sigma_2) = [\gamma_{11}\gamma_{55} - \rho^2\gamma_{15}\gamma_{51}]_{\lambda=0}$, and it takes the form $\widetilde{F}(\rho, k, \sigma_2) = \sigma_2^2 F(\rho^2, k, \sigma_2^2)$. The minimum of F is attained for $\sigma_2^2 = S_2(\rho^2, k) = -\sqrt{2}\,(k - k_0) - \frac{c_1}{2}\rho^2 + \ldots$, and testing for negativity leads to the instability region $\rho^2 < r_{\mathrm{Z}}^{(2)}(k) = \frac{16}{\mathcal{P}+1}(k - k_0)^2 + \mathcal{O}(|k - k_0|^3)$. Thus, we have zigzag unstable rolls in the region

$$\widetilde{\mathcal{R}}(k, 0) < \mathcal{R} < \mathcal{R}^{(2)}_{\text{zigzag}}(k) = \mathcal{R}_0 + \left(1 + \tfrac{\mathcal{P}^2}{4(\mathcal{P}+1)}\right) 2\sqrt{3}\,(k - k_0)^2 + \mathcal{O}(|k - k_0|^3) \text{ and } k < k_0.$$

This is again a detached sideband instability.

We summarize the results in the case $\sigma = (0, \sigma_2)$ as follows.

Theorem 5.5
(Zizag instability) *The roll solution $z_{\rho,k}$ is unstable with respect to zigzag perturbations (that is, $v(t, x) = e^{i\sigma_2 x_2} W(kx_1, x_3)$ with $W \in Z$) if $\widetilde{\mathcal{R}}(k, \rho^2) < \max\{\mathcal{R}^{(1)}_{\text{zigzag}}(k), \mathcal{R}^{(2)}_{\text{zigzag}}(k)\}$.*

5.4 Skew–varicose instabilities: $\sigma_1\sigma_2 \neq 0$

Finally we investigate instabilities which arise due to fully three dimensional perturbations, that is $\sigma_1 \neq 0 \neq \sigma_2$. The discussion of $\Lambda(\rho, k, \sigma, \lambda) = 0$ is now very complicated and not completely understood. To derive necessary conditions for instability the following procedure is used. We introduce a scaling

$$\rho = \varepsilon^{\nu_1}\widetilde{\rho},\ k - k_0 = \varepsilon^{\nu_2}\widetilde{\kappa},\ \sigma_1 = \varepsilon^{\nu_3}\widetilde{\sigma}_1,\ \sigma_2 = \varepsilon^{\nu_4}\widetilde{\sigma}_2, \lambda = \varepsilon^{\nu_5}\widetilde{\lambda},\ \text{with } \nu_j \in \mathbb{N},$$

where ε is a small positive parameter and $\widetilde{\rho}, \widetilde{\kappa}, \widetilde{\sigma}$, and $\widetilde{\lambda}$ are considered to be of order one compared to ε. Inserting this scaling into Λ gives

$$\Lambda(\varepsilon^{\nu_1}\widetilde{\rho}, k_0 + \varepsilon^{\nu_2}\widetilde{\kappa}, \varepsilon^{\nu_3}\widetilde{\sigma}_1, \varepsilon^{\nu_4}\widetilde{\sigma}_2, \varepsilon^{\nu_5}\widetilde{\lambda}) = \varepsilon^{\nu^*}\Lambda^*(\widetilde{\rho}, \widetilde{\kappa}, \widetilde{\sigma}, \widetilde{\lambda}) + \mathcal{O}(\varepsilon^{\nu_6}),$$

where ν^* is chosen such that Λ^* is nontrivial and $\nu^* < \nu_6$. Note that Λ^* is a polynomial.

Assume now that $\Lambda^*(\ldots) = 0$ has a solution $(\widehat{\rho}, \widehat{\kappa}, \widehat{\sigma}, \widehat{\lambda})$ such that $\widehat{\sigma} \neq 0$, $\mathrm{Re}\,\widehat{\lambda} > 0$, and $\widehat{\lambda}$ is a simple root of $\Lambda^*(\widehat{\rho}, \widehat{\kappa}, \widehat{\sigma}, \cdot)$. Then by standard perturbation arguments there exists an $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ the equation $\Lambda(\varepsilon^{\nu_1}\widehat{\rho}, k_0 + \varepsilon^{\nu_2}\widehat{\kappa}, \varepsilon^{\nu_3}\widehat{\sigma}_1, \varepsilon^{\nu_4}\widehat{\sigma}_2, \lambda) = 0$ has a simple root $\lambda = \lambda(\varepsilon)$ which has the expansion $\lambda(\varepsilon) = \varepsilon^{\nu_5}(\widehat{\lambda} + \mathcal{O}(\varepsilon^{\nu_6 - \nu^*}))$. From $\mathrm{Re}\,\widehat{\lambda} > 0$ we thus conclude sideband instability of the rolls $z_{\rho,k}$ along the curve $(\rho, k) = (\varepsilon^{\nu_1}\widehat{\rho}, k_0 + \varepsilon^{\nu_2}\widehat{\kappa})$, $\varepsilon \in (0, \varepsilon_0)$.

It is important to realize that for different scaling vectors $\nu = (\nu_1, \ldots, \nu_5)$ one may obtain different instability criteria. The results on the Eckhaus and zigzag instabilities are obtained with $\nu = (2, 2, 2, 1, 4)$, which is also the scaling used in [ZS83].

Here we follow the approach in [BuB84, Be94] with $\nu = (1, 2, 1, 1, 2)$, or explicitly

$$\rho = \varepsilon\widetilde{\rho},\ k - k_0 = \varepsilon^2\widetilde{\kappa},\ \sigma_1 = \varepsilon\widetilde{\sigma}_1,\ \sigma_2 = \varepsilon\widetilde{\sigma}_2, \lambda = \varepsilon^2\widetilde{\lambda}. \tag{5.11}$$

Using the information on the functions $\gamma_{jk}(\rho, k, \sigma_1^2, \sigma_2^2, \lambda)$ from above we obtain the following expansion for the reduced stability matrix $b^* =$

$$\begin{pmatrix} \varepsilon^4 b_{11}^0 + \mathcal{O}(\varepsilon^6) & \varepsilon^3\sigma_1\sigma_2\rho(d_0 + \mathcal{O}(\varepsilon^2)) & -i\varepsilon^4\sigma_2\rho\Big(b_{13}^0 + \mathcal{O}(\varepsilon^2)\Big) \\ \varepsilon^3\sigma_1\sigma_2\rho(d_3 + \mathcal{O}(\varepsilon^2)) & \varepsilon^2 b_{22}^0 + \mathcal{O}(\varepsilon^4) & -i\varepsilon^3\sigma_1(b_{23}^0 + \mathcal{O}(\varepsilon^2)) \\ i\varepsilon^2\sigma_2\rho[\frac{-3(\mathcal{P}+1)}{4\sqrt{2}} + \mathcal{O}(\varepsilon^2)] & i\sigma_1\mathcal{O}(\varepsilon^3) & \varepsilon^2 b_{33}^0 + \mathcal{O}(\varepsilon^4) \end{pmatrix}, \tag{5.12}$$

where

$$\begin{aligned} b_{11}^0 &= |\widetilde{\sigma}|^2(\widetilde{\lambda} + |\widetilde{\sigma}|^2) + c_1\widetilde{\rho}^2\widetilde{\sigma}_2^2, \\ b_{13}^0 &= -c_1\sqrt{2}\widetilde{\lambda} + d_1\widetilde{\sigma}_1^2 + d_2\widetilde{\sigma}_2^2, \\ b_{22}^0 &= \tfrac{3(\mathcal{P}+1)}{4}\widetilde{\lambda} + 3\widetilde{\sigma}_1^2 + 2c_4\widetilde{\rho}^2, \\ b_{23}^0 &= \tfrac{\sqrt{2}}{15}(31\widetilde{\sigma}_1^2 + 45\widetilde{\sigma}_2^2 + 9(\mathcal{P}+1)\lambda + 45\sqrt{2}\,\widetilde{\kappa}) + d_4\widetilde{\rho}^2, \\ b_{33}^0 &= \tfrac{3(\mathcal{P}+1)}{4}\widetilde{\lambda} + 3\widetilde{\sigma}_1^2 \ = \ b_{22}^0 - 2c_4\widetilde{\rho}^2. \end{aligned}$$

There are four new coefficients d_j which are needed for the subsequent stability analysis. They are calculated in Appendix C:

$$d_0 = 1, \qquad d_1 = \tfrac{\sqrt{2}}{6}, \qquad d_2 = 1/\sqrt{2}, \qquad d_3 = -\tfrac{3\mathcal{P}+8}{10}, \qquad d_4 = \tfrac{11\sqrt{2}\,\mathcal{P}^2}{80}.$$

The scaling of Λ gives $\nu^* = 8$, $\nu_6 = 10$, and

$$\Lambda^*(\widetilde{\rho}, \widetilde{\kappa}, \widetilde{\sigma}, \widetilde{\lambda}) = b_{33}^0[b_{11}^0 b_{22}^0 - \widetilde{\sigma}_1^2\widetilde{\sigma}_2^2\widetilde{\rho}^2 d_0 d_3] + \tfrac{3(\mathcal{P}+1)}{4\sqrt{2}}\widetilde{\sigma}_2^2\widetilde{\rho}^2[b_{22}^0 b_{13}^0 - \widetilde{\sigma}_1^2 d_0 b_{23}^0].$$

The components of the sideband vector $\widetilde{\sigma}$ and the amplitude $\widetilde{\rho}$ appear only as squares, hence we let $r = \widetilde{\rho}^2$, $\widetilde{\sigma}_1^2 = s(1-\delta)$ and $\widetilde{\sigma}_2^2 = s\delta$ where $|\widetilde{\sigma}| = \sqrt{s}$ and $\delta \in [0,1]$. After dividing $\Lambda^* = 0$ by s we obtain the following equation

$$0 = p(\widetilde{\lambda}) \stackrel{\text{def}}{=} p_3\widetilde{\lambda}^3 + p_2(r,\delta,s)\widetilde{\lambda}^2 + p_1(r,\delta,s)\widetilde{\lambda} + p_0(r,\kappa,\delta,s),$$

where $\kappa = 48\sqrt{2}\widetilde{\kappa}/(r\mathcal{P}^2)$. The coefficients p_j have the explicit representation

$$\begin{aligned}
p_3 &= \tfrac{9}{16}(\mathcal{P}+1)^2,\\
p_2 &= \tfrac{9(\mathcal{P}+1)}{64}[4(\mathcal{P}+9-8\delta)s + \mathcal{P}^2 r],\\
p_1 &= \tfrac{9}{2}(1-\delta)(\mathcal{P}+3-2\delta)s^2\\
&\quad+\tfrac{1}{64}[24(\mathcal{P}+1)(2\mathcal{P}+1)\delta^2 - 2\mathcal{P}(33\mathcal{P}^2+18\mathcal{P}+2)\delta + 9\mathcal{P}^2(\mathcal{P}+5)]rs,\\
p_0 &= 9(1-\delta)^2 s^3 + \tfrac{1}{16}(1-\delta)[8(\mathcal{P}-2)\delta^2 - 2(13\mathcal{P}+1)\delta + 9\mathcal{P}^2]rs^2\\
&\quad+\tfrac{3\mathcal{P}^2(\mathcal{P}+1)}{128}[\delta(10\delta-7) - 2\kappa(\delta-\delta^2)]r^2 s\,.
\end{aligned}$$

The Hurwitz criterion [Hu895] states that $p(\widetilde{\lambda}) = 0$ has all roots in the left complex halfplane including the imaginary axis if and only if the coefficients satisfy the inequalities

$$p_j \geq 0 \quad \text{for } j = 0,\ldots,3 \qquad \text{and } p_1p_2 - p_0p_3 \geq 0.$$

It is easy to see that p_3 and p_2 are nonnegative for all $s, r \geq 0$ and $\delta \in [0,1]$.

The discussion of the remaining conditions is more difficult. In the case the coefficient p_0 changes sign a simple real eigenvalue crosses the imaginary axis which is called *monotonic skew–varicose* (MSV) instability. If p_0 remains positive but p_1 or $p_1p_2 - p_0p_3$ become negative, a pair of complex conjugate eigenvalues crosses the imaginary axis which is called *oscillatory skew–varicose* (OSV) instability. In the two lemmata below we give regions where we can guarantee that these instabilities occur. A schematic view of the regions in the $(\mathcal{P},\kappa)$–plane is given in Figure 5.1.

Lemma 5.6
(Monotonic skew–varicose instability) Let $\mathcal{P}_{\text{MSV}} = (3-\sqrt{8})(1+\sqrt{7+\sqrt{32}}) \approx 0.78197$, *then* $p_0(r,\kappa,s,\delta)$ *attains negative values for some* $(s,\delta) \in (0,\infty)\times[0,1]$ *if and only if* $(\mathcal{P},\kappa)$ *satisfies*

$$\mathcal{P} \in (0,\mathcal{P}_{\text{MSV}}) \quad \text{or} \quad \kappa > -7/2.$$

Proof: Note that p_0 has the form $s(a_2 s^2 + a_1 rs + a_0 r^2)$ which remains positive as long as $a_2 \geq 0$, $a_0 \geq 0$, and $a_1 + 2\sqrt{a_0 a_2} \geq 0$. From the definition of a_2 and a_0 it is easy to see that a_2 is always nonnegative and that a_0 is nonnegative for all $\delta \in [0,1]$ and $\mathcal{P} > 0$ if and only if $\kappa \leq -7/2$.

The condition $a_1 + 2\sqrt{a_0 a_2} \geq 0$ is analyzed only for $\kappa \leq -7/2$. We rewrite the problem in the form

$$\alpha(P,\delta) \stackrel{\text{def}}{=} \frac{8(\mathcal{P}-2)\delta^2 - (26\mathcal{P}+2)\delta + 9\mathcal{P}^2}{6\mathcal{P}\sqrt{6(\mathcal{P}+1)}} \geq \beta(\kappa,\delta) \stackrel{\text{def}}{=} -\sqrt{\delta[(10+2\kappa)\delta - (7+2\kappa)]}.$$

Here β is independent of $\mathcal{P}$ and α is a quadratic polynomial in δ which independent of κ. For $\mathcal{P} \leq 2$ the function α is concave in δ while β is convex. Hence the condition is satisfied if and only if $\alpha(P,0) \geq \beta(\kappa,0) = 0$ and $\alpha(P,1) \geq \beta(\kappa,1) = -\sqrt{3}$. The inequality at $\delta = 0$ is always valid while the condition at $\delta = 1$ holds exactly for $\mathcal{P} \geq \mathcal{P}_{\text{MSV}}$.

The case $\mathcal{P} > 2$ is easily estimated, since α is now convex and lies above its tangent $T(P,\delta) = \alpha(P,1) + \partial_\delta \alpha(P,1)(\delta - 1)$ in the point $\delta = 1$. Moreover, $\beta(\kappa,\delta) \leq \beta(-7/2,\delta) = -\sqrt{3}\,\delta$, and an easy check yields $T(\mathcal{P},\delta) \geq -\sqrt{3}\,\delta$ for $\mathcal{P} > 2$ and $\delta \in [0,1]$. $\square$

From the proof of this lemma we see that the instability for $\kappa > -7/2$ occurs for $\delta \in (0,\delta_0)$ and $s \in (0, s_0 r)$, which is an attached sideband instability with $|\sigma_2| \ll |\sigma_1| = \mathcal{O}(\rho)$ (like the Eckhaus instability). The case $\kappa < -7/2$ and $\mathcal{P} < \mathcal{P}_{\text{MSV}}$ leads to negative values for $p_0(r,\kappa,\delta,s)$ when $\delta \in (\delta_1, 1)$ and $s \in (s_1 r, s_2 r)$ with $0 < s_1 < s_2$ fixed. Thus, this is a detached sideband instability with $|\sigma_1| \ll |\sigma_2|$ and $|\sigma_1| \in (\sqrt{s_1}\rho, \sqrt{s_2}\rho)$ (like the zigzag instabilities).

Lemma 5.7
(Oscillatory skew–varicose instability)
(a) There is a value $\mathcal{P}_3 \approx 0.1654$ such that $p_1(r,s,\delta)$ attains negative values for some $(s,\delta) \in (0,\infty) \times [0,1]$ if and only if $\mathcal{P} \in [0,\mathcal{P}_3)$.
(b) There is a value $\mathcal{P}_{\text{OSV}} \approx 0.050712$ and a function $\kappa^* = \kappa^*(\mathcal{P})$ such that the function $p_1 p_2 - p_0 p_3$ attains negative values for some $(s,\delta) \in (0,\infty) \times [0,1]$ if and only if $(\mathcal{P},\kappa)$ satisfies $\kappa < \kappa^*(\mathcal{P})$. Letting

$$\kappa_{\text{OSV}}(\mathcal{P}) = -\frac{1}{2(\mathcal{P}+1)^2}\Big((\mathcal{P}+3)(3\mathcal{P}^2 + 2\mathcal{P} + 2) + 3\mathcal{P}^2\sqrt{(\mathcal{P}+1)(\mathcal{P}+5)}\Big),$$

we have $\kappa^*(\mathcal{P}) = \kappa_{\text{OSV}}(\mathcal{P})$ for $\mathcal{P} \geq \mathcal{P}_{\text{OSV}}$. In the region $0 < \mathcal{P} < \mathcal{P}_{\text{OSV}}$ we have $\kappa^*(\mathcal{P}) > \kappa_{\text{OSV}}(\mathcal{P})$ and $\kappa^*(\mathcal{P}) = \gamma/\mathcal{P}^2 + \mathcal{O}(1/\mathcal{P})$ for $\mathcal{P} \to 0$ with $\gamma \approx 0.0006665$.

Proof: (a) For $p_1 = a_2 s^2 + a_1 rs \geq 0$ we need $a_2, a_1 \geq 0$. Obviously, $a_2 \geq 0$ and a_1 is a quadratic polynomial in δ which attains its minimum value $\widetilde{a}(\mathcal{P})$ for some $\delta \in (0,1)$. From $\widetilde{a}(\mathcal{P}) = (48\mathcal{P}^5 + 191\mathcal{P}^4 + 252\mathcal{P}^3 + 62\mathcal{P}^2 - 12\mathcal{P} - 1)/(8(\mathcal{P}+1)(2\mathcal{P}+1))$ we find the desired result.

(b) As in the previous lemma we may write $p_1p_2 - p_0p_3 = \frac{4096}{27}s(b_2s^2 + b_1rs + b_0r^2)$ which is nonnegative as long as b_0, b_2, and $b_1 + 2\sqrt{b_0b_2}$ are nonnegative. We have

$$
\begin{aligned}
b_2 &= 384(1-\delta)(\mathcal{P}+5-4\delta)^2,\\
b_1 &= \tfrac{4(\mathcal{P}+1)}{3}[-32(\mathcal{P}+1)(11\mathcal{P}+8)\delta^3 + 16(3\mathcal{P}^3+65\mathcal{P}^2+55\mathcal{P}+20)\delta^2\\
&\qquad -2(87\mathcal{P}^3+605\mathcal{P}^2+109\mathcal{P}+23)\delta + 9\mathcal{P}^2(\mathcal{P}+5)(\mathcal{P}+13)],\\
b_0 &= \mathcal{P}^2(\mathcal{P}+1)[4(\mathcal{P}+1)^2(\delta-\delta^2)\kappa - 4(\mathcal{P}+1)(\mathcal{P}+3)\delta^2\\
&\qquad -4(2\mathcal{P}^2-4\mathcal{P}-3)\delta + 3\mathcal{P}^2(\mathcal{P}+5)],
\end{aligned}
$$

which immediately gives $b_2 \geq 0$. Moreover, we see that $b_0(\mathcal{P},\kappa,\delta)$ increases monotonically with κ and satisfies

$$b_0(\mathcal{P},\kappa,0) = 3\mathcal{P}^4(\mathcal{P}+1)(\mathcal{P}+5), \quad b_0(\mathcal{P},\kappa,1) = 3\mathcal{P}^4(\mathcal{P}+1)^2$$

For sufficiently large κ the minimum of b_0 over $\delta \in [0,1]$ will be positive and there is a unique $\kappa = \kappa_{\mathrm{OSV}}(\mathcal{P})$ such that the minimum is negative if and only if $\kappa < \kappa_{\mathrm{OSV}}(\mathcal{P})$. The value is characterized by a double zero of the quadratic polynomial $b_0(\mathcal{P},\kappa,\cdot)$. Using the values of b_0 at $\delta \in \{0,1\}$, this double zero has to occur at $\widetilde{\delta} = \sqrt{\mathcal{P}+5}/(\sqrt{\mathcal{P}+1}+\sqrt{\mathcal{P}+5})$. Equating $b_0(\mathcal{P},\kappa,\widetilde{\delta})$ to zero, we find exactly the given expression $\kappa = \kappa_{\mathrm{OSV}}(\mathcal{P})$.

It remains to consider the third condition $b_3 = b_1 + 2\sqrt{b_0b_2} \geq 0$. As a first estimate consider b_1 divided by $\mathcal{P}+1$ as a polynomial in $\mathcal{P}$ and minimize each coefficient over $\delta \in [0,1]$. This gives $b_1(\mathcal{P},\delta) \geq \frac{4}{3}(\mathcal{P}+1)(9\mathcal{P}^4 + 36\mathcal{P}^3 + 63\mathcal{P}^2 - 15\mathcal{P} - 2)$ which implies that b_1, and hence b_3, is positive for all $(\mathcal{P},\delta) \in [0.3,\infty) \times [0,1]$. However, at $\mathcal{P} = 0$ we obtain $b_3(0,\kappa,\delta) = -8\delta(23 - 128\delta + 160\delta^2)/3$, which attains negative values for some δ and every κ. Hence, κ^* has to grow to infinity for $\mathcal{P} \to 0$. We substitute $\kappa = \mu/\mathcal{P}^2$ into b_3 and obtain

$$\widetilde{b}_3(\mathcal{P},\delta,\mu) = -8\delta(23 - 128\delta + 160\delta^2)/3 + 32\sqrt{6\mu\delta}(1-\delta)(5-4\delta) + \mathcal{O}(\mathcal{P}) \text{ for } \mathcal{P} \to 0.$$

Thus, $\widetilde{b}_3$ has a positive minimum if and only if $\mu \geq \widetilde{\mu}(\mathcal{P}) = \gamma + \mathcal{O}(\mathcal{P})$. The constant γ can be found numerically by looking for a double zero.

The point where the curves $\kappa_{\mathrm{OSV}}(\mathcal{P})$ and $\kappa^*(\mathcal{P})$ separate can again be found numerically by searching for a double zero in δ of the function $b_3(\mathcal{P},\kappa_{\mathrm{OSV}}(\mathcal{P}),\delta)$ which leads to the value $\mathcal{P}_{\mathrm{OSV}}$. □

We note that $\kappa_{\mathrm{OSV}}(\mathcal{P}) \leq -7/2$ if and only if $\mathcal{P} \geq \mathcal{P}_2 \approx 0.54398$.

We may translate these stability boundaries back into the $(\mathcal{R},k)$–plane by using $\mathcal{R} = \mathcal{R}_0 + 2\sqrt{3}(k-k_0)^2 + \frac{\sqrt{3}\mathcal{P}^2}{16}\rho^2 + \dots$ and (5.11), viz. $\rho = \varepsilon\sqrt{r}$ and $k = k_0 + \varepsilon^2\frac{\mathcal{P}^2}{48\sqrt{2}}\kappa r$. Without loss of generality we may assume $r = 1$ which means that $\varepsilon = \rho$. Thus, we arrived at the main instability result.

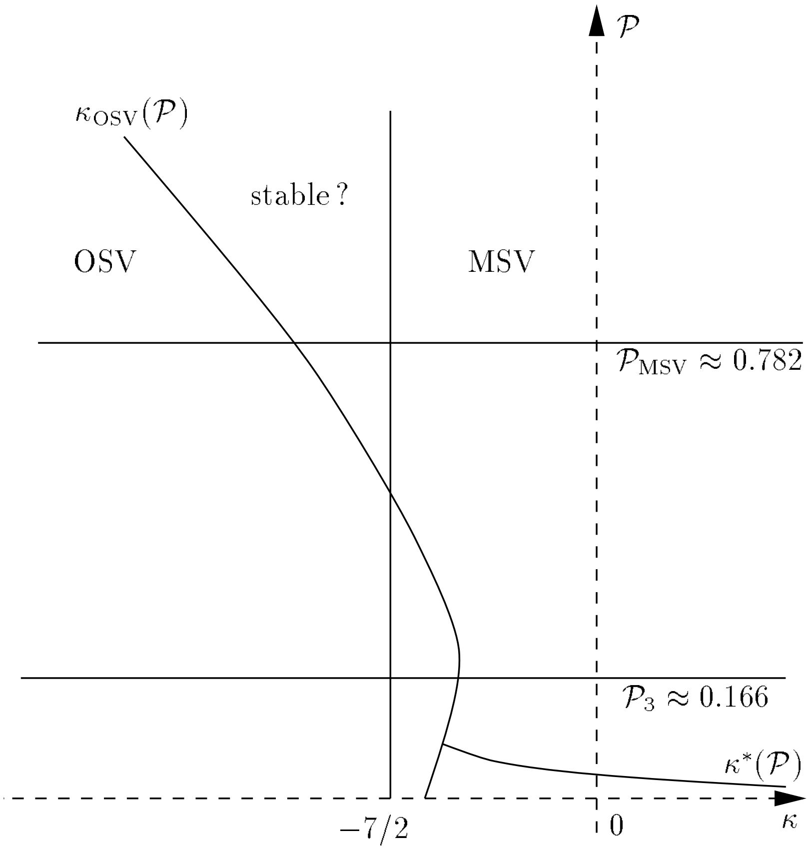

Figure 5.1. The skew varicose stability regions as functions of the Prandtl number $\mathcal{P}$, where $\kappa = 48\sqrt{2}(k-k_0)/(\mathcal{P}^2\rho^2)$.

Theorem 5.8
(Skew–varicose instability) *Assume that*

$$(\mathcal{P},\kappa) \in \Gamma \stackrel{\mathrm{def}}{=} \{\, (\mathcal{P},\kappa) \in (0,\infty)\times\mathbb{R} \,:\, \mathcal{P} < \mathcal{P}_{\mathrm{MSV}},\ \text{or } \kappa > -7/2,\ \text{or } \kappa < \kappa_{\mathrm{OSV}}(\mathcal{P}) \,\}.$$

Then, there exists a number $\rho_0 = \rho_0(\mathcal{P},\kappa) > 0$ such that for all $\rho \in (0,\rho_0)$ the roll solution $z_{\rho,k(\rho)}$, where $k(\rho) = k_0 + \frac{\mathcal{P}^2}{48\sqrt{2}}\kappa\rho^2$, is unstable in the sense of Definition 3.1. (Here, $\rho_0(\mathcal{P},\kappa)$ can be chosen such that it remains bounded away from zero on any compact subsets of Γ.)

The lines $(\rho, k(\rho))$ given in the theorem correspond to curves in the $(\mathcal{R},k)$–plane which have the form $k = k_0 + \kappa(\mathcal{R}-\mathcal{R}_0)/(3\sqrt{6}) + \mathcal{O}(|\mathcal{R}-\mathcal{R}_0|^2)$. Hence, for a given $\mathcal{P} > \mathcal{P}_{\mathrm{MSV}}$ we may vary κ suitably to obtain a family of curves which fill a region

$$A_{\mathrm{SV}}(\mathcal{P}) = \{\, (\widetilde{\mathcal{R}}(k,\rho^2),k) \,:\, \kappa = \tfrac{48\sqrt{2}(k-k_0)}{\mathcal{P}^2\rho^2} \in \mathbb{R}\setminus[\kappa_{\mathrm{OSV}}(\mathcal{P}),-\tfrac{7}{2}] \text{ and } \rho \in (0,\rho_0(\mathcal{P},\kappa)) \,\},$$

which is the set of parameters $(\mathcal{R}, k)$ for which we have proved instability. It consists of two drop shaped regions, one for the MSV and one for the OSV instability. The drops have their tip in the point $(\mathcal{R}_0, k_0)$ with tangents $\mathcal{R} = \mathcal{R}_0$ and $\mathcal{R} = \mathcal{R}_0 + \frac{3\sqrt{6}}{\kappa}(k - k_0)$, where $\kappa = -7/2$ and $\kappa = \kappa_{\mathrm{OSV}}(\mathcal{P})$ for MSV and OSV, respectively. See also Figure 6.1.

This result is somewhat weaker than in the case of the Eckhaus and the zigzag instabilities. There we obtained exact curves $\mathcal{R} = \mathcal{R}_{\mathrm{Eckhaus}}(k)$ and $\mathcal{R} = \mathcal{R}^{(j)}_{\mathrm{zizag}}(k)$ which bound the region of unstable rolls. Here the boundary of the instablity region $A_{\mathrm{SV}}(\mathcal{P})$ has no algebraic characterization, it depends on $\rho_0(\mathcal{P}, \kappa)$ which can only be estimated.

6 Discussion

We have developed a mathematically rigorous and computationally efficient method for the study of sideband instabilities of small periodic patterns bifurcating from a homogeneous state. The theory can detect all instabilities which arise through perturbations having small sideband vectors; however, other instabilities need not be seen. The instability theory is based on a Liapunov–Schmidt reduction (LSR) for the linear spectral problem which yields a complete analysis of the unfolding of the critical eigenvalues as functions of the sideband vector and other bifurcation parameters.

We believe that most of the formal methods for the study of sideband instabilities using multiple scalings or amplitude equations are in fact formally equivalent to our method, i.e., the method presented here leads to computations which are completely analogous to those arising in the other formal theories. The major difference here is our mathematically rigorous background which leads to several principle advantages even without discussing the necessity of proofs:

(1) For more involved problems it will be almost impossible to get a secure result without having a clean algorithm. Our method provides a simple standardized approach for the nonlinear bifurcation problem as well as for the sideband instability theory. The LSR can be implemented as numerical code or as a computer algebra tool. Using these tools the analysis of sideband instabilities can be made very efficient.

(2) Moreover, the knowledge that expansions obtained formally with a few terms are really an approximation of a nice smooth function allows us to apply perturbations arguments which show that effects seen in the lower order terms really persist for the full problem.

(3) Applying our method we are forced to set up a proper functional analytic framework. As a reward we obtain rigorous statements on the spectral properties of the linearization around the basic periodic pattern.

The usefulness of the method is established by the application to the three–dimensional Bénard problem. We were led to a LSR with a five–dimensional bifurcation equation, where the mean flows, the mean pressure and the two convective modes are taken into account. We note that the whole theory can be restricted to the two–dimensional case,

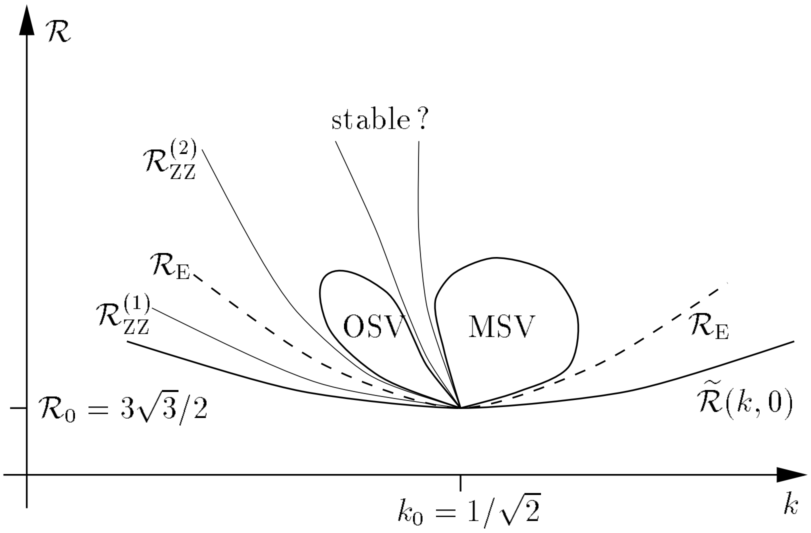

Figure 6.1. Instabilities for the Bénard convetion rolls $\mathcal{P} > 17$. Eckhaus instability $\mathcal{R}_{\mathrm{E}}$, zigzag instability $\mathcal{R}_{\mathrm{ZZ}}^{(j)}$, skew–varicose instabilities: monotonic $\mathcal{R}_{\mathrm{MSV}}$ and oscillatory $\mathcal{R}_{\mathrm{OSV}}$.

then leading to a four–dimensional reduced stability problem (just delete the second row and column in $\widetilde{b}$ and set $\sigma_2 = 0$). There the only instability is the so–called Eckhaus instability which we have studied in Subsection 5.2. A more complete treatment is given in [KvW96] where also stability is shown.

Further instabilites arise through the presence of three–dimensional perturbations of the form $v(t,x) = e^{\lambda t + i(\sigma_1 x_1 + \sigma_2 x_2)} W(kx_1, x_3)$ where $\sigma_2 \neq 0$. We have two types of zigzag instabilities ($\sigma_1 = 0$), the monotonic and the oscillatory skew–varicose instability. For large Prandtl numbers ($\mathcal{P} \geq 17$) our instability regions are depicted in Figure 6.1.

We note that all bifurcating rolls are unstable for $\mathcal{P} < \mathcal{P}_{\mathrm{MSV}} \approx 0.782$ while for larger values for $\mathcal{P}$ there is a sector in the $(\mathcal{R}, k)$–plane where the rolls might be stable. We do not claim stability since we only derived sufficient conditions for *instability*.

Another aspect of our analysis is that the Galileian invariance obtains a central rôle, compare Lemma 5.3. It guarantees that all roll solutions have, for $\sigma = 0$, a three–fold eigenvalue zero. The understanding of this degeneracy is necessary for the proper unfolding of the critical eigenvalues as functions of σ. The complexity of the instability problem for the Bénard roll is mainly due to this fact.

Our results compare nicely with the previous work on the formal level as was derived in [ZS83] and [BuB84, Be94]. In fact, our work reconciles the discrepancies between the first and the two latter works. In both cases, only sufficient conditions for instability were derived, and our analysis implies that the union of both instability criteria is correct.

The instability for $\mathcal{P} < \mathcal{P}_{\mathrm{MSV}}$ was not detected in [BuB84] and [Be94] since there the analysis of p_0 was restricted to the case where $|\sigma| \ll \rho$. Thus, only attached sideband instabilities where detected whereas the instability for $\mathcal{P} < \mathcal{P}_{\rm MSV}$ is due to a detached sideband instability with $|\sigma_1| \ll |\sigma_2|$ and $|\sigma_1| \in (\sqrt{s_0}\rho, \sqrt{s_1}\rho)$.

In [ZS83] the analysis is based on the scaling

$$\rho = \varepsilon^2\widetilde{\rho},\ \ k - k_0 = \varepsilon^2\widetilde{\kappa},\ \ \sigma_1 = \varepsilon^2\widetilde{\sigma}_1,\ \ \sigma_2 = \varepsilon\widetilde{\sigma}_2, \lambda = \varepsilon^4\widetilde{\lambda} \tag{6.1}$$

which is the classical scaling for the Newell–Whitehead–(Segel) equation (cf. [ZS83, NPL93, Mi96]), which in turn was developed to study slow modulations of roll patterns. The resulting scaled eigenvalue problem $\Lambda^*_{ZS}(\widetilde{\rho}, \widetilde{\kappa}, \widetilde{\sigma}, \widetilde{\lambda}) = 0$ is then a quadratic polynomial in $\widetilde{\lambda}$, where unstable eigenvalues are always real. Thus, no oscillatory skew–varicose instability was detected. The reason is that the eigenvalues λ associated with the oscillatory instability have an imaginary part of order $\mathcal{O}(\rho|\sigma_2|)$ which is incompatible with the scaling (6.1), compare equation (3.8) and (3.9) in [BuB84].

A The nonlinear problem

In order to present the calculations of the coefficients $c_1, \ldots, c_5$ in (4.11) most efficiently, we just give the result which is obtained from applying the LSR inductively on the terms which are needed. We define the order parameter $\varepsilon = \sqrt{\mathcal{R} - \mathcal{R}_0} + |k - k_0| + |\eta|$ and note that it is sufficient to calculate $\mathcal{Z}(\mathcal{R}, k, \sum \eta_j\xi_j)$ up to terms of order $\mathcal{O}(\varepsilon^m)$ in order to find all coefficients of the bifurcation equation $\widetilde{n}(\mathcal{R}, k, \eta) = 0$ up to terms of order $\mathcal{O}(\varepsilon^{m+1})$. This is a general fact which holds since the term $A_0\mathcal{Z}$ does not enter into $\widetilde{n}$ since $PA_0\mathcal{Z} = 0$.

Define the vectors

$$Z_{14} = \begin{pmatrix} \frac{38-2\mathcal{P}}{75}\sin y\cos x_3 \\ 0 \\ \frac{7+47\mathcal{P}}{75\sqrt{2}}\cos y\sin x_3 \\ \frac{(17+7\mathcal{P})\sqrt{3}}{75\sqrt{2}}\cos y\sin x_3 \\ \frac{24\mathcal{P}-6}{75\sqrt{2}}\cos y\cos x_3 \end{pmatrix}, \quad Z_{44} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{-\mathcal{P}\sqrt{3}}{8}\sin(2x_3) \\ \frac{8+9\mathcal{P}}{32}\cos(2x_3) - \frac{1}{2}\cos(2y) \end{pmatrix},$$

$$Z_{4k} = \begin{pmatrix} \frac{2}{15}\cos y\cos x_3 \\ 0 \\ \frac{16\sqrt{2}}{15}\sin y\sin x_3 \\ \frac{2\sqrt{6}}{5}\sin y\sin x_3 \\ \frac{4\sqrt{2}}{5}\sin y\cos x_3 \end{pmatrix}, \quad Z_{4\mathcal{R}} = \begin{pmatrix} \frac{2\sqrt{6}}{75}\cos y\cos x_3 \\ 0 \\ \frac{-28\sqrt{3}}{75}\sin y\sin x_3 \\ \frac{-18}{25}\sin y\sin x_3 \\ \frac{-26\sqrt{3}}{75}\sin y\cos x_3 \end{pmatrix}, \quad Z_{11} = 0,$$

then all Z_{jl} lie in $Z_1 = (I-Q)Z$. (For the calculation of the coefficients c_j the vector $Z_{4\mathcal{R}}$ is not needed but we use it in Appendix C.) By the symmetry arguments in Section 4 it is enough to consider the case $\eta = (\eta_1, 0, 0, \eta_4, 0)$, and we let

$z^* = \eta_1\xi_1 + \eta_4\xi_4 + \mathcal{Z}^*$ with $\mathcal{Z}^* = \eta_1^2 Z_{11} + \eta_1\eta_4 Z_{14} + \eta_4^2 Z_{44} + \eta_4(k-k_0)Z_{4k} \in Z_1 = (I-Q)Z$

and define $\mathcal{N}^* = \mathcal{N}(\mathcal{R}, k, z^*)$. We obtain

$$\begin{aligned} g_1\eta_1 &= \langle \mathcal{N}^*, \xi_1\rangle + \mathcal{O}(\varepsilon^4) = \tfrac{-3(\mathcal{P}+1)}{20}\eta_1\eta_4^2 + \mathcal{O}(\varepsilon^4), \\ g_4\eta_4 &= \langle \mathcal{N}^*, \xi_4\rangle + \mathcal{O}(\varepsilon^4) \\ &= \Big(-\tfrac{\sqrt{3}}{2}(\mathcal{R}-\mathcal{R}_0) + 3(k-k_0)^2 - \tfrac{7\mathcal{P}^2+34\mathcal{P}-23}{200}\eta_1^2 + \tfrac{3\mathcal{P}^2}{32}\eta_4^2\Big)\eta_4 + \mathcal{O}(\varepsilon^4), \\ g_5\eta_1\eta_4 &= \langle \mathcal{N}^*, \xi_5\rangle = -\Big(\tfrac{3\sqrt{2}(\mathcal{P}+1)}{8} + \tfrac{27(\mathcal{P}+1)}{20}(k-k_0)\Big)\eta_1\eta_4 + \mathcal{O}(\varepsilon^4), \end{aligned} \tag{A.1}$$

where all g_j are evaluated at $(\mathcal{R}, k, \eta_1^2, \eta_4^2)$.

To check that the given approximation $\mathcal{Z}^*$ of $\mathcal{Z}(\mathcal{R}, k, z_0)$ is in fact sufficient we have to calculate the error $\mathcal{M} = \mathcal{N}^* - \langle \mathcal{N}^*, \xi_1\rangle\xi_1 - \frac{4}{15}(\langle \mathcal{N}^*, \xi_4\rangle\xi_4 + \langle \mathcal{N}^*, \xi_5\rangle\xi_5)$. By explicit calculation we find that $\mathcal{M} = \mathcal{O}(\varepsilon^3)$ which is enough to guarantee $\mathcal{Z}(\mathcal{R}, k, \eta_1\xi_1 + \eta_4\xi_4) = \mathcal{Z}^* + \mathcal{O}(\varepsilon^3)$.

Using the equivariance $\mathcal{Z}(\mathcal{R}, k, T_\tau z_0) = T_\tau \mathcal{Z}(\mathcal{R}, k, z_0)$ the expansion with $\eta_5 \neq 0$ is

$$\begin{aligned} \mathcal{Z}(\mathcal{R}, k, \textstyle\sum_{l=1}^5 \eta_l\xi_l) &= \eta_1\Big(\eta_4 Z_{14} + \eta_5 Z_{15}\Big) + \eta_4^2 Z_{44} + \eta_4\eta_5 Z_{45} + \eta_5^2 Z_{55} \\ &\quad + (k-k_0)\Big(\eta_4 Z_{4k} + \eta_5 Z_{5k}\Big) + \mathcal{O}(\varepsilon^3), \end{aligned}$$

where $Z_{15} = T_{\pi/2}Z_{14}$, $Z_{55} = T_{3\pi/2}Z_{44}$, $Z_{5k} = T_{\pi/2}Z_{4k}$, and

$$Z_{45} = T_{5\pi/4}Z_{44} - T_{7\pi/4}Z_{44} = (0,0,0,0,-\sin(2y))^T.$$

B The linear problem

Here we establish the form (5.9) and (5.10) of $\widetilde{b}(0, k, \sigma, \lambda)$. To this end we use the abstract Lemma 2.4. First consider the functions in the span of ξ_1, ξ_2, and ξ_3. Since $\mathcal{B}(0, k, \sigma, \lambda)$ leaves this three–dimensional subspace invariant we find that the LSR is just the restriction to this subspace. Thus, the form of $\widetilde{b}^{(123)}$ is clear.

For the reduction in the case of ξ_4 and ξ_5 we can use the fact that $\mathcal{B}(0, k, \sigma, \lambda)$ leaves the subspace $Z_{(1,1)} = \{\, e^{iy}\varphi_1(\beta) \in Z \,:\, \beta \in \mathbb{C}^5 \,\}$ invariant, where

$$\mathcal{B}(0, k, \sigma, \lambda)e^{iy}\varphi_1(\beta) = e^{iy}\varphi_1(F_1(\widetilde{\mathcal{R}}(k,0), \sigma, \lambda)\beta)$$

and φ_1 and F_1 are defined in (4.1) and (4.2), respectively.

We first calculate the LSR of the matrix F_1 by solving $F_1(\widetilde{\mathcal{R}}(k,0), \sigma, \lambda)[\chi_4 + V] = \alpha\chi_4$ under the side condition $\chi_4^T\overline{V} = 0$, where $\chi_4 = (\sqrt{2}, 0, -i, -i\sqrt{3}, 3i)^T$ is chosen such that $\zeta = \xi_4 + i\xi_5 = \varphi_1(\chi_4)$. From this linear problem for $(\alpha, V) \in \mathbb{C}^6$ we obtain

$$\begin{aligned} \alpha = {} & \tfrac{\mathcal{P}+1}{5}\lambda + \tfrac{4}{5}\sigma_1^2 + \tfrac{4\sqrt{2}}{5}(k-k_0)\sigma_2^2 + \tfrac{4}{15}\sigma_1^2\sigma_2^2 + \tfrac{2}{5}\sigma_2^4 + \tfrac{16(\mathcal{P}+1)}{75}\sigma_2^2\lambda \\ & + \tfrac{4\sqrt{2}}{45}\left(5\sqrt{2}(k-k_0) + \tfrac{31}{9}\sigma_1^2 + 5\sigma_2^2 + (\mathcal{P}+1)\lambda\right)\sigma_1 + \mathcal{O}(\widetilde{\varepsilon}^4), \end{aligned}$$

and

$$V = \begin{pmatrix} \frac{2}{15}\sigma_1 + \frac{\sqrt{2}}{225}(15\sqrt{2}(k-k_0) - 224\sigma_1^2 - 180\sigma_2^2 + (6\mathcal{P}-114)\lambda) \\ 2\sigma_2 - \frac{28\sqrt{2}}{15}\sigma_1\sigma_2 \\ \frac{-16i\sqrt{2}}{15}\sigma_1 - \frac{i}{225}(240\sqrt{2}(k-k_0) + 346\sigma_1^2 + 270\sigma_2^2 + (141\mathcal{P}+21)\lambda) \\ \frac{-2i\sqrt{6}}{5}\sigma_1 - \frac{i\sqrt{3}}{225}(90\sqrt{2}(k-k_0) + 196\sigma_1^2 + 120\sigma_2^2 + (21\mathcal{P}+51)\lambda) \\ \frac{-4i\sqrt{2}}{5}\sigma_1 - \frac{2i}{25}(10\sqrt{2}(k-k_0) + 9\sigma_1^2 + 5\sigma_2^2 + (4\mathcal{P}-1)\lambda) \end{pmatrix} + \mathcal{O}(\widetilde{\varepsilon}^{9/4}).$$

where $\widetilde{\varepsilon}^2 = |k-k_0| + \sigma_1^2 + |\sigma_2|^{3/2} + |\lambda|$. The apparently strange scaling defined through $\widetilde{\varepsilon}$ stems from the fact that we have to use a special scaling in Subsection 5.3 where the zigzag instability is considered. There the term $\frac{2}{5}\sigma_2^4$ occuring in the function α is needed.

According to Lemma 2.4 we find the functions ν and $\widetilde{\delta}$ in (5.10) to be given by

$$\nu(k,\sigma,\lambda) = \frac{15}{8}[\alpha(k,\sigma,\lambda) + \overline{\alpha(k,-\sigma,\overline{\lambda})}], \quad \widetilde{\delta}(k,\sigma,\lambda) = \frac{15}{8\sigma_1}[\alpha(k,\sigma,\lambda) - \overline{\alpha(k,-\sigma,\overline{\lambda})}]. \quad \text{(B.1)}$$

The additional factor 15/4 appears because $\langle \xi_4, \xi_4 \rangle = \langle \xi_5, \xi_5 \rangle = 15/4$. Moreover, the function $\mathcal{W}_j(0,k,\sigma,\lambda)$ satisfying $(I-P)\mathcal{B}(0,k,\sigma,\lambda)[\xi_j + \mathcal{W}_j] = 0$ can be calculated explicitly:

$$\begin{aligned} \mathcal{W}_j(0,k,\sigma,\lambda) &= 0 \quad \text{for } j = 1,2,3, \\ \mathcal{W}_4(0,k,\sigma,\lambda) &= \tfrac{1}{2}\Big[e^{iy}\varphi(V(k,\sigma,\lambda)) + e^{-iy}\varphi(\overline{V(k,-\sigma,\overline{\lambda})})\Big], \\ \mathcal{W}_5(0,k,\sigma,\lambda) &= \tfrac{-i}{2}\Big[e^{iy}\varphi(V(k,\sigma,\lambda)) - e^{-iy}\varphi(\overline{V(k,-\sigma,\overline{\lambda})})\Big]. \end{aligned} \quad \text{(B.2)}$$

C The stability problem

It remains to calculate the coefficients $d_0, \ldots d_4$ which are important in studying the zigzag and the skew–varicose instability. To find the expansion of b^* as given in (5.12) we recall the definition in (5.7) and then we find

$$\begin{aligned} d_0 &= \gamma_{14}(0,k_0,0,0,0) - \gamma_{24}(0,k_0,0,0,0), \\ d_1 &= \partial_{(\sigma_1^2)}\gamma_{15}(0,k_0,0,0,0) + \gamma_{25}(0,k_0,0,0,0), \\ d_2 &= \partial_{(\sigma_2^2)}\gamma_{15}(0,k_0,0,0,0), \\ d_3 &= \gamma_{42}(0,k_0,0,0,0) - \gamma_{41}(0,k_0,0,0,0), \\ d_4 &= -\partial_{(\rho^2)}\gamma_{45}(0,k_0,0,0,0). \end{aligned}$$

As in Subsection 5.4 we let $\varepsilon = \rho + |\sigma| + \sqrt{|\lambda| + |k-k_0|}$ and find that we have to calculate the columns of $\widetilde{b}$ up to different orders in ε: namely up to ε^2 for the columns 1,2, and 4, and up to ε^3 for column 5. Before we treat these cases separately we recall that Theorem 2.2 allows us to reduce the computational effort for the solution of $(I-P)\mathcal{B}(\rho,k,\sigma,\lambda)[\xi_j + \mathcal{W}_j] = 0$ considerably. In fact, we will employ the results from

Appendix A and B in the calculations below to find the relevant terms in the expansions of $\mathcal{W}_j$ for either $(\sigma,\lambda)=(0,0)$ or $\rho=0$, respectively. Recall also the expansion (4.12):

$$z_{\rho,k} = \rho\xi_4 + \rho^2 Z_{44} + \rho^3 \Big(Z_{444} - \frac{c_4}{c_2} Z_{4\mathcal{R}}\Big) + \rho(k-k_0)Z_{4k} + \mathcal{O}(\varepsilon^4).$$

To find $\widetilde{b}(\rho,k,\sigma,\lambda)e_j$ up to order $\mathcal{O}(\varepsilon^m)$, we calculate $\mathcal{W}_j$ up to order $\mathcal{O}(\varepsilon^{m-1})$. Moreover, we have to be aware that $z_{\rho,k}$ enters the problem linearly from the derivative of the quadratic nonlinearity and it is multiplied with the order 1 term ξ_j. Hence, we use an expansion of $z_{\rho,k}$ up to order $\mathcal{O}(\varepsilon^m)$.

Columns 1 to 4 of $\widetilde{b}$

We only need to calculate the terms up to and including order ε^2. Thus, it is sufficient to expand only the term of order ε in $\mathcal{W}_j$, $j=1,\dots,4$, namely

$$\sum_{j=1}^{4} \omega_j \mathcal{W}_j = \omega_1(\rho Z_{14} + \mathcal{O}(\varepsilon^2)) + \omega_2 \mathcal{O}(\varepsilon^2) + \omega_3 \mathcal{O}(\varepsilon^2) + \omega_4(2\rho Z_{44} + i\sigma_1 W^0_{41} + i\sigma_2 W^0_{42} + \mathcal{O}(\varepsilon^2)),$$

where W^0_{4j} is the corresponding lowest order expansion coefficent of $\mathcal{W}_4(0,k,\sigma,\lambda)$ from Appendix B:

$$W^0_{41} = \frac{\sqrt{2}}{15}\begin{pmatrix} \sqrt{2}\sin y\cos x_3 \\ 0 \\ -16\cos y \sin x_3 \\ -6\sqrt{3}\cos y\sin x_3 \\ -12\cos y\cos x_3 \end{pmatrix}, \quad W^0_{42} = \begin{pmatrix} 0 \\ 2\sin y\cos x_3 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

For the roll solution we use the approximation $z_{\rho,k} = \rho\xi_4 + \rho^2 Z_{44} + \mathcal{O}(\varepsilon^3)$. Inserting all this into $\widetilde{b}_{jl} = \langle \mathcal{B}(\rho,k,\sigma,\lambda)[\xi_l + \mathcal{W}_l], \xi_j\rangle$, we find

$$\widetilde{b} = \begin{pmatrix} \lambda+|\sigma|^2 - \frac{3(\mathcal{P}+1)}{20}\rho^2 & 0 & i\sigma_1 & i\sigma_1\rho & * \\ 0 & \lambda+|\sigma|^2 & i\sigma_2 & 0 & * \\ -i\sigma_1 & -i\sigma_2 & 0 & 0 & 0 \\ \frac{3\mathcal{P}+8}{10} i\sigma_1\rho & 0 & 0 & \frac{3(\mathcal{P}+1)}{4}\lambda + 3\sigma_1^2 + \frac{3\mathcal{P}^2}{16}\rho^2 & * \\ -\frac{3(\mathcal{P}+1)}{4\sqrt{2}}\rho & 0 & 0 & 0 & * \end{pmatrix} + \mathcal{O}(\varepsilon^3),$$

which gives $d_0 = 1$ and $d_3 = -(3\mathcal{P}+8)/10$. Moreover, an explicit calculation yields

$$\mathcal{B}(\dots)[\xi_l + \mathcal{W}(\dots)_l] - \textstyle\sum_{j=1}^5 \frac{\widetilde{b}_{jl}(\dots)}{\langle \xi_j,\xi_j\rangle}\xi_j = \mathcal{O}(\varepsilon^2), \quad \text{for } l=1,\dots,4,$$

as desired.

Column 5 of $\widetilde{b}$

We now need a higher order approximation. To avoid the calculation of Z_{444} we replace (4.12) by

$$z_{\rho,k} = \rho\xi_4 + \rho^2 Z_{44} + \rho(k-k_0)Z_{4k} + \mathcal{O}(\rho^3+\varepsilon^4).$$

For $\mathcal{W}_5$ we have

$$\begin{aligned}\mathcal{W}_5 \;=\; & \rho Z_{45} + i\sigma_1 W_{51}^0 + i\sigma_2 W_{52}^0 + \rho^2(\tfrac{2c_4}{\sqrt{3}}Z_{5\mathcal{R}} + Z_{445}) + i\rho(\sigma_1 W_{51}^1 + \sigma_2 W_{52}^1) \\ & +\lambda W_{5\lambda}^0 + (k-k_0)W_{5k}^0 - \sigma_1^2 W_{511}^0 - \sigma_1\sigma_2 W_{512}^0 - \sigma_2^2 W_{522}^0 + \mathcal{O}(\varepsilon^3).\end{aligned}$$

Here the superscript l in W^l stands for the associated power of ρ, hence all coefficients W^0 can be easily obtained from V and (B.2). Further we have $W_{52}^1 = 0$, $Z_{5\mathcal{R}} = T_{\pi/2}Z_{4\mathcal{R}}$,

$$Z_{445} = \frac{\mathcal{P}^2}{600}\begin{pmatrix} 2\sqrt{2}\sin y\cos x_3 \\ 0 \\ -47\cos y\sin x_3 \\ -7\sqrt{3}\cos y\sin x_3 \\ -24\cos y\cos x_3 \end{pmatrix} + \frac{\mathcal{P}^2}{27328}\begin{pmatrix} 27\sqrt{2}\sin y\cos(3x_3) \\ 0 \\ -9\cos y\sin(3x_3) \\ -361\sqrt{3}\cos y\sin(3x_3) \\ 513\cos y\cos(3x_3) \end{pmatrix},$$

and

$$W_{51}^1 = \frac{1}{240}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 44\mathcal{P}\sqrt{6}\sin(2x_3) \\ 16\sqrt{2}\cos(2y) - (99\mathcal{P}+128)\sqrt{2}\cos(2x_3) \end{pmatrix}.$$

Hence, $\widetilde{b}_{j5} = \langle \mathcal{B}(\ldots)[\xi_5 + \mathcal{W}_5], \xi_j\rangle$ leads to

$$\begin{pmatrix} \sqrt{2}\Big(\frac{2}{3}\sigma_1^2 + \frac{1}{2}\sigma_2^2 + \frac{3}{20}(\mathcal{P}+1)\lambda\Big)\rho \\ \frac{\sqrt{2}}{2}\rho\sigma_1\sigma_2 \\ 0 \\ \frac{-i\sqrt{2}}{240}\Big(33\mathcal{P}^2\rho^2 + 720\sqrt{2}(k-k_0) + 496\sigma_1^2 + 720\sigma_2^2 + 144(\mathcal{P}+1)\lambda\Big)\sigma_1 \\ \frac{3(\mathcal{P}+1)}{4}\lambda + 3\sigma_1^2 \end{pmatrix} + \mathcal{O}(\rho^3+\varepsilon^4).$$

From our general considerations in Section 5 (cf. (5.8)) we know, that the last column of $\widetilde{b}$ cannot contain a term ρ^3. Thus, we have established the desired expansion and found the coefficients $d_1 = \sqrt{2}/6$, $d_2 = 1/\sqrt{2}$, and $d_4 = 11\sqrt{2}\mathcal{P}^2/80$. It may be checked that

$$\mathcal{B}(\ldots)[\xi_5 + \mathcal{W}(\ldots)_5] - \textstyle\sum_{j=1}^5 \frac{\widetilde{b}_{j5}(\ldots)}{\langle\xi_j,\xi_j\rangle}\xi_j = \mathcal{O}(\varepsilon^3),$$

when the given approximations for $z_{\rho,k}$ and $\mathcal{W}_5$ are inserted.

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References

[Be94] **A.J. Bernoff.** Finite amplitude convection between stress–free boundaries; Ginzburg–Landau equations and modulation theory. *European J. Applied Math.*, **5**, 267–282, 1994.

[BM95] **T.J. Bridges, A. Mielke.** A proof of the Benjamin–Feir instability. *Archive Rational Mech. Analysis.* **133** (1995) 145–198.

[BM96] **T.J. Bridges, A. Mielke.** Instability of spatially–periodic states for a family of semilinear PDE's on an infinite strip. *Math. Nachrichten*, **179** (1996) 5–25.

[BoB85] **E.W. Bolton, F.H. Busse.** Stabilites of convection rolls in a layer with stress–free boundaries. *J. Fluid Mech.*, **150** (1984) 487–498.

[Bu71] **F.H. Busse.** Stability regions of cellular fluid flow. In "*Instability of Continuous Systems*, H. Leipholz (edr), Proc. of the IUTAM Symposium in Bad Herrenalb 1969, Springer–Verlag 1971." pp. 41–47.

[BuB84] **F.H. Busse, E.W. Bolton.** Instabilites of convection rolls with stress–free boundaries near threshold. *J. Fluid Mech.*, **146** (1984) 115–125.

[CE90] **P. Collet, J.–P. Eckmann.** *Instabilites and Fronts in Extended Systems.* Princeton University Press 1990.

[CH88] **S.–N. Chow, J.K. Hale.** *Methods of Bifurcation Theory.* Springer–Verlag, 1985.

[Da91] **K. Damerow.** A sufficient condition for the validity of the principle of reduced stability. *Math. Nachrichten* **154** (1991) 243–252.

[DP94] **W. Decker, W. Pesch.** Order parameter and amplitude equations for the Rayleigh–Bénard convection. *J. Phys. II France* **4** (1994) 419–438.

[Eck65] **W. Eckhaus.** *Studies in Non–Linear Stability Theory.* Springer–Verlag. Springer Tracts in Nat. Phil. Vol. **6**, 1965.

[GS85] **M. Golubitsky, D.G. Schaeffer.** *Singularities and Groups in Bifurcation Theory Vol. I.* New York: Springer–Verlag 1985.

[GSS88] **M. Golubitsky, I. Stewart, D.G. Schaeffer.** *Singularities and Groups in Bifurcation Theory Vol. II.* New York: Springer–Verlag 1988.

[Hu895] **A. Hurwitz.** Über die Bedingungen unter welchen eine Gleichung nur Wurzeln mit negativen reellen Teilen besitzt. *Math. Annalen* **46** (1895) 273–284.

[KL83] **H. Kielhöfer, R. Lauterbach.** On the principle of reduced stability. *J. Functional Analysis* **53** (1983) 99–111.

[Ku96] **M. Kuwamura.** The stability of roll solutions of the 2–D Swift–Hohenberg equation and the phase diffusion equation. *SIAM J. Math. Anal.* 1996. Submitted.

[KvW96] **Y. Kagei, W. von Wahl.** Stability of convection roll solutions of Boussinesq equations in two space dimensions. Preprint Universität Bayreuth 1996.

[Mi95] **A. Mielke.** A new approach to sideband instabilities using the principle of reduced instability. In "*Nonlinear Dynamics and Pattern Formation in the Natural Environment.* A. Doelman, A. van Harten (eds), Pitman Research Notes in Mathematics Series **335**. Longman 1995." Pages 206–222.

[Mi96] **A. Mielke.** Instability and stability of rolls in the Swift–Hohenberg equation. Preprint Universität Hannover, 1996. Submitted to *Nonlinear Diff. Eqns. Appl.*

[MT84] **B. Malomed, M.I. Tribel'skii.** Bifurcations in distributed kinetic systems with aperiodic instability. *Physica D,* **14** (1984) 67–87.

[NPL93] **A.C. Newell, T. Passot, J. Lega.** Order parameter equations for patterns. *Annual Reviews Fluid Mech.,* **25** (1993) 399–453.

[Re89] **L. Recke** On linear stability of bifurcating equilibria. *Math. Nachrichten* **140** (1989) 59–68.

[RR92] **M. Renardy, R.C. Rogers.** *An Introduction to Partial Differential Equations.* Springer–Verlag 1992.

[SdP78] **J.T. Stuart, R.C. di Prima.** The Eckhaus and the Benjamin–Feir resonance. *Proc. Royal Soc. London*, **A 362**, 27–41, 1978.

[SvW92] **B.J. Schmitt, W. von Wahl.** Decomposition of solenoidal fields into poloidal, toroidal and the mean flow. Applications to the Boussinesq–equation. In *"The Navier–Stokes Equation II — Theory and Numerical Methods.* Proceedings Oberwolfach 1991. J.G. Heywood, K. Masuda, R. Rautmann, S.A. Solonnnikov (eds). Lecture Notes in Math. Vol. **1530**, Springer–Verlag 1992," pp. 291–305.

[Va84] **A. Vanderbauwhede.** Stability of bifurcating equilibria and the principle of reduced stability. In "*Bifurcation Theory and Applications*, Proceedings of the CIME meeting in Montecatini 1983, edr. L. Salvadori, Springer Lecture Notes Math. **1057** (1984)", pp. 209–223.

[ZS83] **A. Zippelius, E.D. Siggia.** Stability of finite–amplitude convection. *Phys. Fluids* **26** (1983) 2905–2915.