A Proof of the Benjamin-Feir Instability

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Abstract

The existence and linear stability problem for the Stokes periodic wavetrain on fluid of finite depth is formulated in terms of the spatial and temporal Hamiltonian structure of the water-wave problem. A proof, within the Hamiltonian framework, of instability of the Stokes periodic wavetrain is presented. A Hamiltonian center manifold analysis reduces the linear stability problem to an ordinary differential eigenvalue problem on $\mathbb{R}^4$. A projection of the reduced stability problem onto the tangent space of the 2-manifold of periodic Stokes waves is used to prove the existence of a dispersion relation $\Lambda(\lambda, \sigma, I_1, I_2) = 0$ where $\lambda \in \mathbb{C}$ is the stability exponent for the Stokes wave with amplitude I_1 and mass flux I_2 and σ is the "sideband" or spatial exponent. A rigorous analysis of the dispersion relation proves the result, first discovered in the 1960's, that the Stokes gravity wavetrain of sufficiently small amplitude is unstable for $F \in (0, F_o)$ where $F_o \approx 0.8$ and F is the Froude number.

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1. Introduction

In this paper the linear stability problem for the Stokes periodic travelling wave is formulated rigorously. The analysis is based on an inverse formulation, posing both the existence problem and the linear stability problem in terms of spatial dynamical system on the real line with a Hamiltonian structure. In this way, the serious difficulties associated with characterization of the function spaces for evolution of the time-dependent problem are avoided. A proof is then given of the result, discovered in the mid 1960's (BENJAMIN-FEIR [1967], BENJAMIN [1967], WHITHAM [1967]), that the Stokes' periodic gravity wave travelling in fluid of finite depth, and of sufficiently small amplitude, is unstable when the Froude number, F, is less than F_o where $F_o \approx 0.8$. Much more (effect of finite amplitude, three-dimensional effects, etc.) is known, from numerical calculations and the analysis of modulation equations, about the instability of the Stokes wavetrain and related waves (cf. BENNEY & ROSKES [1969], MCLEAN [1982], SAFFMAN & YUEN [1985]) and our formulation extends to some of these cases.

As far as we are aware a rigorous proof of the Benjamin-Feir instability has not previously been given nor even formulated. The above–mentioned papers do not provide a proof in the mathematical sense. The formal framework introduced by BENJAMIN–FEIR [1967] and WHITHAM [1967] as well as the models based on the Nonlinear Schrödinger equation are not yet understood well enough in order to derive exact results in the sense that the spectrum of the linearization around the Stokes wave can be controlled. Moreover, all the hypotheses under which the theory is valid are not immediately evident nor precisely stated. The advantage of the present formulation is manifold: *(i)* we give a precise definition of the Benjamin–Feir instability, *(ii)* we list the precise hypotheses under which the theorem can be proved, and *(iii)* we give a complete proof. Additionally there are other novelties: *(iv)* the whole proof is carried out in a Hamiltonian setting, and *(v)* the general method extends to other problems where even formal methods have difficulties.

The starting point for the analysis is the classic governing equations for water waves at the surface of an inviscid, irrotational and constant density fluid. The problem is restricted to two space dimensions with coordinates (x, y). The velocity field is given by the gradient of a velocity potential $\phi(x, y, t)$, the free surface is specified by $y = \eta(x, t)$ and the horizontal bottom is at $y = -h$ with $h > 0$. The governing equations are recorded in Appendix A. Abstractly, for the present purposes, the most important point about the governing equations is that they can be formulated in the following way

$$\mathbf{M}Z_t + \mathbf{K}(u_s)Z_x = \nabla S(Z) \quad \text{with} \quad Z = \begin{pmatrix} \Phi \\ \eta \\ \phi \\ u \end{pmatrix} \tag{1.1}$$

where $\Phi = \phi|_{y=\eta}$, $u = \phi_x$ and $u_s = u|_{y=\eta}$ (cf. Appendix A and BRIDGES [1994a,b,c]). The operators $\mathbf{M}$ and $\mathbf{K}(u_s)$, defined in (A.5), are skew-symmetric. The term $\nabla S(Z)$ is the gradient, with respect to the inner product (A.7) of a functional equivalent to the flow-force for the water-wave problem. In Appendix A it is verified that (1.1) along with the side conditions $\phi_y\big|_{y=-h} = 0$ and $\phi\big|_{y=\eta} = \Phi$ recovers the specified equations governing the water-wave problem.

Let $E(Z)$ be the total energy for the water-wave problem. Then, with respect to a suitable inner product including integration over x, it is easily shown that $\nabla E(Z) = \nabla S(Z) - \mathbf{K}(u_s)Z_x$ and so (1.1) can be written

$$\mathbf{M}Z_t = \nabla E(Z)\,. \tag{1.2}$$

The operator $\mathbf{M}$ is singular (see (A.5)). But, restriction of (1.2) to the complement of the kernel of $\mathbf{M}$ (which is equivalent to imposing Laplace's equation in the Z-coordinates as a constraint) recovers a well-defined evolution equation which is in fact the ZAKHAROV [1968] Hamiltonian formulation for the time-dependent problem. However to analyze the stability of periodic waves, the formulation (1.2) does not seem to be a profitable way to proceed. First, the time-evolution properties of the water-wave problem are not well understood. Although the Stokes wave is spatially periodic and so, in principle, can be studied using Banach spaces of periodic functions, the Stokes wave is unstable to a class of waves that are, in general, spatially quasi-periodic. Moreover, the linearization of (1.2), about a Stokes periodic wave, or even a uniform state, leads to a spectrum, for each fixed spatial wave number or Floquet exponent, with a countable set of eigenvalues along the imaginary axis with the possibility of attendant small divisors. The difficulties associated with the time-evolution problem are avoided by treating the spatial "evolution" part of (1.1) as an organizing center. By analyzing the spatial part as a dynamical system, the function space in the x-direction need not be specified in advance. Our methods cannot capture the complete spectrum of the time-dependent problem nor prove well-posedness of the initial-value problem. However, rigorous results on the existence of a subset of the complete spectrum are obtained and when this subset has an element with positive real part we can conclude, rigorously, a linear instability result.

First transform (1.1) to a moving frame with frame speed $c \in \mathbb{R}$

$$t' = t \quad \text{and} \quad x' = x - ct\,.$$

Then (1.1) becomes (with no confusion, dropping the primes)

$$\mathbf{M}Z_t + (\mathbf{K}(u_s) - c\mathbf{M})Z_x = \nabla S(Z) \tag{1.3}$$

and therefore, formally, uniformly travelling states are orbits of the spatial dynamical system $(\mathbf{K}(u_s) - c\mathbf{M})Z_x = \nabla S(Z)$. In fact the spatial evolution equation, relative to the moving frame, has the equivalent representation

$$\mathbf{K}(u_s)Z_x = \nabla S(Z) + c\nabla Q(Z) \tag{1.4}$$

where $Q(Z)$ is the mass flux functional: $Q(Z) = \int_{-h}^{\eta} u\,dy$. The fact that $\mathbf{M}Z_x$ and $\nabla Q(Z)$ are related is due to the relation between the mass flux and the impulse for the water-wave problem. The idea of treating uniformly travelling states of the water-wave problem as orbits of a spatial Hamiltonian system has been a subject of recent and active interest (Benjamin [1984], Mielke [1991, Chapter 9], Baesens & MacKay [1992], Bridges [1992a,b,1994a]).

The analysis of the existence problem for the Stokes wave begins with equation (1.4). Note that the special form of (1.4) leads immediately to the characterization of uniform flows (states that are independent of x and t) as critical points of S (the flow force) on level sets of Q (the mass flux). The details of this characterization are in Appendix B which lead to Figure 1.

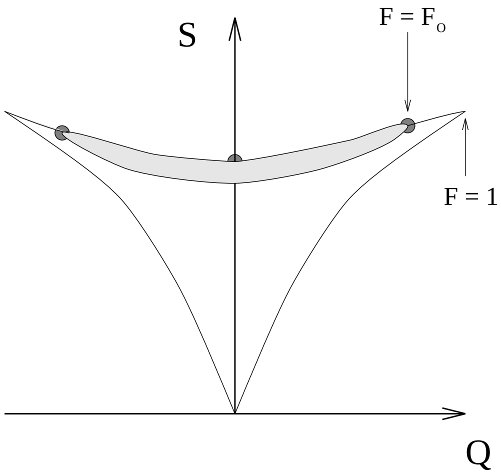

Figure 1. Uniform states and unstable periodic Stokes' waves of small amplitude as a function of the flow force (S) and the mass flux (Q). The boundary of the swallowtail corresponds to uniform flows and the two small shaded regions correspond to unstable periodic Stokes' waves (cf. Theorem 2).

In Figure 1 the uniform flows correspond to the boundary of the swallowtail (in this picture the Bernoulli constant is fixed). The cusp points correspond to Froude number $F = 1$ where $F = c/\sqrt{gh}$. In the definition of the Froude number, c is the frame speed, g the gravitational constant and h the uniform flow depth – which is determined by the Bernoulli constant (see Appendix B). Away from the boundary of the swallowtail, steady nonuniform flows may exist. Related to this paper is that fact that along the upper branch of the swallowtail, between the two cusp points, the uniform flow is subcritical and it is near this branch that the periodic Stokes wave is found. Since our theory is local we do not consider the existence or stability of steady states in other parts of the (S, Q)-diagram (cf. remarks after Theorem 2 in introduction).

Perturbing about the uniform flow, $Z_0 = (0, 0, 0, -c)$, with

$$Z = Z_0 + Z'\,, \quad S + cQ = S_0 + cQ_0 + S' \quad \text{and} \quad Q = Q_0 + Q'$$

(cf. Appendix B) leads to the spatial dynamical system (where without confusion the primes are dropped)

$$\mathbf{K}(u_s - c)Z_x = \nabla S(Z)\,. \tag{1.5}$$

The system (1.5) is an infinite-dimensional Hamiltonian system with Hamiltonian functional S (the flow force) and symplectic form

$$\omega_Z = -\mathbf{d}\alpha_Z \quad \text{with} \quad \alpha_Z = \int_{-h}^{\eta} (u - c)\mathbf{d}\phi\, dy$$

(cf. Bridges [1992a,1994a]). The symplectic manifold associated with (1.5) will be characterized in Section 2.

The equation (1.5) is ill-posed as an evolution equation and one method of analysis, to find all bounded states, is to use spatial center manifold theory. The use of spatial center manifold theory for proving the existence of uniformly travelling waves was introduced by Kirchgässner [1982] with further developments and applications to water waves in Mielke [1988] and Iooss & Kirchgässner [1992]. The Hamiltonian structure of spatial center manifolds, when the original equation has a Hamiltonian structure, is introduced in Mielke [1991]. In the present paper we extend this theory to include linear stability results for waves with a spatial center manifold. The approach is quite general and not restricted to water waves (cf. Bridges & Mielke [1994] and remarks in Section 7) but in the present paper we restrict attention throughout to the water wave problem.

The spatial center manifold reduction proceeds as follows. After a suitable transformation of the dependent variables (see Section 2) the time-independent states of (1.3) satisfy the spatial dynamical system $Z_x = \mathcal{F}(Z)$ where $\mathcal{F}(Z) \stackrel{\text{def}}{=} K(Z)^{-1}\nabla S(Z)$ is the composition of a skew-symmetric operator with the gradient of the functional S. For the range of parameters of interest, namely $0 < F < 1$, the spectrum of $D\mathcal{F}(0)$ consists of a pair of zero eigenvalues (forced by symmetry), a pair of pure imaginary eigenvalues and a countable set of eigenvalues whose real parts are bounded away from zero. The properties of the center subspace are considered in Section 3. In Section 4, we prove that $D\mathcal{F}(0)$ satisfies the required resolvent estimate and that $\mathcal{F}(Z)$ is a well-defined quasi-linear mapping on a linear Hilbert space. Therefore the center manifold theorem for quasi-linear mappings (MIELKE [1988]) can be applied in order to prove the existence of a four-dimensional center manifold containing all local bounded states.

The Hamiltonian structure of the problem is then used to prove that the flow on the center manifold has a (finite-dimensional) Hamiltonian structure

$$\frac{d\alpha}{dx} = \widetilde{J}(\alpha)\nabla\widetilde{S}(\alpha) \qquad \text{for } \alpha \in \mathcal{U} \subset \mathbb{R}^4$$

where $\widetilde{S}$ is the restriction of the flow force to the center manifold and $\widetilde{J}$ is the co-symplectic operator (inverse of symplectic operator) restricted to the center manifold.

The equivariant Darboux Theorem (see Theorem 4.3 and MIELKE [1991, p. 54]) applies to flatten the symplectic form and then a normal form transformation is used to simplify the Hamiltonian function. The transformed system on the center manifold then takes the form

$$\beta_x = J_1 \nabla H(\beta) \quad \text{with} \quad J_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \beta \subset \mathbb{R}^4 \tag{1.6}$$

and

$$H(\beta) = \widehat{H}(I_1, I_2) = -k_o I_1 + \tfrac{1}{2} I_2^2 + \varepsilon_1 I_1 I_2 + \varepsilon_2 I_1^2 + \cdots \tag{1.7}$$

where

$$I_1 = \tfrac{1}{2}(\beta_1^2 + \beta_2^2) \quad \text{and} \quad I_2 = \beta_4\,. \tag{1.8}$$

Due to symmetry, $H(\beta)$ is independent of β_3 and $\beta_4 = I_2$ is an exact spatial invariant. The symmetry corresponds to adding a constant to the velocity potential and the invariant I_2 is related to the restriction of the mass flux to the center-manifold. The reduced system is a completely integrable Hamiltonian system and the Hamiltonian function $H(\beta)$ can be expressed as a function of I_1 and I_2 by normal form theory. We say that the integrable reduced system is non-degenerate if

$$\Delta_K = \det\begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \neq 0 \quad \text{where} \quad H_{ij} = \frac{\partial^2 \widehat{H}}{\partial I_i \partial I_j}\,. \tag{1.9}$$

The determinant Δ_K is similar to the KAM determinant although the flow of the reduced system is not on a torus. It consists of a center in the (β_1, β_2) plane crossed with a cylinder (which is related to the mean flow in the full system). One can in principle factor out the symmetry and reduce the system to a Hamiltonian system on

$\mathbb{R}^2$. However, I_2 is no longer an invariant for the linear stability problem and therefore it is important to retain the full four-dimensional system in preparation for the linear stability analysis. Further analysis of the reduced system is given in Section 4 with the following main result. (Recall the notation $(\mathcal{M}, \omega, H)$ for a Hamiltonian system where $\mathcal{M}$ is a manifold, ω is a symplectic form – a non-degenerate closed two-form – and $H : \mathcal{M} \to \mathbb{R}$ is a Hamiltonian function.)

Theorem 1 (Stokes' manifold). *Suppose*

$$F \in (0,1)\,, \quad c \neq c_g \quad \text{and} \quad \Delta_K \neq 0\,.$$

Then every bounded state $\widehat{Z}$ of the system (1.5) with $\|\widehat{Z}\|_\infty$ sufficiently small is in one to one correspondence with an orbit of the reduced Hamiltonian system

$$(\mathcal{M}_R, \mathbf{d}\beta_1 \wedge \mathbf{d}\beta_2 + \mathbf{d}\beta_3 \wedge \mathbf{d}\beta_4, H) \quad \text{with } H \text{ in (1.7) and } \mathcal{M}_R \subset \mathbb{R}^4\,.$$

Moreover, for every fixed $F \in (0,1)$, the reduced Hamiltonian system has a two manifold of spatially periodic states (the Stokes' travelling wave) which can be characterized by the integrals I_1 and I_2 in (1.8).

The existence of the Stokes periodic wave for sufficiently small amplitude, goes back to the early part of this century and work of Levi-Civita, Struik and Nekrasov. The new points here are *(i)* the proof is given completely within a Hamiltonian context, *(ii)* the proof yields a cylinder of periodic orbits in the reduced phase space including the parameterization by the mass flux (represented by I_2) which is important for the linear stability theory and *(iii)* the nature of the proof, with minor additional effort, extends to include linear instability results. The condition $F \in (0,1)$ is imposed and for gravity waves it is easy to verify that $c > c_g$ where c_g is the group velocity and that $\Delta_K < 0$ (cf. equations (3.8) and (4.10)).

In Section 5 the linear stability problem is formulated as follows. After subtracting off the uniform flow, the system (1.3) can be written

$$\mathbf{M}Z_t + \mathbf{K}(u_s - c)Z_x = \nabla S(Z)\,. \tag{1.10}$$

Note that $\mathbf{M}$ is independent of Z whereas $\mathbf{K}$ depends on Z via $u_s - c$. As we want to linearize around a nontrivial state $Z = \widehat{Z}(x)$ it is advantageous to perform a coordinate transformation $Z = \mathcal{T}(\widetilde{Z})$ such that $\mathbf{K}$ is constant on the nontrivial state while $\mathbf{M}$ will be dependent on the new coordinates $\widetilde{Z}$. This leads to the transformed system

$$\widehat{\mathbf{M}}(\widetilde{Z})\widetilde{Z}_t + \widehat{\mathbf{K}}(\widetilde{Z})\widetilde{Z}_x = \nabla \widehat{S}(\widetilde{Z}), \tag{1.11}$$

where $\widehat{S} = S \circ \mathcal{T}$, $\widehat{\mathbf{M}}(\widetilde{Z}) = D\mathcal{T}^*\mathbf{M}D\mathcal{T}$ and similarly for $\widehat{\mathbf{K}}$. Let $\widehat{Z}$ represent a basic time-independent state, periodic in x, satisfying (1.11). Linearizing about $\widehat{Z}$, letting $Z = \widehat{Z} + widehatV$, results in

$$\widehat{M}(x)\widehat{V}_t + \widehat{\mathbf{K}}_0\widehat{V}_x = D^2\widetilde{S}(\widehat{Z}(x))\widehat{V} \tag{1.12}$$

where $\widehat{M}(x)$ is $\widehat{\mathbf{M}}(\widetilde{Z})$ evaluated at $\widehat{Z}(x)$ and $\widehat{\mathbf{K}}_0 = \widehat{\mathbf{K}}(\widehat{Z}(x))$ is constant. Since the problem is linear and autonomous in t we may look for solutions of the form $\widehat{V}(t,x) = V(x)e^{\lambda t}$ in which case (1.12) becomes

$$\widehat{\mathbf{K}}_0 V_x = \left[D^2\widetilde{S}(\widehat{Z}(x)) - \lambda\widehat{M}(x)\right]V\,. \tag{1.13}$$

The problem (1.13) is a linear partial differential equation and an eigenvalue problem for $\lambda \in \mathbb{C}$. Formally, the linear stability problem can be stated as follows. If there exists a nontrivial bounded solution V of (1.13) for some $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$ then the basic state is unstable. Our concept of linear instability is in fact what is usually called *spectral* instability, as we show that unstable spectral points exist.

To analyze the linear stability problem, for a special solution $\beta = \widehat{\beta}(I,x)$ of (1.6), a second center manifold analysis is applied to (1.13) with the non-autonomous term $\widehat{\mathbf{K}}_0^{-1}(D^2\widetilde{S}(\widehat{Z}(x)) - D^2\widetilde{S}(0))$ and λ sufficiently small. There is a close connection between the center-manifold reduction for the stability problem in Section 5 and the center-manifold reduction for the steady wave problem in Section 4. When $\lambda = 0$ the linearized stability problem (1.13) is the linearization of the steady problem. An important consequence of the theory of Section 5 is that the reduced instability problem retains this property; that is, the reduced system for (1.13), when $\lambda = 0$, is precisely the – linearized – reduced system for the steady problem. In Section 5 we prove that the reduction of (1.13) results in a linear differential equation on the center manifold of the form

$$K_1\frac{d}{dx}V_1 = D^2H(\widehat{\beta}(I,x))V_1 - \lambda M_1(\lambda,I,x)V_1 \quad \text{for} \quad V_1 \in \mathbb{C}^4 \tag{1.14}$$

where $K_1 = J_1^{-1}$ with J_1 defined in (1.6) and $H(\beta)$ is the reduced Hamiltonian function (1.7). When $\lambda = 0$, the system (1.14) is just the linearization of (1.6).

Corollary 5.3 ensures that any bounded solution of (1.14) with $|\lambda|$ sufficiently small and $\mathrm{Re}(\lambda) > 0$ will result in a linearly unstable solution of the full problem (1.13). Therefore Section 6 is devoted to an analysis of the reduced system (1.14). Floquet's Theorem is immediately applicable to (1.14) since $D^2H(\widehat{\beta}(I,x))$ and $M(\lambda,I,x)$ are periodic in x in which case every bounded solution of (1.14) is of the form $V_1 = Ue^{i\sigma x}$ with $\sigma \in \mathbb{R}$ and U periodic and satisfying

$$\mathcal{L}_I U \stackrel{\text{def}}{=} K_1U_x - D^2H(\widehat{\beta}(I,x))U = -\lambda M_1(\lambda,I,x)U - i\sigma K_1 U\,. \tag{1.15}$$

When $\lambda = \sigma = 0$ the problem (1.15) can be completely analyzed at any point on the two-manifold of periodic states given by Theorem 1. The tangent space to the two-manifold of basic states is in the kernel of the linear operator (1.14) when $\lambda = \sigma = 0$. Therefore the Lyapunov-Schmidt theory can be used to project the problem (1.15) onto the tangent space (cf. BRIDGES & ROWLANDS [1994], BRIDGES [1994b,c]). The projection yields a "dispersion relation" $\Lambda(\lambda, \sigma, I) = 0$ and the zeros of $\Lambda(\lambda, \sigma, I)$ are the *temporal* eigenvalues $\lambda \in \mathbb{C}$ as a function of the sideband spatial exponent σ and the parameters I_1 and I_2. A rigorous analysis of the dispersion relation leads to sufficient conditions for instability. In particular, in Sections 5 and 6 we prove the following.

Theorem 2 (Benjamin-Feir instability). *Suppose*

$$h > 0 \text{ and finite}, \quad gh - c_g^2 \neq 0 \quad \text{and} \quad \dot{c}_g = \frac{\partial c_g}{\partial k} \neq 0 .$$

Then there exists a function $\nu(F)$ with $\nu(F_o) = 0$ where $F_o \approx 0.8$ and $\nu(F) > 0$ for $F \in (0, F_o)$ such that for $\sigma \neq 0$ but sufficiently small $\Lambda(\lambda, \sigma, I)$ has the pair of zeros

$$\lambda_{1,2} = i(c - c_g)\sigma \pm \varepsilon_7 \sqrt{2I_1\nu(F)}\,\sigma + \mathcal{O}(\sigma^2 + I_1)$$

where $\varepsilon_7 > 0$. In particular, for $F \in (0, F_o)$ the manifold of periodic states given by Theorem 1 consists of linearly unstable solutions of the time-dependent problem.

An explicit expression for $\nu(F)$ will be a consequence of the theory in Section 6. We remark that the function $\nu(F)$ is precisely the function $X(k)$ in BENJAMIN [1967, p. 68], the function defined in equations (57)-(58) of WHITHAM [1967, p. 409] and the critical coefficient ν that appears in the nonlinear Schrödinger equation model for gravity wave instability (HASIMOTO & ONO [1972, equation (4.7)]).

One of the hypotheses of Theorems 1 and 2 is that h be finite. In the limit as $h \to \infty$ the linear operator $\mathbf{A} = D\mathcal{F}(0)$ has a continuous spectrum covering the real axis and so the hypotheses of the center manifold theory are invalidated.

Instabilities with $\sigma \neq 0$ but sufficiently small are called sideband instabilities. The present theory captures the sideband instability only and does not predict the converse; linear stability. In other words when $\nu(F) < 0$ we cannot conclude linear stability. This is because a hypothesis of the center manifold reduction for the linear stability problem (1.13) is that $|\lambda|$ is sufficiently small. The theory does not rule out the possibility that $\mathrm{Re}(\lambda)$ and σ lie outside of a neighborhood of zero.

There has been much discussion in the literature about the relation between the Benjamin-Feir instability and the Eckhaus instability theories (STUART & DIPRIMA [1978]). Therefore we hasten to point out that our result does not apply to the Eckhaus theory. The Benjamin-Feir instability theory arises in conservative systems and the Eckhaus theory in dissipative systems. In the Eckhaus theory there is a bifurcation parameter (a Reynolds number for example) and a critical band of stable wave numbers (not to be confused with the sideband) whose width is measured by a universal number ($1/\sqrt{3}$) and neither of these features appear in the conservative case. However the formulation of the above analysis – spatial center manifold theory applied to the linear stability problem and geometric analysis of the reduced ODE – is in principle also applicable to the Eckhaus instability (cf. MIELKE [1994]).

The consequences of Theorem 2 are illustrated in the (Q, S) plane in Figure 1. The flow force S for the Stokes wave, for small amplitude, is less than the value for the uniform flow and therefore the Stokes wave lies in the interior of the swallowtail. There are two points along the top branch of the swallowtail associated with $F = F_o$ and one point associated with $F = 0$. The periodic Stokes waves between these points, in an interior neighborhood of the subcritical uniform flow, are linearly unstable.

2. The spatial Hamiltonian system

In this section the properties of the Hamiltonian system (1.5) are studied. The main technical step is characterizing the symplectic manifold for the system. The basic components of the spatial Hamiltonian system are the one-form

$$\widetilde{\alpha}_z = \int_{-h}^{\tilde{\eta}} (\widetilde{u} - c)\mathbf{d}\widetilde{\phi}\, d\widetilde{y}\,,$$

from which the two form $\widetilde{\omega}_z = -\mathbf{d}\widetilde{\alpha}_z$ is formally obtained, and the Hamiltonian function

$$\widetilde{S} = \tfrac{1}{2}\int_{-h}^{\tilde{\eta}} (\widetilde{u}^2 - \widetilde{\phi}_{\widetilde{y}}^2) d\widetilde{y} - \tfrac{1}{2} g\widetilde{\eta}^2\,.$$

Note that $\mathbf{d}\widetilde{\phi}$ is a differential one-form and $\mathbf{d}$ represents exterior differentiation; the image of $\mathbf{d}\widetilde{\phi}$ is in the tangent space of the symplectic manifold whereas $d\widetilde{y}$ is a measure for the integration over the $\tilde{y}$ cross section.

The coordinates for the Hamiltonian structure are $\widetilde{Z} = (\widetilde{\Phi}, \widetilde{\eta}, \widetilde{\phi}, \widetilde{u})^T$. Here a tilde is used to distinguish the above variables from the variables on a chart to be introduced shortly. The above objects are defined on the manifold

$$\begin{aligned}\mathcal{M} = \{\, \widetilde{Z} = (\widetilde{\Phi}, \widetilde{\eta}, \widetilde{\phi}, \widetilde{u}) \ :\ & \widetilde{\Phi},\, \widetilde{\eta} \in \mathbb{R},\ \ h + \eta > 0\,,\\ & \widetilde{\phi} \in \mathbb{H}^1((-h, \widetilde{\eta})),\ \widetilde{u} \in \mathbb{H}^0((-h, \widetilde{\eta})) \text{ and } \widetilde{\phi}|_{y=\tilde{\eta}} = \widetilde{\Phi}\ \}.\end{aligned}$$

Here $\mathbb{H}^p(a, b)$ is the Sobolev space of functions $f : (a, b) \to \mathbb{R}$ whose $0, \ldots, p^{th}$ derivatives are square integrable over the region (a, b) with norm $\|f\|_p^2 = \sum_{j=0}^p \|f^{(j)}\|^2$, where $\|f\|_0^2 = \int_a^b f(y)^2 dy$ is the norm in $\mathbb{L}_2(a, b) = \mathbb{H}^0(a, b)$.

The governing equations for the above system are given by

$$\widetilde{Z}_{\tilde{x}} = \widetilde{\mathbf{X}}_S \tag{2.1a}$$

where, with suitable restriction of $\widetilde{Z}$ to a manifold domain $\mathcal{M}_D \subset \mathcal{M}$, the vector field $\widetilde{\mathbf{X}}_S$ is defined by

$$\widetilde{\omega}_Z(\widetilde{\mathbf{X}}_S, \xi) = \langle\!\langle \nabla \widetilde{S}(\widetilde{Z}), \xi \rangle\!\rangle_Z \tag{2.1b}$$

using the $\mathbb{L}_2$ inner-product defined in (A.7). The concept of a manifold domain is a generalization of the concept of the domain of a linear operator to the case where a vector field is only densely defined on an infinite-dimensional manifold (cf. MIELKE [1991, p. 10] and references therein). For our analysis the linear theory with $\mathcal{M}_D$ given in (2.6) will be sufficient.

The representation (2.1) is a useful abstract representation of the Hamiltonian system. However, because the inner product is position dependent, through $\tilde{\eta}$, $\widetilde{\mathbf{X}}_S$ cannot be explicitly obtained. To study further the properties of the vector field $\widetilde{\mathbf{X}}_S$ we introduce a coordinate chart for the manifold $\mathcal{M}$. Consider the following transformation

$$(Z; x, y) = \mathcal{E}(\widetilde{Z}; \tilde{x}, \tilde{y})$$

where

$$x = \widetilde{x}, \quad y = \rho(\widetilde{y}, \eta) \quad \text{with inverse } \widetilde{y} = \chi(y, \eta)$$

and

$$\rho(-h, \eta) = -h\,, \quad \rho(\eta, \eta) = 0 \quad \text{and} \quad \partial\rho/\partial\widetilde{y} > 0\,.$$

For definiteness we use the following transformation, $\rho(\widetilde{y}, \eta) = \frac{h}{h+\eta}(\widetilde{y} - \eta)$ and $\chi(y, \eta) = \eta + (1 + \frac{\eta}{h})y$ where η here is identified with $\widetilde{\eta}$. Under the transformation $\mathcal{E}$ the dependent variables become

$$Z = (\Phi, \eta, \phi, u)^T \tag{2.2}$$

with

$$\begin{aligned} &\phi(x, y) = \widetilde{\phi}(\widetilde{x}, \chi(y, \eta))\,, \quad \Phi(x) = \phi(x, y)|_{y=0} \\ &u(x, y) = \widetilde{u}(\widetilde{x}, \chi(y, \eta)) \text{ and } u_s(x) = u(x, y)|_{y=0}\,. \end{aligned}$$

Let $\Omega_h = \{\, y \,:\, -h < y < 0 \,\}$, then the image of the manifold $\mathcal{M}$ under the transformation $\mathcal{E}$ is the chart $\mathcal{U}$ with

$$\mathcal{U} = \mathcal{E}(\mathcal{M}) = \{\, Z = (\Phi, \eta, \phi, u) \in \mathbb{X}_0 \;:\; h + \eta > 0 \,\} \tag{2.3}$$

where

$$\mathbb{X}_0 = \{\, Z \in \mathbb{R}^2 \times \mathbb{H}^1(\Omega_h) \times \mathbb{H}^0(\Omega_h) \;:\; \phi|_{y=0} = \Phi \,\}.$$

The chart $\mathcal{U}$ is an open subset of the linear space $\mathbb{X}_0$. Further on we use only $\mathbb{X}_0$ and assume everything is locally defined, i.e., close to $Z = 0$.

The Hamiltonian functional on $\mathbb{X}_0$ is given by

$$\begin{aligned} S(Z) = \widetilde{S} \circ \mathcal{E} &= \tfrac{1}{2} \int_{-h}^{0} \left[u^2 - (\phi_y/\chi_y)^2\right] \chi_y \, dy - \tfrac{1}{2}\eta^2 \\ &= \tfrac{1}{2}(1 + \tfrac{\eta}{h})\|u\|_0^2 - \tfrac{1}{2}\frac{h}{h+\eta}\|\phi'\|_0^2 - \tfrac{1}{2}g\eta^2 \end{aligned} \tag{2.4}$$

where the Sobolev norms are now defined with respect to the set Ω_h. It is clear from the above expression that S is well-defined on $\mathbb{X}_0$.

On $\mathbb{X}_0$, the one form $\tilde{\alpha}_{_Z}$ is transformed to

$$\alpha_{_Z} = \mathcal{E}^* \tilde{\alpha}_{_Z} = (1 + \tfrac{\eta}{h}) \int_{-h}^{0} (u - c)\mathbf{d}\phi \, dy - \left(\int_{-h}^{0} (u - c)\phi_y (1 + \tfrac{y}{h}) \, dy \right) \mathbf{d}\eta$$

and defines via $\omega_{_Z} = -\mathbf{d}\alpha_{_Z}$ the exact symplectic form on $\mathbb{X}_0$,

$$\begin{aligned} \omega_{_Z}(U, V) = (1 + \tfrac{\eta}{h}) \int_{-h}^{0} (V_3 W_4 - V_4 W_3) dy + \frac{1}{h} \int_{-h}^{0} (u - c)(V_3 W_2 - V_2 W_3) dy \\ + \int_{-h}^{0} (1 + \tfrac{y}{h}) \Big[\phi_y (V_4 W_2 - V_2 W_4) + (u - c)(V_{3y} W_2 - V_2 W_{3y}) \Big] dy. \end{aligned} \tag{2.5}$$

It is clear that $\omega_{_Z}$ is well-defined on $\mathbb{X}_0$. It remains to be shown that ω is (weakly) non-degenerate, i.e., $\omega_{_Z}(V, W) = 0$ for all W implies $V = 0$. The symplectic form is not strongly non-degenerate in the sense of MARSDEN AND HUGHES [1983], p. 253. However the weakness of the symplectic form will not be important in the sequel as the main object of the analysis (cf. Sections 4-5) will be to reduce the problem to a finite-dimensional submanifold.

The non-degeneracy proof will be done indirectly by introducing a manifold domain, which is also necessary to set up the associated vector field for the spatial problem. The definition of manifold domain is given in MARSDEN & HUGHES [1983] and MIELKE [1991, p. 10]. When considered with respect to the chart $\mathbb{X}_0$ the manifold domain is the linear space

$$\mathcal{D}(\mathbf{A}) = \mathbb{X}_0 \cap \{ \phi \in \mathbb{H}^2(\Omega_h), \ u \in \mathbb{H}^1(\Omega_h), \ \phi_y|_{y=-h} = 0 \}. \tag{2.6}$$

The notation $\mathcal{D}(\mathbf{A})$ anticipates the functional analysis of Sections 3 and 4. The linear operator $\mathbf{A}$ is the linearization of the vector field (cf. equation (2.19)).

On $\mathcal{D}(\mathbf{A})$ we can rewrite the symplectic form by a partial integration as follows:

$$\omega_{_Z} = (1 + \tfrac{\eta}{h}) \int_{-h}^{0} (\mathbf{d}\phi \wedge \mathbf{d}u) \, dy + \left\{ (u_s - c)\mathbf{d}\Phi - \int_{-h}^{0} (1 + \tfrac{y}{h}) \begin{vmatrix} \mathbf{d}\phi & \mathbf{d}u \\ \phi_y & u_y \end{vmatrix} dy \right\} \wedge \mathbf{d}\eta, \tag{2.7}$$

where $u_s = u|_{y=0}$. Moreover, on $\mathbb{X}_0$ we introduce the inner product

$$\langle V, W\rangle = V_1 W_1 + V_2 W_2 + \int_{-h}^{0} (V_3 W_3 + V_4 W_4)\, dy\,. \tag{2.8}$$

Then the symplectic form (2.7) evaluated on arbitrary tangent vectors V, $W \in \mathcal{D}(\mathbf{A})$ becomes

$$\omega_{Z}(V, W) = \langle V, \mathbf{K}(Z) W\rangle\,. \tag{2.9}$$

For $Z \in \mathcal{D}(\mathbf{A})$ the symplectic operator $\mathbf{K}(Z)$ is formally invertible with inverse $\mathbf{J}(Z) = \mathbf{K}(Z)^{-1}$ given by

$$\mathbf{J}(Z) = \begin{pmatrix} 0 & \frac{1}{u_s - c} & \frac{1}{u_s - c}\int_{-h}^{0} \frac{\chi_\eta}{\chi_y}\phi_y \cdot\, dy & \frac{1}{u_s - c}\int_{-h}^{0} \frac{\chi_\eta}{\chi_y} u_y \cdot\, dy \\ -\frac{1}{u_s - c} & 0 & 0 & 0 \\ -\frac{1}{u_s - c}\frac{\chi_\eta}{\chi_y}\phi_y & 0 & 0 & \frac{1}{\chi_y} \\ -\frac{1}{u_s - c}\frac{\chi_\eta}{\chi_y} u_y & 0 & -\frac{1}{\chi_y} & 0 \end{pmatrix} \tag{2.10}$$

where

$$\chi_\eta = 1 + \frac{y}{h} \quad \text{and} \quad \chi_y = 1 + \frac{\eta}{h}\,.$$

The system (1.5) has been transformed to the following spatial Hamiltonian system

$$Z_x = \mathbf{J}(Z)\, \nabla S(Z) \tag{2.11}$$

where the gradient of $S(Z)$ using the inner product (2.8) is

$$\nabla S(Z) = \begin{pmatrix} \delta S/\delta \Phi \\ \delta S/\delta \eta \\ \delta S/\delta \phi \\ \delta S/\delta u \end{pmatrix} = \begin{pmatrix} -[\phi_y/\chi_y]|_{y=0} \\ \frac{1}{h}\int_{-h}^{0} [\frac{1}{2}u^2 + \frac{1}{2}(\phi_y/\chi_y)^2] dy - g\eta \\ (\phi_y/\chi_y)_y \\ \chi_y u \end{pmatrix}\,. \tag{2.12}$$

Both $\nabla S(Z)$ and the co-symplectic operator $\mathbf{J}(Z)$ require restriction to the manifold domain $\mathcal{D}(\mathbf{A}) \subset \mathbb{X}_0$ in order to be well defined.

There is an interesting identity that arises due to the tangent space constraint $\mathbf{d}\phi|_{y=0} = \mathbf{d}\Phi$. Imposing the constraint on the image of $\mathbf{J}(Z)\nabla S(Z)$ in (2.11) results in

$$\begin{aligned} 0 &= \mathbf{d}\phi|_{y=0} - \mathbf{d}\Phi \\ &= \left(-\frac{1}{u_s - c}\frac{\phi_y}{\chi_y}\frac{\delta S}{\delta \Phi} + \frac{1}{\chi_y}\frac{\delta S}{\delta u}\right)\Bigg|_{y=0} - \frac{1}{u_s - c}\int_{-h}^{0} \frac{\chi_\eta}{\chi_y}\left(\phi_y \frac{\delta S}{\delta \phi} + u_y \frac{\delta S}{\delta u}\right) dy \\ &= \frac{1}{u_s - c}\mathbf{B}(\eta, \phi, u) \end{aligned}$$

where

$$\mathbf{B}(\eta, \phi, u) = \tfrac{1}{2}u_s^2 - c u_s + \tfrac{1}{2}(\phi_y/\chi_y)^2|_{y=0} + g\eta\,. \tag{2.13}$$

We name the above functional $\mathbf{B}(\eta, \phi, u)$ the Bernoulli functional since Bernoulli's Theorem at the fluid surface requires that $\mathbf{B}(\eta, \phi, u)$ equals a constant, which is chosen to be zero in the present case. The Bernoulli functional will be important for the analysis in Section 4.

The functional

$$Q = (1 + \tfrac{\eta}{h}) \int_{-h}^{0} u \, dy - c\,\eta \tag{2.14}$$

is an additional independent spatial invariant. It will play an important role in the linear stability theory and the analysis in Section 4. Physically, the functional Q is the mass flux relative to the moving frame and is an absolute spatial invariant; that is $dQ/dx = 0$ for states satisfying (2.11).

The invariance of the functional Q can be verified by relating it to an underlying symmetry group of the problem. Consider the translation group $\Gamma = \mathbb{R}$ with action

$$\Gamma \cdot Z = Z + \gamma V \quad \text{with } V = (1, 0, 1, 0)^T \quad \text{and } \gamma \in \Gamma = \mathbb{R}. \tag{2.15}$$

This action perturbs the velocity potential by a constant, a seemingly trivial action, but fundamentally important as the *flow* along the group orbit affects the linear stability of the Stokes' wavetrain. Using Noether's Theorem, adapted to the spatial setting (cf. Bridges [1994a, App. A]), the relation between the Γ symmetry and the mass flux is as follows

$$\mathbf{K}(Z)V = \nabla Q = \begin{pmatrix} 0 \\ \frac{1}{h}\int_{-h}^{0} (u - c) dy \\ 0 \\ (1 + \frac{\eta}{h}) \end{pmatrix} \tag{2.16}$$

We have proved the following.

Lemma 2.1. *Time-independent states of the inviscid, irrotational gravity-wave problem, defined in (A.1)-(A.3), are in one-to-one correspondence with orbits of an infinite-dimensional spatial Hamiltonian system* $(\mathcal{M}, \widetilde{\omega}_Z, \widetilde{S})$. *Moreover there exists a second independent invariant* Q *for the system with associated symmetry group* Γ *satisfying* $S(\Gamma_\gamma \cdot Z) = S(Z)$ *and* $\Gamma_\gamma^* \omega = \omega$ *for all* $\gamma \in \Gamma$.

The spatial dynamical system (2.11) takes an elegant form when expressed in terms of the Poisson bracket. The Poisson bracket is defined using

$$\{F, G\}_Z = \omega_Z(X_F, X_G)$$

(see MARSDEN & HUGHES [1983, p. 263, definition 3.5]). We find

$$\begin{aligned}\{F,G\}_Z = \frac{1}{\chi_y}\int_{-h}^{0}\left(\frac{\partial F}{\partial \phi}\frac{\partial G}{\partial u} - \frac{\partial F}{\partial u}\frac{\partial G}{\partial \phi}\right) dy + \frac{1}{(u_s - c)}\left(\frac{\partial F}{\partial \Phi}\frac{\partial G}{\partial \eta} - \frac{\partial F}{\partial \eta}\frac{\partial G}{\partial \Phi}\right)\\ + \frac{1}{(u_s-c)}\int_{-h}^{0}\frac{\chi_y}{\chi_\eta}u_y\left(\frac{\partial F}{\partial \Phi}\frac{\partial G}{\partial u} - \frac{\partial F}{\partial u}\frac{\partial G}{\partial \Phi}\right) dy\\ + \frac{1}{(u_s-c)}\int_{-h}^{0}\frac{\chi_y}{\chi_\eta}\phi_y\left(\frac{\partial F}{\partial \Phi}\frac{\partial G}{\partial \phi} - \frac{\partial F}{\partial \phi}\frac{\partial G}{\partial \Phi}\right) dy\end{aligned}$$

in terms of which the vector field (2.11) takes the elegant form

$$\begin{aligned}\Phi_x &= \{\Phi, S\}, \quad \eta_x = \{\eta, S\},\\ \phi_x &= \{\phi, S\}, \quad u_x = \{u, S\}.\end{aligned} \tag{2.17}$$

The spatial dynamical system is also reversible with respect to the x-variable. The action of the reverser on the extended phase space is given by

$$\mathcal{R}\cdot(x;\Phi,\eta,\phi,u) = (-x;-\Phi,\eta,-\phi,u). \tag{2.18}$$

It is easily verified that $\mathcal{R}\mathbf{K}(\mathcal{R}Z)\mathcal{R} = -\mathbf{K}(Z)$ and $S\circ\mathcal{R} = S$. The reversibility will not play an essential role in the theoretical part of this work, but from the computational point of view it leads to some simplifications.

The vector field in (2.11) can be decomposed into a linear and nonlinear part;

$$Z_x = \mathbf{A}Z + \mathcal{N}(Z) \stackrel{\text{def}}{=} \mathcal{F}(Z)$$

with

$$\mathbf{A}Z \stackrel{\text{def}}{=} \begin{pmatrix} \frac{g}{c}\eta \\ -\frac{1}{c}\phi'|_{y=0} \\ u \\ -\phi'' \end{pmatrix} \tag{2.19}$$

where $\mathbf{A} : \mathcal{D}(\mathbf{A}) \subset \mathbb{X} \to \mathbb{X}$.

The nonlinear part of the vector field is

$$\mathcal{N}(Z) = \begin{pmatrix} \frac{1}{u_s - c}\left[\frac{1}{2}u_s + \frac{1}{2}(\phi'/\chi_y)^2 - \frac{g}{c}u_s\eta\right]\Big|_{y=0} \\ \frac{1}{u_s-c}\left[\frac{u_s}{c} - \frac{\eta}{h} + \frac{\eta}{h}\frac{u_s}{c}\right]\frac{\phi'}{\chi_y}\Big|_{y=0} \\ \frac{1}{u_s-c}\frac{\chi_\eta}{\chi_y^2}\phi'|_{y=0}\,\phi' \\ -(1-\frac{1}{\chi_y^2})\phi'' + \frac{1}{u_s-c}\frac{\chi_\eta}{\chi_y^2}\phi'|_{y=0}\,u' \end{pmatrix}. \tag{2.20}$$

3. Hamiltonian structure of the center subspace

The linearization of the spatial dynamical system (2.11) about the uniform state $Z = 0$ is an infinite-dimensional Hamiltonian system with symplectic form

$$\omega_0 = c\mathbf{d}\eta \wedge \mathbf{d}\Phi + \int_{-h}^{0} (\mathbf{d}\phi \wedge \mathbf{d}u)\, dy \tag{3.1}$$

which, when evaluated on tangent vectors ξ_1 and ξ_2 becomes

$$\omega_0(\xi_1, \xi_2) = \langle \mathbf{K}_0\xi_1, \xi_2\rangle \quad \text{with} \quad \mathbf{K}_0 = \mathbf{K}(0) = \begin{pmatrix} 0 & c & 0 & 0 \\ -c & 0 & 0 & 0 \\ 0 & 0 & 0 & -I \\ 0 & 0 & I & 0 \end{pmatrix}.$$

The Hamiltonian function for the linearized system is

$$S_0(Z) = \int_{-h}^{0} \tfrac{1}{2}(u^2 - \phi_y^2)\, dy - \tfrac{1}{2} g\eta^2 \tag{3.2}$$

and the linear operator $\mathbf{A}$ has the decomposition $\mathbf{A} = \mathbf{J}_0 D^2 S_0$ where $\mathbf{J}_0 = \mathbf{J}(0) = \mathbf{K}_0^{-1}$ (cf. (2.10) and (2.19)).

The eigenvalue problem $\mathbf{A}Z = \lambda Z$ for $Z \in \mathcal{D}(\mathbf{A})$ results in the following transcendental equation for the eigenvalue λ

$$hF^2\lambda^2 \cos\lambda h = \lambda \sin\lambda h\,. \tag{3.3}$$

When λ is pure imaginary this expression recovers the classic dispersion relation for gravity waves. The properties of this eigenvalue relation when $\lambda \in \mathbb{C}$ are well understood. For $0 < F < 1$ the spectrum consists of a pair of pure imaginary eigenvalues $\pm ik_o$, a pair of zero eigenvalues and a countable number of eigenvalues whose real parts are bounded away from zero. The purely imaginary pair of eigenvalues corresponds to the linear Stokes' gravity wave and k_o satisfies, for given F and h, the dispersion relation

$$F^2 = \frac{c^2}{gh} = \frac{\tanh k_o h}{k_o h}\,. \tag{3.4}$$

For each $F \in (0,1)$ and $h > 0$ there is a unique positive value of k_o satisfying equation (3.4). The pair of zero eigenvalues is forced by symmetry. In fact the geometric eigenvector corresponding to the zero eigenvalue is precisely the infinitesimal action V of the symmetry group defined in equation (2.15) (compare V in (2.15) with the eigenfunction ξ_3 below).

The four-dimensional center subspace is spanned by $\xi_1,\ldots,\xi_4$ with

$$\xi_1 = \frac{k_o}{a}\begin{pmatrix} 0 \\ c/g \\ 0 \\ \mathrm{ch}(y) \end{pmatrix}, \ \xi_2 = \frac{1}{a}\begin{pmatrix} 1 \\ 0 \\ \mathrm{ch}(y) \\ 0 \end{pmatrix}, \ \xi_3 = \frac{1}{b}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \ \xi_4 = \frac{1}{b}\begin{pmatrix} 0 \\ c/g \\ 0 \\ 1 \end{pmatrix}. \tag{3.5}$$

In particular, the eigenvectors satisfy

$$\mathbf{A}\xi_1 = k_o\xi_2\,, \quad \mathbf{A}\xi_2 = -k_o\xi_1\,, \quad \mathbf{A}\xi_3 = 0 \quad \text{and} \quad \mathbf{A}\xi_4 = \xi_3\,. \tag{3.6}$$

The coefficients and function $\mathrm{ch}(y)$ in (3.5) are defined by

$$a = F\sqrt{k_o h\left(\frac{c-c_g}{c}\right)}\,, \quad b = \sqrt{h(1-F^2)} \quad \text{and} \quad \mathrm{ch}(y) = \frac{\cosh k_o(y+h)}{\cosh k_o h}\,. \tag{3.7}$$

The linear problem is non-degenerate when $a\cdot b \neq 0$ with a and b as defined in (3.7). The condition $F \in (0,1)$ is a hypothesis and therefore $b \neq 0$. For gravity waves the group velocity, c_g, is

$$c_g = \tfrac{1}{2}c\left(1 + \frac{1}{F^2}\mathrm{sech}^2 k_o h\right)\,. \tag{3.8}$$

From (3.8) we deduce that $a \neq 0$ since

$$\frac{c-c_g}{c} = \tfrac{1}{2}(1 - \tfrac{1}{F^2}\mathrm{sech}^2 k_o h) > 0 \quad \text{for} \quad F \text{ satisfying (3.4)}\,.$$

The coefficients a and b have alternative characterizations. From Appendix B, the mass flux evaluated on the uniform flow, as a function of c, has slope ${Q_0}'(c) = h(1-F^2)$ and so

$$b = [{Q_0}'(c)]^{1/2}\,.$$

The coefficient a is related to the slope of the dispersion relation (3.4) since, treating c as a function of k, $dc/dk = -(c-c_g)/k$ and so

$$a = \frac{k_o F}{c}\left(-ch\frac{dc}{dk}\right)^{1/2}\,.$$

For gravity waves the slope dc/dk is negative. Although capillary-gravity waves are not considered here, it is noteworthy that, for capillary-gravity waves, there is a critical point on the dispersion curve at which $c = c_g$.

Henceforth, in this section and in the later analysis on the center manifold, we will be working in the neighborhood of the trivial equilibrium point. Therefore we will be working on the basic linear space $\mathbb{X}$ with the symmetric bilinear form $\langle\cdot,\cdot\rangle$ as defined in (2.8). The adjoint operator $\mathbf{A}^* : \mathcal{D}(\mathbf{A}^*) \to \mathbb{X}$, satisfying $\langle W, \mathbf{A}Z\rangle = \langle \mathbf{A}^*W, Z\rangle$, is

$$\mathbf{A}^*W \stackrel{\text{def}}{=} \begin{pmatrix} W_4'|_{y=0} \\ \frac{g}{c}W_1 \\ -W_4'' \\ W_3 \end{pmatrix}$$

with domain

$$\mathcal{D}(\mathbf{A}^*) = \left\{ \; W \in \mathbb{R}^2 \times \mathbb{H}^1 \times \mathbb{H}^2 \;\; : \;\; W_4'|_{y=-h} = 0 \quad \text{and} \quad W_4|_{y=0} = -\frac{1}{c} W_2 \; \right\} .$$

The adjoint eigenvectors $\xi_1^*, \ldots, \xi_4^*$ satisfy

$$\mathbf{A}^* \xi_1^* = -k_o \xi_2^* \, , \;\; \mathbf{A}^* \xi_2^* = k_o \xi_1^* \, , \;\; \mathbf{A}^* \xi_3^* = \xi_4^* \text{ and } \mathbf{A}^* \xi_4^* = 0 \, .$$

Expressions for the adjoint eigenvectors follow easily from the symplectic structure,

$$\xi_1^* = -\mathbf{K}_0 \xi_2 \, , \;\; \xi_2^* = \mathbf{K}_0 \xi_1 \, , \;\; \xi_3^* = -\mathbf{K}_0 \xi_4 \text{ and } \xi_4^* = \mathbf{K}_0 \xi_3 \, . \tag{3.9}$$

The eigenvectors are normalized so that

$$\langle \xi_i^*, \xi_j \rangle = \delta_{ij} \quad \text{for} \quad i, j = 1, \ldots, 4 \, . \tag{3.10}$$

Decompose the space $\mathbb{X} = \mathbb{X}_1 \oplus \mathbb{X}_2$ with $\mathbb{X}_1$ the four dimensional center subspace spanned by $\xi_1, \ldots, \xi_4$ and $\mathbb{X}_2$ the hyperbolic complement. The projection operator $\mathbf{P} : \mathbb{X} \to \mathbb{X}_1$ is given explicitly by $\mathbf{P} = \sum_{j=1}^4 \langle \xi_j^*, \cdot \rangle \xi_i$ and satisfies $\mathbf{P}\,\mathbf{A} = \mathbf{A}\,\mathbf{P}$, $\mathbb{X}_2 = (I - \mathbf{P})\mathbb{X}$. Let $\alpha \in \mathbb{R}^4$ be coordinates on the center subspace. Then any element $Z \in \mathbb{X}_1$ has the representation

$$Z = \alpha_1 \xi_1 + \alpha_2 \xi_2 + \alpha_3 \xi_3 + \alpha_4 \xi_4 \, . \tag{3.11}$$

The symplectic form restricted to the center subspace is obtained as follows. Express $U, \; V \in \mathbb{X}_1$ in terms of the given basis: $U = \alpha \cdot \xi$ and $V = \widetilde{\alpha} \cdot \xi$ where $\alpha \cdot \xi = \sum_{j=1}^4 \alpha_j \xi_j$. Then, using (3.9) and the biorthogonality conditions (3.10),

$$\begin{aligned} \omega_0(U, V) &= \langle \alpha \cdot \xi, \mathbf{K}_0 \widetilde{\alpha} \cdot \xi \rangle = \langle \alpha \cdot \xi, \widetilde{\alpha}_1 \xi_2^* - \widetilde{\alpha}_2 \xi_1^* + \widetilde{\alpha}_3 \xi_4^* - \widetilde{\alpha}_4 \xi_3^* \rangle \\ &= \alpha_2 \widetilde{\alpha}_1 - \alpha_1 \widetilde{\alpha}_2 + \alpha_4 \widetilde{\alpha}_3 - \alpha_3 \widetilde{\alpha}_4 = (\alpha, K_1 \widetilde{\alpha}) \quad \text{with } K_1 = J_1^{-1} \, . \end{aligned}$$

Therefore,

$$\omega_0|_{\mathbb{X}_1} = \mathbf{d}\alpha_1 \wedge \mathbf{d}\alpha_2 + \mathbf{d}\alpha_3 \wedge \mathbf{d}\alpha_4 \, . \tag{3.12}$$

Similarly, the Hamiltonian function on $\mathbb{X}_1$ is obtained as follows. For $Z \in \mathcal{D}(\mathbf{A})$ we have

$$S_0(Z) = \tfrac{1}{2} \int_{-h}^0 (u^2 - \phi_y^2) \, dy - \tfrac{1}{2} g \eta^2 = \tfrac{1}{2} \langle Z, \mathbf{K}_0 \mathbf{A} \, Z \rangle = -\tfrac{1}{2} \langle \mathbf{K}_0 Z, \mathbf{A} Z \rangle \, .$$

But, for $Z \in \mathbb{X}_1$, using (3.11), we have $\mathbf{A} Z = \sum_{j=1}^4 \alpha_j \mathbf{A} \xi_j = k_o \alpha_1 \xi_2 - k_o \alpha_2 \xi_1 + \alpha_4 \xi_3$ and $\mathbf{K}_0 Z = \alpha_1 \xi_2^* - \alpha_2 \xi_1^* + \alpha_3 \xi_4^* - \alpha_4 \xi_3^*$, and so we arrive at

$$\widetilde{S}_0(\alpha) \stackrel{\text{def}}{=} S_0(\alpha \cdot \xi) = -\tfrac{1}{2} k_o (\alpha_1^2 + \alpha_2^2) + \tfrac{1}{2} \alpha_4^2 \, . \tag{3.13}$$

The restriction of the linear system to the center subspace results in a Hamiltonian system on $\mathbb{R}^4$ with the standard symplectic form and vector field

$$\frac{d}{dx}\alpha = J_1 \nabla \widetilde{S}_0(\alpha) = \begin{pmatrix} -k_o \alpha_2 \\ k_o \alpha_1 \\ \alpha_4 \\ 0 \end{pmatrix} \tag{3.14}$$

The flow consists of a center in the (α_1, α_2) plane crossed with a cylinder associated with a drift along the group orbit in the (α_3, α_4) plane.

For further reference we note that the reduced action of the reverser $\mathcal{R}$ from (2.18) on the center subspace is given by

$$\mathcal{R}_0 \alpha = (\alpha_1, -\alpha_2, -\alpha_3, \alpha_4)^T . \tag{3.15}$$

4. Hamiltonian center manifold for gravity waves

In this section we prove the existence of a four-dimensional Hamiltonian center manifold, the Stokes' manifold, for the nonlinear problem governing uniformly-travelling gravity waves on a fluid layer of finite depth.

The Hilbert space $\mathbb{X}$ decomposes into the two closed $\mathbf{A}$-invariant subspaces $\mathbb{X}_1$ and $\mathbb{X}_2$. Define $\mathbf{A}_j = \mathbf{A}|_{\mathbb{X}_j} : \mathcal{D}(\mathbf{A}) \cap \mathbb{X}_j \to \mathbb{X}_j$ for $j = 1, 2$ and write $Z = Z_1 + Z_2$ with $Z_j \in \mathbb{X}_j$ for $j = 1, 2$. Then, system (2.19)-(2.21) can be rewritten as the linearly decoupled but nonlinearly coupled pair of equations

$$\begin{aligned} \frac{d}{dx} Z_1 &= \mathbf{A}_1 Z_1 + \mathbf{P}\,\mathcal{N}(Z_1 + Z_2) \\ \frac{d}{dx} Z_2 &= \mathbf{A}_2 Z_2 + (\mathbf{I} - \mathbf{P})\,\mathcal{N}(Z_1 + Z_2)\,. \end{aligned} \tag{4.1}$$

The spectrum of $\mathbf{A}_2$ is hyperbolic: $\sigma(\mathbf{A}_2) \cap i\mathbb{R} = \emptyset$ and $\sigma(\mathbf{A}_1) = \{0, \pm i k_o\}$. The center-manifold theory will be used to prove that the second equation of (4.1) can be replaced by $Z_2 = \mathcal{H}(Z_1)$, called the center manifold, and that the first equation of (4.1) then reduces to an ordinary differential equation governing all, local, bounded states of the full nonlinear problem.

The construction of the center manifold cannot be done directly for (2.19)-(2.21) because the operator $\mathbf{A} : \mathcal{D}(\mathbf{A}) \to \mathbb{X}$ is not densely defined since

$$\mathbb{X}_0 = \overline{\mathcal{D}(\mathbf{A})}^{\mathbb{X}} = \{\ Z \in \mathbb{X}\ :\ \Phi = \phi|_{y=0}\ \}$$

is a proper subspace of $\mathbb{X}$ where $\mathbf{A}$ is as defined in (2.19) and $\mathcal{D}(\mathbf{A})$ in (2.6). This problem arises through the incompatibility of the right-hand sides associated with Φ_x and ϕ_x. For the image of the operator $\mathbf{A}Z + \mathcal{N}(Z)$ to be in $\mathbb{X}_0$ it is required that $\phi_x|_{y=0} = \Phi_x$ which in turn requires the Bernoulli functional $\mathbf{B}(u,\phi,\eta)$ to vanish (see equation (2.13)). Therefore define

$$\mathcal{B} = \{\ Z \in \mathcal{D}(\mathbf{A})\ :\ \mathbf{B}(\eta,\phi,u) = 0\ \},$$

then $Z \in \mathcal{B}$ implies $\mathbf{A}Z+\mathcal{N}(Z) \in \mathbb{X}_0$. Note that the vanishing of the Bernoulli functional is a compatibility condition. The first equation of (2.11) when written out is

$$\Phi_x = \frac{1}{u_s - c}\left[\tfrac{1}{2}u_s^2 + \tfrac{1}{2}\left(\phi_y/\chi_y\right)^2 - g\eta\right]$$

which is precisely Bernoulli's equation at the interface. The requirement that $\mathbf{B}(\eta,\phi,u) = 0$ therefore only ensures compatibility with the governing equations. It is essential because it ensures also that the image of the vector field in (2.11) is in the tangent space of the symplectic manifold.

There is a second natural constraint given by the mass flux (defined in equation (2.14)). The mass flux $Q(Z)$ is constant along solutions. We will use the invariant Q also in the construction of the center-manifold.

Recall $\mathbb{X}_1 = \mathrm{span}\{\xi_1,\ldots,\xi_4\} \subset \mathbb{X}_0$ and introduce the cylinder

$$\mathcal{U}_1^\epsilon = \{\ \alpha_1\xi_1 + \alpha_2\xi_2 + \alpha_3\xi_3 + \alpha_4\xi_4\ :\ \alpha_j \in \mathbb{R} \text{ and } |(\alpha_1,\alpha_2,\alpha_4)| \le \epsilon\ \} \subset \mathbb{X}_1\,.$$

Theorem 4.1. *Let $j \in \mathbb{N}$ be arbitrary but finite. Then there exists an $\epsilon > 0$ and a reduction function $\mathcal{H} = \mathcal{H}(Z_1) \in \mathcal{C}^j(\mathcal{U}_1^\epsilon, \mathcal{D}(\mathbf{A}))$ with $\mathcal{H}(0) = 0$, $D\mathcal{H}(0) = 0$ and $\mathcal{H}(Z_1 + \alpha_3\xi_3) = \mathcal{H}(Z_1)$ for every $\alpha_3 \in \mathbb{R}$. Moreover, the graph of $\mathcal{H}$,*

$$\mathbb{M} = \{\ Z_1 + \mathcal{H}(Z_1) \in \mathcal{D}(\mathbf{A})\ :\ Z_1 \in \mathcal{U}_1^\epsilon\ \},$$

is contained in $\mathcal{B}$ and is a local center manifold for (4.1).

Proof. We introduce a new local coordinate system in $\mathcal{D}(\mathbf{A})$; $\widehat{Z} = \mathcal{S}(Z) \stackrel{\text{def}}{=} (\Phi,\delta,\phi,\widehat{u})$. For notational simplicity we omit Φ further on as $\Phi = \phi|_{y=0}$. The new coordinate δ equals the mass flux $Q(Z)$, the variable η will be eliminated. The main point is that the Bernoulli functional $\mathbf{B}$ will take the linear form $hF^2\widehat{u}(0) = [\widehat{u}] \stackrel{\text{def}}{=} \int_{-h}^0 \widehat{u}\,dy$.

From $0 = Q(Z) - \delta = \left(1 + \frac{\eta}{h}\right)[u] - c\eta - \delta$ we find $\eta = \widetilde{q}(\delta,[u])$. Here and an in the remainder of the proof all the functions with a tilde denote real-valued functions on $\mathbb{R}^m$, $m \in \mathbb{N}$, whereas all other functions are between Banach spaces. On the other hand

$$0 = \mathbf{B}(\eta,\phi,u) = -cu(0) + \tfrac{1}{2}(u(0)^2 + (\phi_y(0)/\chi_y)^2) + g\eta$$

can be solved for $\eta = \widetilde{B}(\phi_y(0), u(0))$. Now the Bernoulli functional takes the form

$$0 = \widetilde{B}(\phi_y(0), u(0)) - \widetilde{q}([u], \delta) = \frac{c}{g}u(0) - \frac{1}{c}[u] + \frac{1}{c}\delta + h.o.t..$$

Therefore the new coordinate $\widehat{u}$ is defined implicitly as $u = \widehat{u} + \widehat{a}(\widehat{Z}) = \widehat{u} + \widehat{a}(\delta, \phi, \widehat{u})$ such that

$$\widetilde{B}(\phi_y(0), \widehat{u}(0) + \widehat{a}(\widehat{Z})(0)) - \widetilde{q}([\widehat{u} + \widehat{a}(\widehat{Z})], \delta) = \frac{c}{g}\widehat{u}(0) - \frac{1}{c}[\widehat{u}]\,. \tag{4.2}$$

One possible construction of $\widehat{a}$ is as follows. Denote by $\widetilde{d} = \widetilde{d}(\alpha, \beta)$ the solution of $\widetilde{B}(\alpha, \beta + \widetilde{d}) = \frac{c}{g}\beta$, let $\widehat{B}(\widehat{Z})(y) = \widetilde{d}(\phi_y(y), \widehat{u}(y))$ and $\widehat{a}(\widehat{Z})(y) = \widehat{B}(\widehat{Z})(y) + \frac{2y}{h^2}\widehat{C}(\widehat{Z})$ with $\widehat{C}(\widehat{Z}) \in \mathbb{R}$. By construction we have $\widetilde{B}(\phi_y(0), \widehat{u}(0) + \widehat{a}(\widehat{Z})(0)) = \frac{c}{g}\widehat{u}(0)$. Hence, requiring (4.2) $\widehat{C}$ is uniquely defined by

$$[\widehat{u}] = c\widetilde{q}([\widehat{u} + \widehat{a}(\widehat{Z})], \delta) = c\widetilde{q}([\widehat{u} + \widehat{B}(\widehat{Z})] + \widehat{C}(\widehat{Z}), \delta)\,.$$

Thus, we find $\widehat{C}(\widehat{Z}) = \widetilde{r}([\widehat{u}], [\widehat{B}(\widehat{Z})], \delta) = \delta + h.o.t\,.$ This completes the construction of the smooth mapping $\mathcal{S} : \mathcal{D}(\mathbf{A}) \to \mathcal{D}(\mathbf{A})$; $Z \mapsto \widehat{Z}$.

Note that $D\mathcal{S}(Z) \in \mathcal{L}(\mathcal{D}(\mathbf{A}), \mathcal{D}(\mathbf{A}))$ can be extended to a bounded linear operator $\widetilde{D\mathcal{S}}(Z) \in \mathcal{L}(\mathbb{X}_0, \mathbb{X}_0)$ which depends smoothly on $Z \in \mathcal{D}(\mathbf{A})$. This follows from the special form of $\widehat{a}$ which guarantees that $D\widehat{a}(\widehat{Z})$ can be extended to $\mathcal{L}(\mathbb{X}_0, \mathbb{L}_2)$ with smooth dependence on $\widehat{Z} \in \mathcal{D}(\mathbf{A})$. Introducing the inverse $\Sigma : \mathcal{D}(\mathbf{A}) \to \mathcal{D}(\mathbf{A})$ of $\mathcal{S}$ we find the transformed differential equation

$$\widehat{Z}_x = \widetilde{D\mathcal{S}}(\Sigma(\widehat{Z}))(\mathbf{A}\Sigma(\widehat{Z}) + \mathcal{N}(\widehat{Z})) = \widehat{\mathbf{A}}\widehat{Z} + \widehat{\mathcal{N}}(\widehat{Z})\,. \tag{4.3}$$

By construction the right-hand side is a smooth mapping from $\mathcal{D}(\widehat{\mathbf{A}}) = \{\widehat{Z} \in \mathcal{D}(\mathbf{A}) \ : \ hF^2\widehat{u}(0) = [\widehat{u}]\ \}$ into $\mathbb{X}_0$. A simple calculation shows that

$$\widehat{\mathbf{A}}\widehat{Z} = \begin{pmatrix} \frac{g}{c^2}[\widehat{u}] \\ 0 \\ \widehat{u} + \frac{2y}{h^2}\delta \\ -\phi_{yy} \end{pmatrix}$$

and Lemma 4.2 below provides the resolvent estimate $\|(\widehat{\mathbf{A}} - i\xi)^{-1}\|_{\mathbb{X}_0 \to \mathbb{X}_0} = \mathcal{O}(|\xi|^{-1})$ for $\xi \in \mathbb{R}$ and $\xi \to \infty$. As $D(\widehat{\mathbf{A}})$ is dense in $\mathbb{X}_0$ the center manifold theorem in Mielke [1988, Theorem 1] is applicable to (4.3) proving the existence of a four-dimensional local center manifold $\widehat{\mathbb{M}}$. Now $\mathbb{M} = \Sigma(\widehat{\mathbb{M}})$ is the desired manifold in the original Z-coordinates. ■

Lemma 4.2. *Let $\mathbb{X}_0$, $\mathcal{D}(\widehat{\mathbf{A}})$ and $\widehat{\mathbf{A}}$ be as above. Then the operator $\widehat{\mathbf{A}}$ is densely defined, closed and has a compact resolvent. Moreover, there are real-valued constants $\xi_0 > 0$ and $C > 0$ such that $(\widehat{\mathbf{A}} - i\xi) : \mathcal{D}(\widehat{\mathbf{A}}) \to \mathbb{X}_0$ is an isomorphism for $\xi \in \mathbb{R}$ with $|\xi| \geq \xi_0$ and*

$$\|(\widehat{\mathbf{A}} - i\xi)^{-1}\|_{\mathbb{X}_0 \to \mathbb{X}_0} \leq \frac{C}{1 + |\xi|} .$$

Proof. The only nonstandard part is the estimate on the resolvent. The resolvent problem reads

$$-i\xi\delta = \rho_0 , \quad \phi'(-h) = 0 , \quad hF^2 u(0) = [u] ;$$

$$\frac{2y}{h^2}\delta + u - i\xi\phi = \rho_1 \in \mathbb{H}^1 , \quad -\phi'' - i\xi u = \rho_2 \in \mathbb{L}_2 \quad \text{for } y \in (-h, 0) .$$

Using the fact that $\phi'(0) = [\phi''] = -[\rho_2 + i\xi u] = -[\rho_2] - i\xi hF^2 u(0)$ we obtain

$$|\xi u(0)\overline{\phi'(0)}| \leq \frac{3}{2hF^2}|\phi'(0)|^2 + \frac{1}{2hF^2}[\rho_2]^2 \leq \frac{3}{hF^2}\|\phi'\|\,\|\phi''\| + \frac{1}{2F^2}\|\rho_2\|^2 ,$$

where $\|f\|^2 = \int_{-h}^0 f(y)^2 dy$. Eliminating δ and abbreviating $\widetilde{\rho}_1 = \rho_1 - \frac{2y}{\xi h^2} i\rho_0$ allows for the following estimates

$$\begin{aligned}
\|\widetilde{\rho}_1\|^2 + h^2\|\widetilde{\rho}_1'\|^2 + h^2\|\rho_2\|^2 &= \|u - i\xi\phi\|^2 + h^2\|u' - i\xi\phi'\|^2 + h^2\|\phi'' + i\xi u\|^2 \\
&= \|u\|^2 + \xi^2\|\phi\|^2 - 2\xi\,\mathrm{Im}[u\overline{\phi}] \\
&\quad + h^2\left\{\|\phi''\|^2 + \|u'\|^2 + \xi^2\|\phi'\|^2 + \xi^2\|u\|^2 - 2\,\mathrm{Im}[\xi u(0)\overline{\phi'(0)}]\right\} \\
&\geq (1 - h|\xi| + h^2\xi^2)\|u\|^2 + (\xi - \tfrac{|\xi|}{h})\|\phi\|^2 \\
&\quad + h^2\{\|\phi''\|^2 + \xi^2\|\phi'\|^2 - \tfrac{6}{hF^2}\|\phi'\|\,\|\phi''\|\} - \tfrac{h^2}{F^2}\|\rho_2\|^2 \\
&\geq \tfrac{1}{2}\xi^2\{\|\phi\|^2 + h^2\|\phi'\|^2 + h^2\|u\|^2\} - \tfrac{h^2}{F^2}\|\rho_2\|^2
\end{aligned}$$

for $|\xi| \geq \xi_0 = \max\{\frac{2}{h}, \frac{5}{hF^2}\}$. Recalling $\delta = i\rho/\xi$ and $\widetilde{\rho}_1 = \rho_1 - 2iy\rho_0/(\xi k^2)$ we find

$$\|(\phi(0), \delta, \phi, u)^T\|_{\mathbb{X}_0} \leq \left\| (\widehat{\mathbf{A}} - i\xi)^{-1} \begin{pmatrix} \rho_1(0) \\ \rho_0 \\ \rho_1 \\ \rho_2 \end{pmatrix} \right\|_{\mathbb{X}_0} \leq \frac{C}{1 + |\xi|}\|(\rho_1(0), \rho_0, \rho_1, \rho_2)^T\|_{\mathbb{X}_0}$$

for $|\xi| \geq \xi_0$ which is the desired result. ■

The standard consequences of the center manifold theory (cf. Mielke [1991, p. 14]) follow:

(i) For every $j \in \mathbb{N}$ there exists a $\mathcal{C}^j$ reduction function but henceforth we fix j in the theory to be sufficiently large but finite (e.g., $j = 6$ will do).

(ii) In α-coordinates the reduced system is given by ($\alpha \cdot \xi = \alpha_1\xi_1 + \ldots + \alpha_4\xi_4$)

$$\alpha_x = \mathbf{A}_1\alpha + \mathbf{P}\,\mathcal{F}(\alpha \cdot \xi + \mathcal{H}(\alpha \cdot \xi)), \qquad \alpha \in \mathcal{U}_1^\epsilon\,. \tag{4.4}$$

(iii) Note that α_3 does not appear in the right-hand side of (4.4) which manifests the Γ symmetry. The restricted action of Γ on the center manifold is given by

$$\Gamma_\gamma\big|_{\mathbb{X}_1} \stackrel{\text{def}}{=} \widetilde{\Gamma}_\gamma : \alpha \mapsto \alpha + \gamma(0,0,1,0)^T \quad \text{for } \gamma \in \Gamma = \mathbb{R}\,. \tag{4.5}$$

(iv) In addition to the translation symmetry we also retain the reversibility with respect to the reverser $\mathcal{R}_0$ defined in (3.15).

Now we show that the reduced system (4.4) is in fact a Hamiltonian system. Moreover, we construct a coordinate system $\beta = \Upsilon(\alpha)$ such that the reduced system takes the form of a canonical Hamiltonian system which is in exact normal form. The first property relies on a general theory developed in Mielke [1991] whereas the last property derives from the fact that the system is completely integrable due to the Γ symmetry and the related conserved quantity associated to the mass flux.

Theorem 4.3. *The vector field on the center manifold, defined in (4.4), is a Hamiltonian system $(\mathcal{M}_R, \omega_R, H)$ with $\mathcal{M}_R \subset \mathcal{U}_1^\epsilon \subset \mathbb{R}^4$. Moreover, there exist coordinates $\beta = \Upsilon(\alpha) = \alpha + \mathcal{O}(|(\alpha_1,\alpha_2,\alpha_4)|^2)$ such that the Hamiltonian system is canonical and in exact normal form. This means that the symplectic structure takes the form $\omega_R = \mathbf{d}\beta_1 \wedge \mathbf{d}\beta_2 + \mathbf{d}\beta_3 \wedge \mathbf{d}\beta_4$ and that the restriction of S in (2.4) onto $\mathbb{M}$, denoted H, is given as*

$$H(\beta) = \widehat{H}(I_1, I_2) = -k_o I_1 + \tfrac{1}{2} I_2^2 + \varepsilon_1 I_1 I_2 + \varepsilon_2 I_1^2 + \mathcal{O}(I_1^3 + |I_2|^3)$$

where $I_1 = \frac{1}{2}(\beta_1^2 + \beta_2^2)\,, \quad I_2 = \beta_4.$

Proof. The Hamiltonian form of (4.4) is obtained by restriction of the Hamiltonian functional S and the symplectic form ω, given by Lemma 4.1, to the center manifold (cf. Mielke [1991]). Therefore (4.4) has the Hamiltonian formulation

$$\alpha_x = \widetilde{J}(\alpha)\,\nabla \widetilde{S}(\alpha) \quad \text{for} \quad \alpha \in \mathcal{U}_1^\epsilon$$

where $\widetilde{J}(\alpha)$ is the co-symplectic operator (inverse of symplectic operator) restricted to $\mathbb{M}$ and $\widetilde{S}(\alpha) \stackrel{\text{def}}{=} S(\alpha \cdot \xi + \mathcal{H}(\alpha \cdot \xi))$ with $\alpha \in \mathcal{U}_1^\epsilon$. The reduced system is a Hamiltonian system $(\mathcal{U}_1^\epsilon, \widetilde{\omega}, \widetilde{S})$ with $\widetilde{\omega}_\alpha(\zeta_1, \zeta_2) = (\zeta_1, \widetilde{J}(\alpha)^{-1}\zeta_2)$ for any $\zeta_1, \zeta_2 \in \mathbb{R}^4$.

For a canonical symplectic structure we apply the Equivariant Darboux Theorem for non-compact group actions (Mielke [1991, p. 54]). The reduced system is Γ-equivariant with action defined in (4.5). The Darboux transformation can be chosen to respect the Γ symmetry in which case the transformed Hamiltonian is independent of the third coordinate.

Let $\widetilde{\beta} \subset \mathcal{U}_1^\epsilon$ be the new coordinates determined by the Darboux transformation. According to Theorems 5.7 and 5.15 in MIELKE [1991] the coordinates $\widetilde{\beta}$ can be chosen such that $\widetilde{\beta} = \widetilde{\Upsilon}(\alpha) = \alpha + \mathcal{O}(|(\alpha_1, \alpha_2, \alpha_4)|^2)$. Thus, the transformation of $\alpha \in \mathcal{U}_1^\epsilon$ acts on the cross-section of the cylinder $\mathcal{U}_1^\epsilon$ and perhaps requires a shrinking of the ϵ cross-section of the cylinder. Therefore the transformed reduced system is again a Γ-equivariant Hamiltonian system with $(\mathcal{U}_1^{\widetilde{\epsilon}}, \widetilde{\Omega}, \widetilde{H})$, with $\widetilde{\epsilon} < \epsilon$, $\widetilde{\Omega} = \mathbf{d}\widetilde{\beta}_1 \wedge \mathbf{d}\widetilde{\beta}_1 + \mathbf{d}\widetilde{\beta}_3 \wedge \mathbf{d}\widetilde{\beta}_4$, and Hamiltonian function $\widetilde{H}(\widetilde{\beta})$ satisfying $\partial\widetilde{H}/\partial\widetilde{\beta}_3 = 0$. Reversibility according to (3.15) tells us that $\widetilde{H}$ is even in $\widetilde{\beta}_2$.

It remains to bring the system into normal form. We do this first for the restriction to the $(\widetilde{\beta}_1, \widetilde{\beta}_2)$-system, where $\widetilde{\beta}_4$ is considered as a parameter. For each small fixed $I_2 = \widetilde{\beta}_4$ the Hamiltonian system

$$\frac{d}{dx}\widetilde{\beta}_1 = \frac{\partial}{\partial\widetilde{\beta}_2}\widetilde{H}(\widetilde{\beta}_1, \widetilde{\beta}_2, I_2), \quad \frac{d}{dx}\widetilde{\beta}_2 = -\frac{\partial}{\partial\widetilde{\beta}_1}\widetilde{H}(\widetilde{\beta}_1, \widetilde{\beta}_2, I_2),$$

has a center on the $\widetilde{\beta}_1$-axis close to the origin. The linearization has eigenvalues close to $\pm ik_o$. Hence, the system can be brought into exact normal form, i.e., there exists a near-identity canonical change of coordinates

$$\beta_j = \widehat{\Upsilon}_j(\widetilde{\beta}_1, \widetilde{\beta}_2, I_2), \quad \text{for } j = 1, 2\,,$$

such that the transformed system has the Hamiltonian $H(\beta_1, \beta_2, I_2) = \widehat{H}(I_1, I_2)$ with $I_1 = \frac{1}{2}(\beta_1^2 + \beta_2^2)$ (cf. ARNOLD [1988] page 112 or HENRARD [1993]). In fact, $\widehat{\Upsilon}_j$ is uniquely given under the reversibility constraints that it is even in $\widetilde{\beta}_2$ for $j = 1$ and odd for $j = 2$.

We now have to show that a full four dimensional canonical mapping $\beta = \widehat{\Upsilon}(\widetilde{\beta})$ can be found. It will have the form

$$\widehat{\Upsilon}(\widetilde{\beta}) = (\widehat{\Upsilon}_1(\widetilde{\beta}), \widehat{\Upsilon}_2(\widetilde{\beta}), \widetilde{\beta}_3 + \overline{\Upsilon}(\widetilde{\beta}_1, \widetilde{\beta}_2, \widetilde{\beta}_4), \widetilde{\beta}_4).$$

Note that $I_2 = \beta_4 = \widetilde{\beta}_4$. Using the abbreviation $\widehat{\Upsilon}_{j,n} = \partial\widehat{\Upsilon}_j/\partial\widetilde{\beta}_n$ we have by construction

$$\widehat{\Upsilon}_{1,1}\widehat{\Upsilon}_{2,2} - \widehat{\Upsilon}_{2,1}\widehat{\Upsilon}_{1,2} = 1\,. \tag{4.6}$$

The other conditions for $\widehat{\Upsilon}$ being canonical reduce to

$$\begin{pmatrix} \overline{\Upsilon}_{,1} \\ \overline{\Upsilon}_{,2} \end{pmatrix} = \begin{pmatrix} \widehat{\Upsilon}_{2,1} & -\widehat{\Upsilon}_{1,1} \\ \widehat{\Upsilon}_{2,2} & -\widehat{\Upsilon}_{1,2} \end{pmatrix} \begin{pmatrix} \widehat{\Upsilon}_{1,4} \\ \widehat{\Upsilon}_{2,4} \end{pmatrix}.$$

The solvability condition for this problem is satisfied due to some cancellation and employing the derivative of (4.6) with respect to $\widetilde{\beta}_4$. Thus, $\overline{\Upsilon}$ can be found such that it is odd in $\widetilde{\beta}_2$.

The composition $\Upsilon = \widehat{\Upsilon} \circ \widetilde{\Upsilon}$, where $\widehat{\Upsilon}$ represents the normal form transformation and $\widetilde{\Upsilon}$ represents the Darboux transformation, is the desired coordinate change, and the theorem is proved. ■

In the remainder of this section we will illuminate some of the properties of the reduced system and their relation with the water-wave problem.

The vector field associated with the reduced system is

$$\frac{d}{dx}\beta = J_1 \nabla H(\beta) = \begin{pmatrix} \frac{\partial H}{\partial \beta_2} \\ -\frac{\partial H}{\partial \beta_1} \\ \frac{\partial H}{\partial \beta_4} \\ -\frac{\partial H}{\partial \beta_3} \end{pmatrix} = \begin{pmatrix} H_1(I)\beta_2 \\ -H_1(I)\beta_1 \\ H_2(I) \\ 0 \end{pmatrix} . \tag{4.7}$$

Here and in the sequel we use the abbreviations

$$H_1(I) \stackrel{\text{def}}{=} \frac{\partial \widehat{H}}{\partial I_1} = -k_o + \varepsilon_1 I_2 + 2\varepsilon_2 I_1 + \mathcal{O}(|I|^2), \quad H_2(I) \stackrel{\text{def}}{=} \frac{\partial \widehat{H}}{\partial I_2} = I_2 + \varepsilon_1 I_1 + \mathcal{O}(|I|^2) .$$

Explicit expressions for the normal form coefficients ε_1 and ε_2 are constructed in Appendix C resulting in

$$\begin{aligned} \varepsilon_1 &= -\frac{tk_o}{2bca^2}(3-t^2) \\ \varepsilon_2 &= -\frac{k_o^4}{2ga^4}\left[\frac{9-10t^2+9t^4}{8t^2} - \tfrac{1}{4}(1-t^2)^2 - \tfrac{1}{4}(\tfrac{F^2}{1-F^2})(3-t^2)^2\right] \end{aligned} \tag{4.8}$$

where $t = \tanh k_o h$ and a, b are defined in (3.7).

The flow of the vector field consists of a center in the (β_1, β_2) plane representative of the Stokes periodic travelling wave and a cylinder in the (β_3, β_4) plane which is related to the mean flow of the full system. To see this consider $Z = \beta \cdot \xi + \cdots$ with $\beta_1 = \beta_2 = 0$ resulting in

$$\begin{pmatrix} \Phi(x) \\ \eta(x) \\ \phi(x,y) \\ u(x,y) \end{pmatrix} = \frac{1}{b}\beta_4^o \begin{pmatrix} x \\ c/g \\ x \\ 1 \end{pmatrix} + \frac{1}{b}\beta_3^o \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \cdots .$$

The flow produced by the (β_3, β_4) coordinates introduces a change in the mean velocity and the mean elevation of the fluid. The orbits of (4.7) are given explicitly by

$$\begin{aligned} \beta_1(x) &= \sqrt{2I_1}\cos(H_1 x + \mu_1), & \beta_2(x) &= -\sqrt{2I_1}\sin(H_1 x + \mu_1) \\ \beta_3(x) &= H_2 x + \mu_2, & \beta_4(x) &= I_2 \end{aligned} \tag{4.9}$$

with constants $(\mu_1, \mu_2) \in \mathbb{R}^2$. Note that the solutions in (4.9) are exact solutions of the completely integrable system (4.7) which describes the flow on the center manifold. For $\mu = 0$ this solution is denoted by $\widehat{\beta}(I, x)$. Mostly, we will drop the argument I in H_j and $\widehat{\beta}$. We call $\widehat{\beta}$ periodic with period $L = 2\pi/(-H_1)$, although this is not strictly true as the β_3 component contains a linear term in x.

According to (3.15) the reversibility leads to $\mathcal{R}_0\widehat{\beta}(x) = \widehat{\beta}(-x)$, for $\mu = 0$.

We say that the above integrable system is non-degenerate if $\Delta_K = H_{11}H_{22} - H_{12}H_{21} \neq 0$ (cf. (1.9)). In terms of ε_1 and ε_2 the "KAM determinant" is

$$\begin{aligned} \Delta_K &= 2\varepsilon_2 - \varepsilon_1^2 + \mathcal{O}(|I|) \\ &= -\frac{k_o^4}{ga^4}\left[\frac{9 - 10t^2 + 9t^4}{8t^2} - \tfrac{1}{4}(1-t^2)^2\right] + \mathcal{O}(|I|)\,, \end{aligned} \tag{4.10}$$

and it is easy to verify that $2\varepsilon_2 - \varepsilon_1^2 < 0$ for gravity waves.

The center in the (β_1, β_2) plane represents the Stokes' periodic wavetrain. The wave number of the nonlinear wave is

$$k \stackrel{\text{def}}{=} -H_1(I) = k_o - \varepsilon_1 I_2 - 2\varepsilon_2 I_1 + \cdots\,.$$

The parameter I_2 is related to the restriction of the mass flux to the center manifold. It is clear that the correction to the wave number is dependent on the assigned mass flux. It is interesting to note that the classical weakly nonlinear change in the wave number (cf. Benjamin [1967, eqn. (14)]) is obtained by setting $H_2 = 0$ which implies $I_2 = -\varepsilon_1 I_1 + \cdots$ and then

$$k = k_o - \Delta_K I_1 + \cdots \tag{4.11}$$

with Δ_K given in (4.10). The result in (4.11) is a single line in the (I_1, I_2) plane. In Section 5 and Section 6, the linear stability of the complete two-parameter (I_1, I_2)-family of periodic states will be studied.

5. Reduction of the linear spectral problem

The time-dependent problem (cf. Appendix A), in terms of the Z-coordinates introduced in Section 2, takes the following form

$$\mathbf{M}Z_t + \mathbf{K}(Z)Z_x = \nabla S(Z)\,. \tag{5.1}$$

The time-independent part is equation (2.11). The operator $\mathbf{K}(Z)$ is defined in (2.9) and $\mathbf{M}(\Phi, \eta, \phi, u)^T = (-\eta, \Phi, 0, 0)^T$, see (A.5). Both operators, $\mathbf{M}$ and $\mathbf{K}$, define closed two-forms on $\mathbb{X}$. For $\mathbf{K}$ this was shown above, and for $\mathbf{M}$ this is immediate as it is constant and skew-symmetric. The interplay of these two structures will play an essential role in the stability analysis.

Our aim is now to study the linearized problem around a given solution $\widehat{Z}(x)$ of (5.1) which was constructed as a solution $\widehat{\beta}(x)$ in the previous section. We encounter the difficulty that the spatial symplectic form $\mathbf{K}$ depends on position. Hence, a linearization leads to terms involving the derivative of $\mathbf{K}$ with respect to Z multiplied by Z_x. For the temporal structure $\mathbf{M}$ this problem does not appear, as the basic state is time independent.

We construct a new coordinate system which will have the following advantages: firstly the center manifold will be completely flat and on the center manifold the old and the new coordinates coincide. Secondly the new spatial symplectic form will be constant along the center manifold, such that it has block structure with respect to the center coordinates and the hyperbolic part in $\mathbb{X}_2$. The existence of such a coordinate transformation can be expected from the generalized version of the Darboux Theorem due to Weinstein (cf. MIELKE [1991, Thm. 5.14]). However in the present context the general theorem is not applicable as the two-form $\mathbf{K}$ is only weakly non-degenerate.

Theorem 5.1. *There is a local change of coordinates $Z = \mathcal{T}(\beta, Y_2) : \mathbb{R}^4 \times (\mathcal{D}(\mathbf{A}) \cap \mathbb{X}_2) \to \mathcal{D}(\mathbf{A})$, such that the following properties hold:*

(1) $\mathcal{T}(\beta, 0) = \Upsilon^{-1}(\beta) \cdot \xi + \mathcal{H}(\Upsilon^{-1}(\beta) \cdot \xi))$, i.e., the center manifold is given as $Y_2 \equiv 0$.

(2) Define the transformed spatial symplectic structure $\widehat{\mathbf{K}}$ via

$$\widehat{\mathbf{K}}(\beta, Y_2) = D\mathcal{T}(\beta, Y_2)^* \mathbf{K}(\mathcal{T}(\beta, Y_2)) D\mathcal{T}(\beta, Y_2)$$

then, it satisfies $\widehat{\mathbf{K}}(\beta, Y_2) = \begin{pmatrix} K_1 + \mathcal{O}_2 & \mathcal{O}_2 \\ \mathcal{O}_2 & K_2 + \mathcal{O}_2 \end{pmatrix}$, where $K_1 \in \mathbb{R}^{4 \times 4}$ is as above, K_2 is constant and the terms $\mathcal{O}_2 = \mathcal{O}(\|Y_2\|^2_{\mathcal{D}(\mathbf{A})})$ are uniform in β.

(3) $\widehat{S}(\beta, Y_2) = H(\beta) + \mathcal{O}(\|Y_2\|^2_{\mathcal{D}(A)})$, for $Y_2 \to 0$.

Proof. To satisfy the first property we introduce the coordinates (γ, Y_2) via

$$Z = \Upsilon^{-1}(\gamma) \cdot \xi + \mathcal{H}(\Upsilon^{-1}(\gamma) \cdot \xi)) + Y_2.$$

Obviously, $Y_2 = 0$ corresponds to the center manifold, and on the center manifold the γ-coordinates are identical to the β-coordinates introduced in Section 4. In these coordinates the transformed spatial structure $\widetilde{\mathbf{K}}(\gamma, Y_2)$ takes the form $\widetilde{\mathbf{K}} = \left(\begin{smallmatrix} \widetilde{K}_{11} & \widetilde{K}_{12} \\ \widetilde{K}_{21} & \widetilde{K}_{22} \end{smallmatrix}\right)$.

We do a further change of coordinates, such that $K_{12} = \mathcal{O}(\|Y_2\|^2)$. Then all remaining properties in 2) will follow. The coordinate change will only involve γ and keep Y_2 fixed. We let $\gamma = B(\beta, Y_2) = \beta + a(\beta) Y_2 + b(\beta)[Y_2, Y_2]$ where the linear operator a and the symmetric bilinear form b have to be found. Let $\widehat{\mathbf{K}} = \left(\begin{smallmatrix} \widehat{K}_{11} & \widehat{K}_{12} \\ \widehat{K}_{21} & \widehat{K}_{22} \end{smallmatrix}\right)$ be the spatial structure in (β, Y_2) coordinates, then $\widehat{K}_{12} = D_\beta B^* \left[\widetilde{K}_{11} D_{Y_2} B + \widetilde{K}_{12} \right]$. As $D_\beta B$ is close to the identity, we have to make the sum in the square bracket of order $\mathcal{O}(\|Y_2\|^2)$. This leads to the following conditions on a and b:

$$\widetilde{K}_{11}(\beta, 0) a(\beta) = -\widetilde{K}_{12}(\beta, 0)\,,$$
$$2\widetilde{K}_{11} b[V_2, W_2] = -\big(D_\beta \widetilde{K}_{11}[aV_2] D_{Y_2} \widetilde{K}_{11}[V_2]\big) a W_2 - \big(D_\beta \widetilde{K}_{12}[aV_2] D_{Y_2} \widetilde{K}_{12}[V_2]\big) W_2$$

for all V_2, $W_2 \in \mathbb{X}_2$. These relations determine a and b uniquely. Note that the symmetry in V_2 and W_2 of the right-hand side in the second equation follows from the closedness of the form $\widetilde{K}$. Hence, the new spatial structure satisfies

$$\widehat{K}_{11}(\beta, Y_2) = K_1 + \mathcal{O}(\|Y_2\|), \quad \widehat{K}_{12}(\beta, Y_2) = \widehat{K}^*_{21}(\beta, Y_2) = \mathcal{O}(\|Y_2\|^2). \tag{5.2}$$

The first follows since $B(\beta, 0) = \beta$ and from the construction in Section 4.

In general, the closedness of the two-form $\widehat{K}$ is equivalent to

$$\begin{aligned}\langle U_1, D_{(\beta,Y_2)}\widehat{K}(\beta, Y_2)[U_2]U_3\rangle + \langle U_2, D_{(\beta,Y_2)}\widehat{K}(\beta, Y_2)[U_3]U_1\rangle \\ + \langle U_3, D_{(\beta,Y_2)}\widehat{K}(\beta, Y_2)[U_1]U_2\rangle = 0\end{aligned}$$

for all $U_j \in \mathbb{R}^4 \times \mathbb{X}_2$. Letting $Y_2 = 0$, $U_j = (u_j, 0)$ for $j = 1, 2$ and $U_3 = (0, V_2)$ we conclude with (5.2) that $D_{Y_2}\widehat{K}_{11}(\beta, 0) = 0$ implying $\widehat{K}_{11} = K_1 + \mathcal{O}(\|Y_2\|^2)$. Similarly, we may take $U_1 = (u, 0)$ and $U_j = (0, V_{2j})$ for $j = 2, 3$, then $D_\beta\widetilde{K}_{22}(\beta, 0) = 0$. Thus, property 2) is established.

It remains to establish property 3). However, this is a consequence of the invariance of the center manifold $Y_2 \equiv 0$. Note that the spatial problem has the form

$$\begin{pmatrix} \widehat{K}_{11} & \widehat{K}_{12} \\ \widehat{K}_{21} & \widehat{K}_{22} \end{pmatrix} \begin{pmatrix} \beta \\ Y_2 \end{pmatrix}_x = \begin{pmatrix} D_\beta\widehat{S}(\beta, Y_2) \\ D_{Y_2}\widehat{S}(\beta, Y_2) \end{pmatrix}.$$

At $Y_2 = 0$ the structure $\widehat{K}$ decouples and $Y_2 = 0$ can only be invariant if $D_{Y_2}\widehat{S}(\beta, 0) = 0$ for all β. ■

Using the transformation of Theorem 5.1 we may now consider the time-dependent problem

$$\widehat{\mathbf{M}}(\beta, Y_2)\begin{pmatrix} \beta \\ Y_2 \end{pmatrix}_t + \widehat{\mathbf{K}}(\beta, Y_2)\begin{pmatrix} \beta \\ Y_2 \end{pmatrix}_x = \nabla\widehat{S}(\beta, Y_2). \tag{5.3}$$

The basic solutions underlying our stability analysis are now given by $(\beta, Y_2) = (\widehat{\beta}(x), 0)$, with $\widehat{\beta}$ as defined in (4.9). To linearize we insert $(\beta, Y_2) = (\widehat{\beta}(x) + \widehat{V}_1, \widehat{V}_2)$ into (5.3) and retain only linear terms in $(\widehat{V}_1, \widehat{V}_2)$. As the resulting linear equation is autonomous in time we may look for solutions of the form $(\widehat{V}_1, \widehat{V}_2)(t, x) = e^{\lambda t}(V_1(x), V_2(x))$, $\lambda \in \mathbb{C}$, leading to the spectral problem

$$\widehat{\mathbf{K}}_0\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}_x = \left[\begin{pmatrix} D^2_\beta H(\widehat{\beta}(x)) & 0 \\ 0 & D^2_{Y_2}\widehat{S}(\widehat{\beta}(x), 0) \end{pmatrix} - \lambda\widehat{\mathbf{M}}(\widehat{\beta}(x), 0)\right]\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}. \tag{5.4}$$

Note that $\widehat{\mathbf{K}}_0 = \left(\begin{smallmatrix} K_1 & 0 \\ 0 & K_2 \end{smallmatrix}\right)$ is constant and has the inverse $\widehat{\mathbf{J}}_0 = \left(\begin{smallmatrix} J_1 & 0 \\ 0 & J_2 \end{smallmatrix}\right)$ with

$$K_1 = J_1^{-1} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in \mathbb{R}^{4\times 4}.$$

Without displaying explicitly, we work, from now on, with the complexified problem: for general $\lambda \in \mathbb{C}$ we have to allow for $V_1 \in \mathbb{C}^4$ and $V_2 \in \mathbb{X}_2 \oplus i\mathbb{X}_2$.

Definition. *A solution $\widehat{Z}(\cdot) = \mathcal{T}(\widehat{\beta}(\cdot), 0)$ of (1.10) or (5.1) is called* **linearly unstable**, *if the spectral problem (5.4) admits a nontrivial bounded solution (λ, V_1, V_2) with* $\mathrm{Re}(\lambda) > 0$.

For fixed $\lambda \in \mathbb{C}$ this spectral problem is again an elliptic problem written in spatial dynamics form. Moreover, for small $|\lambda|$ and small $\|\widehat{\beta}(\cdot)\|_\infty$ it is a small perturbation of the linear problem studied in Section 3. Hence, it admits a linear center manifold. According to the definition of the Stokes family the size of the solutions $\widehat{\beta}$ are controlled by $I = (I_1, I_2)$. Note also, that the solutions $\widehat{\beta}(x)$ are not periodic in x (unless $H_2(I) = 0$). However, the spectral problem has purely periodic coefficients as the nonperiodic part $\widehat{\beta}_3$ does not appear.

Theorem 5.2. *The eigenvalue problem (5.4) has, for sufficiently small $(\lambda, I) \in \mathbb{C} \times \mathbb{R}^2$ a unique linear center manifold $V_2 = g(\lambda, I, x)V_1$ with $g(\lambda, I, x) \in \mathcal{L}(\mathbb{X}_1, \mathbb{X}_2 \cap \mathcal{D}(\mathbf{A}))$ such that every bounded solution of (5.4) for (λ, I) sufficiently small is in one-to-one correspondence with a bounded solution of*

$$K_1 \frac{d}{dx} V_1 = \left[D_\beta^2 H(\widehat{\beta}(x)) - \lambda M_{11}(\widehat{\beta}(x)) - \lambda M_{12}(\widehat{\beta}(x)) g(\lambda, I, x) \right] V_1 \,, \tag{5.5}$$

where $M_{ij}(\beta)$ is defined via $\widehat{\mathbf{M}}(\beta, 0) = \begin{pmatrix} M_{11}(\beta) & M_{12}(\beta) \\ M_{21}(\beta) & M_{22}(\beta) \end{pmatrix}$.

The reduction function $g(\lambda, I, \cdot)$ is periodic in x with the same period as $\widehat{\beta}(I, \cdot)$ and has the expansion $g(\lambda, I, x) = \lambda g_1(I, x) + \mathcal{O}(|\lambda|^2)$, where g_1 satisfies

$$\frac{d}{dx} g_1 + g_1 J_1 D_\beta^2 H(\widehat{\beta}(x)) - J_2 D_{Y_2}^2 \widehat{S}(\widehat{\beta}(x), 0) g_1 = -J_2 M_{21}(\widehat{\beta}(x)) \,. \tag{5.6}$$

Remark. The uniqueness is unusual for a center manifold but is obtained in this case due to linearity. Note that $g(\lambda, I, x)$ depends nonlocally on the function $\widehat{\beta}(I, \cdot)$, so it is not possible to write $g = g(\lambda, \widehat{\beta}(x))$. Notice also that the reduced spectral problem is almost the projection of the full problem onto the V_1 part. The corrections appearing through the rigorous center manifold reduction process appear only in terms of order λ^2. However, for the Stokes waves instability problem we will need information up to this order.

Proof. The method is a variation of the proof of Theorem 4.1. The main technical difficulty is to transform the vector field in such a way that the center manifold theory for non-autonomous vector fields (Mielke [1988]) applies.

In order to be able to appeal to our center manifold construction in the nonlinear case we transform the spectral problem (5.4) back into $\mathbb{X}$ coordinates via $V = V_1 \cdot \xi + V_2 = (\widetilde{\Phi}, \widetilde{\eta}, \widetilde{\phi}, \widetilde{u}) \in \mathbb{X} \oplus i\mathbb{X}$. Then, V satisfies the spectral problem

$$V_x = \mathbf{A}V + B(\lambda, I, x)V, \tag{5.7}$$

where $B(\lambda, I, x) \in \mathcal{L}(\mathcal{D}(\mathbf{A}), \mathbb{X})$ with small norm for (λ, I) small. The functional $Q(Z)$ can no longer be used to eliminate η as Q is no longer a conserved quantity when $\lambda \neq 0$. Nevertheless, we construct a linear x-periodic coordinate transformation $W = \mathbf{T}(\lambda, I, x)V$ such that the transformed system

$$W_x = \left[\frac{\partial \mathbf{T}}{\partial x} + \mathbf{T}(\mathbf{A} + B(\lambda, I, x)) \right] \mathbf{T}^{-1} W \tag{5.8}$$

is of the form $W_x = [\widehat{\mathbf{A}} + \widehat{B}(\lambda, I, x)]W$ where $\widehat{\mathbf{A}}$ is the linear operator defined in Lemma 4.2 and $\widehat{B}(\lambda, I, x) \in \mathcal{L}(\mathcal{D}(\widehat{\mathbf{A}}), \mathbb{X}_0)$. Then Theorem 1 of Mielke [1988] is applicable yielding the linear center manifold for (5.7). The result in the original coordinates is given by inverting $\mathbf{T}$. The smoothness and periodicity of g are obtained from the theory of non-autonomous center manifolds (cf. Mielke [1988]).

We define $W = (\widetilde{\Phi}, \widetilde{\delta}, \widetilde{\phi}, \widetilde{v})$ implicitly via

$$\widetilde{\delta} = [\widetilde{u}] - c\widetilde{\eta}, \quad \widetilde{u} = \widetilde{v} + \frac{2y}{h^2} f(\lambda, I, x)W, \tag{5.9}$$

where $f(\lambda, I, x) \in \mathcal{L}(\mathbb{X}_0, \mathbb{C})$ will be chosen suitably. The compatibility condition $[\mathbf{A} + B(\lambda, I, x)]V \in \mathbb{X}_0$ reads

$$\frac{g}{c}\widetilde{\eta} + \widetilde{u}(0) + b(\lambda, I, x)V = 0, \tag{5.10}$$

where $b(\lambda, I, x) \in \mathcal{L}(\mathbb{X}, \mathbb{C})$ is periodic in x and small. Inserting (5.9) into (5.10) shows that (5.10) is equivalent to $hF^2\widetilde{v}(0) = [\widetilde{v}]$ if and only if

$$\frac{g}{c^2}(\widetilde{\delta} + fW) + b(\lambda, I, x) \begin{pmatrix} \widetilde{\phi}(0) \\ (\widetilde{\delta} - [\widetilde{v}] + fW)/c \\ \widetilde{\phi} \\ \widetilde{v} + \frac{2y}{h^2} fW \end{pmatrix} = 0 .$$

From the smallness of b we deduce that $f(\lambda, I, x)$ is uniquely defined. ■

A corollary of Theorem 5.2 is the "principle of reduced linear instability". It ensures that the linear instability problem *for* $|\lambda|$ *small* can be decided by the analysis of an ordinary differential equation. We cannot make any claim regarding stability as the theory cannot rule out the possibility of unstable modes with $|\lambda|$ outside a neighborhood of the origin.

Corollary 5.3 (Principle of reduced linear instability). *Suppose there exists a nontrivial bounded solution* $V_1 \in \mathbb{C}^4$ *of the finite-dimensional problem (5.5) with* $\mathrm{Re}(\lambda) > 0$. *Then the basic stationary state* $\widehat{Z}$ *of the infinite-dimensional problem (5.1) is linearly unstable.*

In the next section we will show how bounded solution of (5.5) can be constructed by using Floquet theory. The task is to gather sufficient information about the reduced spectral problem (5.5). Our approach will involve a geometrical argument which uses the whole four-dimensional Stokes family $\beta(x) = \widehat{\beta}(I_1, I_2, \mu_1, \mu_2, x)$ as defined in (4.9).

Remark. Formally, one would expect Floquet's Theorem to be applicable to the full infinite-dimensional system (5.4); that is, that every bounded solution can be decomposed into the product of an exponential $e^{i\sigma x}$ with $\sigma \in \mathbb{R}$ and a periodic function with the same period as the basic state. However, (5.4) is an elliptic system, and for such problems Floquet's Theorem has difficulties: it is not true in general that every bounded solution of an elliptic system with periodic coefficients can be decomposed into an (infinite) sum of products of a periodic function (of the same period as the coefficients) and an exponential with purely imaginary exponent. Counterexamples are given in KUCHMENT [1993] and MIELKE [1992]. In order to invoke the Floquet decomposition it is necessary to prove its validity *a priori*, we do this by applying spatial center manifold theory to (5.4). In this way we are able to reduce the eigenvalue problem (5.4), for small λ, to the finite-dimensional ordinary differential equation (5.5) where Floquet's Theorem is immediate and the reduced eigenvalue problem can be completely analyzed.

6. Analysis of the reduced stability problem

The principle of reduced linear instability reduces the analysis of the spectral problem to an ordinary differential equation on $\mathbb{C}^4$, which is given in (5.5). Although the basic state is not strictly L-periodic (cf. equation (4.9)) the Hamiltonian functional is independent of β_3 and therefore the x-dependent coefficients in the spectral problem (5.5) are L-periodic. Floquet's Theorem is therefore applicable to (5.5) resulting in bounded solutions of the form

$$V_1(x) = e^{i\sigma x} U(x) \quad \text{with } \sigma \in \mathbb{R} \quad \text{and} \quad U(x+L) = U(x) \in \mathbb{C}^4$$

where the period L depends on (I_1, I_2) via the definition $L = -2\pi/H_1(I)$.

We denote by $\mathbb{Y}^j \stackrel{\text{def}}{=} \mathcal{C}^j_{\text{per}}([0,L], \mathbb{C}^4)$ for $j = 0, 1$ and introduce the symmetric bilinear pairing

$$[\![U, V]\!] \stackrel{\text{def}}{=} \frac{1}{L} \int_0^L (U(x), V(x))\, dx$$

where $(\cdot,\cdot)$ is the usual inner product on $\mathbb{C}^4$.

It follows from Floquet's Theorem that (5.5) has a bounded solution $(\lambda, V_1(x))$ if and only if the following problem has a nontrivial solution $(\lambda, \sigma, U) \in \mathbb{C} \times \mathbb{R} \times \mathbb{Y}^1$:

$$\mathcal{L}_I U = \mathcal{G}(\lambda, \sigma, I, x) U \tag{6.1}$$

where

$$\mathcal{L}_I U \stackrel{\text{def}}{=} K_1 \frac{d}{dx} U - D_\beta^2 H(\widehat{\beta}(x)) U\,, \quad \mathcal{L}_I : \mathbb{Y}^1 \to \mathbb{Y}^0$$

and

$$\mathcal{G}(\lambda, \sigma, I, x) \stackrel{\text{def}}{=} -i\sigma K_1 - \lambda M_{11}(\widehat{\beta}(x)) - \lambda M_{12}(\widehat{\beta}(x)) g(\lambda, I, x)\,.$$

The symmetry of the operator $\mathcal{L}_I$ with respect to the inner product $[\![\cdot,\cdot]\!]$ follows immediately from the symmetry of D^2H and the antisymmetry of K_1 and d/dx. Fundamental to the analysis is the fact that the linear operator $\mathcal{L}_I$ can be completely analyzed.

Proposition 6.1. *Let $\mathcal{L}_I$ be as defined above and suppose $I_1 \neq 0$.*

(a) The operator $\mathcal{L}_I$ has a two-dimensional kernel spanned by ϕ_1 and ϕ_2 given by

$$\phi_j = \frac{\partial}{\partial \mu_j} \widehat{\beta}(x), \quad \text{namely } \phi_1(x) = -\sqrt{2I_1} \begin{pmatrix} \sin(H_1 x) \\ \cos(H_1 x) \\ 0 \\ 0 \end{pmatrix}, \ \phi_2(x) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

(b) The operator $\mathcal{L}_I$ has a unique small eigenvalue $\mu(I) = \mathcal{O}(I_1)$ with associated eigenfunction

$$\psi_I(x) = \begin{pmatrix} \cos(H_1 x) \\ -\sin(H_1 x) \\ 0 \\ \rho(I) \end{pmatrix} = \frac{1}{\sqrt{2I_1}} K_1 \phi_1 + \rho(I) K_1 \phi_2$$

where

$$\begin{pmatrix} H_{22} & \sqrt{2I_1} H_{12} \\ \sqrt{2I_1} H_{12} & 2I_1 H_{11} \end{pmatrix} \begin{pmatrix} \rho(I) \\ 1 \end{pmatrix} = -\mu(I) \begin{pmatrix} \rho(I) \\ 1 \end{pmatrix}.$$

(c) Let $\mathbb{W}^j = \{U \in \mathbb{Y}^j \ : \ [\![U, \phi_1]\!] = [\![U, \phi_2]\!] = 0\}$. Then, $\mathcal{L}_I U = f$ is solvable if and only if $f \in \mathbb{W}^0$. For all $f \in \mathbb{W}^0$ there is a unique solution $U \in \mathbb{W}^1$, which has the expansion

$$U = \frac{[\![\psi_I, f]\!]}{\mu(I) [\![\psi_I, \psi_I]\!]} \psi_I + G_0 f + \mathcal{O}(\sqrt{2I_1}). \tag{6.2}$$

Here $V = G_0 f$ is the unique solution of

$$\mathcal{L}_{I^*} V = f - \frac{[\![\psi_{I^*}, f]\!]}{[\![\psi_{I^*}, \psi_{I^*}]\!]} \psi_{I^*} \quad \text{and } [\![\psi_{I^*}, V]\!] = 0\,,$$

where $I^* = (0, I_2^*(I))$. I^* *is uniquely defined via* $H_1(0, I_2^*(I)) = H_1(I)$.

Proof. Part (a) and (b) are immediate by direct calculations, except for the uniqueness of the small eigenvalue. This can be seen by considering the limit $I_1 \to 0$, where I_2 is adjusted such that $H_1(I_1, I_2)$ remains fixed ($H_{12}|_{I=0} = \varepsilon_1 \neq 0$).

The operator $\mathcal{L}_{I^*}$ is a constant coefficient problem which has the three-dimensional kernel span$\{\phi_1, \phi_2, \psi_{I^*}\} \subset \mathbb{Y}^1$. As the eigenvalues are discrete and depend continuously on I_1 the uniqueness is established.

The solvability conditions in part (c) for $\mathcal{L}_I U = f$ are a direct consequence of the Fredholm property of the ordinary differential operator. To establish the expansion for small I_1 we remark that it is possible to construct the Green's function for the problem exactly, as the problem can be transformed into a constant coefficient problem by a suitable rotation in the (U_1, U_2) coordinates. ∎

For later reference note the expansions

$$\begin{aligned} \rho(I) &= -\frac{H_{12}}{H_{22}}\sqrt{2I_1} - \frac{\Delta_K H_{12}}{H_{22}^3}(2I_1)^{3/2} + \mathcal{O}(I_1^{5/2}), \\ \mu(I) &= -\frac{\Delta_K}{H_{22}}2I_1 + \frac{\Delta_K H_{12}^2}{H_{22}^3}(2I_1)^2 + \mathcal{O}(I_1^3), \\ \frac{1}{\mu[\![\psi_I, \psi_I]\!]} &= \frac{1}{\mu(1+\rho^2)} = -\frac{H_{22}}{2I_1\Delta_K}\left(1 - 2\frac{H_{12}^2}{H_{22}^2}2I_1\right) + \mathcal{O}(I_1). \end{aligned} \tag{6.3}$$

The analysis proceeds by studying (6.1) using the Lyapunov-Schmidt method. The functions ϕ_1 and ϕ_2 are tangent vectors to the orbit of the basic state $\widehat{\beta}$ and the idea is to project the linear stability problem onto the tangent space of this orbit (cf. Bridges & Rowland [1994], Bridges [1994b,c]). The solution of (6.1) is sought in the form $\tau_1\phi_1 + \tau_2\phi_2 + W$ where $(\tau_1, \tau_2) \in \mathbb{C}^2$ and $W \in \mathbb{W}^1$. The Lyapunov-Schmidt reduction proceeds as follows.

Let $\mathbf{E} : \mathbb{Y}^j \to \mathbb{Y}^j$ be the unique symmetric projection with range $\mathbb{W}^j$. Then $\mathbf{E}\mathcal{L}_I = \mathcal{L}_I\mathbf{E}$ and the spectral problem (6.1) is split into the pair of equations

$$\begin{aligned} 0 =& \mathcal{L}_I W - \mathbf{E}\mathcal{G}(\lambda, \sigma, I, \cdot)[\tau_1\phi_1 + \tau_2\phi_2 + W]; \\ 0 =& (\mathbf{I} - \mathbf{E})\mathcal{G}(\lambda, \sigma, I, \cdot)[\tau_1\phi_1 + \tau_2\phi_2 + W]. \end{aligned} \tag{6.4}$$

The first equation of (6.4) can be solved uniquely, via the implicit function theorem for W as a function of $(\lambda, \sigma, I, \tau)$ and moreover, due to the linearity of (6.1), the dependence of W on τ is linear: $W = \tau_1 W_1(\lambda, \sigma, I) + \tau_2 W_2(\lambda, \sigma, I)$.

The second equation of (6.4) is two-dimensional and contains the solvability conditions for (6.1). Inserting the function W obtained from the first equation these conditions take the form

$$\mathbf{N}(\lambda,\sigma,I)\begin{pmatrix}\tau_1\\ \tau_2\end{pmatrix} = 0, \text{ with } \mathbf{N}_{kl}(\lambda,\sigma,I) = [\![\phi_k, \mathcal{G}(\lambda,\sigma,I,\cdot)[\phi_l + W_l(\lambda,\sigma,I,\cdot)]]\!] \qquad (6.5)$$

where $\mathbf{N}(\lambda,\sigma,I)$ is a 2×2 matrix with complex-valued entries.

To find a nontrivial solution $U = \tau_1(\phi_1 + W_1) + \tau_2(\phi_2 + W_2)$, satisfying (6.1), it remains to find a nontrivial τ satisfying (6.5). Therefore set

$$\Lambda(\lambda,\sigma,I) = \det(\mathbf{N}(\lambda,\sigma,I)) \,. \qquad (6.6)$$

There exists a nontrivial solution of (6.5) if and only if there exists $(\lambda,\sigma,I)\in \mathbb{C}\times\mathbb{R}\times\mathbb{R}^2$, with $(|\lambda| + |\sigma| + |I|)$ sufficiently small, satisfying $\Lambda(\lambda,\sigma,I) = 0$. The smallness of λ and I is required by the center-manifold reduction and the smallness of $|\sigma|$ is required by the Lyapunov-Schmidt reduction.

The equation $\Lambda(\lambda,\sigma,I) = 0$, for given I, has the physical interpretation of a dispersion relation. For given values of the sideband wave number σ we study the temporal eigenvalues $\lambda\in\mathbb{C}$.

Lemma 6.2. *For any fixed I, $\partial_\lambda \mathbf{N}(0,0,I) = \partial_\sigma \mathbf{N}(0,0,I) = 0$.*

Proof. According to the Liapunov-Schmidt reduction we know $W_k = \mathcal{O}(|\lambda| + |\sigma|)$ as the same is true for $\mathcal{G}$. Hence,

$$\begin{aligned} N_{kl} &= [\![\phi_k, \mathcal{G}(\lambda,\sigma,I)[\phi_l + W_l]]\!] \\ &= -i\sigma[\![\phi_k, K_1\phi_l]\!] - \lambda[\![\phi_k, M_{11}(\widehat{\beta}(\cdot))\phi_l]\!] + \mathcal{O}(|\lambda|^2 + \sigma^2). \end{aligned}$$

The σ-term vanishes as K_1 is antisymmetric implying $[\![\phi_j, K_1\phi_j]\!] = 0$ for $j = 1,2$. Moreover, ϕ_1 has zero mean over the interval $[0,L]$, while $K_1\phi_2$ is constant, which results in $[\![\phi_1, K_1\phi_2]\!] = [\![\phi_2, K_1\phi_1]\!] = 0$.

Similarly we find $[\![\phi_j, M_{11}\phi_j]\!] = 0$. To see that the off-diagonal terms are also zero we transform the variables β on the center manifold back into the full problem in Z-coordinates. Let $Z = \mathcal{T}(\beta, Y_2)$ be the coordinate change from Theorem 5.1 such that $\widehat{Z}(x) = \mathcal{T}(\widehat{\beta}(I,\mu,x),0)$ is the Stokes waves in the original coordinates. Then by the definition of M_{11} we have

$$\begin{aligned} [\![\phi_1, M_{11}(\widehat{\beta}(x))\phi_2]\!] &= \frac{1}{L}\int_0^L \langle D\mathcal{T}(\widehat{\beta}(x),0)\phi_1, \mathbf{M}D\mathcal{T}(\widehat{\beta}(x),0)\phi_2\rangle_{\mathbb{X}} dx \\ &= \frac{1}{L}\int_0^L \langle \frac{\partial}{\partial\mu_1}\widehat{Z}(x), \mathbf{M}\frac{\partial}{\partial\mu_2}\widehat{Z}(x)\rangle dx = 0\,, \end{aligned}$$

as $\partial_{\mu_1}\widehat{Z}$ has zero mean whereas $\mathbf{M}\partial_{\mu_1}\widehat{Z}$ is constant. ■

For the analysis of the quadratic (λ,σ)-terms in $\mathbf{N}(\lambda,\sigma,I)$ the derivatives of W_1 and W_2 with respect to λ and σ are needed. Define the functions ϕ_3, ϕ_4, ϕ_5 and ϕ_6 by

$$\mathcal{L}_I\phi_{k+2} = K_1\phi_k \quad \text{and } \mathcal{L}_I\phi_{k+4} = M_{11}(\widehat{\beta})\phi_k \qquad \text{for } k=1,2\,. \tag{6.7}$$

Then with $W = \tau_1 W_1 + \tau_2 W_2$ it is evident that

$$\left.\frac{\partial W_k}{\partial \sigma}\right|_{\lambda=\sigma=0} = -i\phi_{k+2} \quad \text{and} \quad \left.\frac{\partial W_k}{\partial \lambda}\right|_{\lambda=\sigma=0} = -\phi_{k+4} \qquad \text{for } k=1,2\,.$$

The functions ϕ_3 and ϕ_4 have an interesting characterization in terms of the derivative of the basic state $\widehat{\beta}$ with respect to I (cf. Appendix E) and are given explicitly by

$$\phi_3 = -\frac{H_{22}}{2I_1\Delta_K}K_1\phi_1 + \frac{H_{12}}{\Delta_K}K_1\phi_2, \quad \phi_4 = \frac{H_{12}}{2I_1\Delta_K}K_1\phi_1 - \frac{H_{11}}{\Delta_K}K_1\phi_2.$$

Note that these expressions are compatible with part (c) of Proposition 6.1.

The functions ϕ_5 and ϕ_6 are constructed using Proposition 6.1(c) and the expansion of $M_{11}(\widehat{\beta})$ given in Appendix D;

$$\begin{aligned} \phi_5 &= -\frac{H_{22}}{\Delta_K\sqrt{2I_1}}\varepsilon_5\psi_I + \sqrt{2I_1}\widehat{\phi}_5 + \mathcal{O}(2I_1) \\ \phi_6 &= -\frac{H_{22}}{\Delta_K\sqrt{2I_1}}\varepsilon_6\psi_I + \widehat{\phi}_6 + \mathcal{O}(\sqrt{2I_1}) \end{aligned} \tag{6.8}$$

where

$$\begin{aligned} \varepsilon_5 &\stackrel{\text{def}}{=} \frac{1}{\sqrt{2I_1}}[\![\psi_I, M_{11}(\widehat{\beta})\phi_1]\!] = -\frac{1}{c-c_g} + \mathcal{O}(2I_1) \\ \varepsilon_6 &\stackrel{\text{def}}{=} \frac{1}{\sqrt{2I_1}}[\![\psi_I, M_{11}(\widehat{\beta})\phi_2]\!] = -\varepsilon_1\frac{c}{gb^2} + \tfrac{1}{2}\varepsilon_3 + \mathcal{O}(2I_1) \end{aligned} \tag{6.9}$$

and

$$\varepsilon_3 = -\frac{hk_o^2}{ga^2b^3}(1-t^2+2F^2) = \frac{k_o}{gb^3}\left(\frac{2c_g+c}{c-c_g}\right)\,. \tag{6.10}$$

The expression for ε_3 is derived in Appendix D from the expansion of $M_{11}(\widehat{\beta})$. The functions $\widehat{\phi}_5$ and $\widehat{\phi}_6$ are the G_0-part of ϕ_5 and ϕ_6 respectively and are given by

$$\begin{aligned} \widehat{\phi}_5 &= G_0M_{11}(0)\phi_1 = \frac{c}{gk_oab}\begin{pmatrix} 0 \\ 0 \\ 2\sin(H_1x) \\ -k_o\cos(H_1x) \end{pmatrix} \\ \widehat{\phi}_6 &= G_0M_{11}(0)\phi_2 = \frac{c}{gab^2}\begin{pmatrix} b \\ 0 \\ 0 \\ -a \end{pmatrix}\,. \end{aligned} \tag{6.11}$$

We are now in a position to study the quadratic (λ, σ)-terms in the matrix $\mathbf{N}(\lambda, \sigma, I)$.

Lemma 6.3. *Let*

$$\mathbf{N}(\lambda, \sigma, I) = \lambda^2 \mathbf{N}^{(1)}(I) + i\sigma\lambda \mathbf{N}^{(2)}(I) + (i\sigma)^2 \mathbf{N}^{(3)}(I) + \mathcal{O}(|\lambda|^3 + |\sigma|^3),$$

then the matrices $\mathbf{N}^{(j)}$ are real symmetric matrices with

$$\begin{aligned} \mathbf{N}_{kl}^{(1)} &= [\![\phi_k, M_{11}(\widehat{\beta}(\cdot))\phi_{l+4} + M_{12}(\widehat{\beta}(\cdot))V_l]\!]\,, \\ \mathbf{N}_{kl}^{(2)} &= [\![\phi_k, K_1\phi_{l+4}]\!] + [\![\phi_l, K_1\phi_{k+4}]\!]\,; \\ \mathbf{N}^{(3)}(I) &= \begin{pmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{pmatrix}^{-1} = \frac{1}{\Delta_K}\begin{pmatrix} H_{22} & -H_{12} \\ -H_{12} & H_{11} \end{pmatrix}\,; \end{aligned} \tag{6.12}$$

where $V_l \in \mathcal{C}^1_{\text{per}}([0, L], \mathbb{X}_2)$ satisfies

$$K_2 \frac{d}{dx} V_l - D^2_{Y_2} S(\widehat{\beta}(x), 0)V_l = M_{21}(\widehat{\beta}(x))\phi_l\,. \tag{6.13}$$

Proof. The definition of $\mathbf{N}_{kl} = [\![\phi_k, \mathcal{G}(\lambda, \sigma, I, \cdot)[\phi_l + W_l]]\!]$ and $W_l = -i\sigma\phi_{l+2} - \lambda\phi_{l+4} +$ $h.o.t.$ leads to

$$\begin{aligned} \mathbf{N}_{kl}^{(1)} &= [\![\phi_k, M_{11}\phi_{l+4} - M_{12}g_1(I, \cdot)\phi_l]\!]\,, \\ \mathbf{N}_{kl}^{(2)} &= [\![\phi_k, K_1\phi_{l+4} + M_{11}\phi_{l+2}]\!]\,, \qquad \mathbf{N}_{kl}^{(3)} = [\![\phi_k, K_1\phi_{l+2}]\!]\,, \end{aligned}$$

where $g_1(I, x) \in \mathcal{L}(\mathbb{C}^4, \mathbb{X}_2)$ was defined in Theorem 5.2.

The given formulae for $\mathbf{N}^{(3)}$ follows now by an explicit calculation. The coefficients of $\mathbf{N}^{(2)}$ can be rewritten using properties of the spatial and temporal symplectic structures K_1 and M_{11} and the symmetry of the operator $\mathcal{L}_I$:

$$\begin{aligned} [\![\phi_k, M_{11}\phi_{l+2}]\!] &= -\,[\![M_{11}\phi_k, \phi_{l+2}]\!] = -[\![\mathcal{L}_I\phi_{k+4}, \phi_{l+2}]\!] \\ &= -\,[\![\phi_{k+4}, \mathcal{L}_I\phi_{l+2}]\!] = -[\![\phi_{k+4}, K_1\phi_l]\!] = [\![\phi_l, K_1\phi_{k+4}]\!]\,. \end{aligned}$$

It remains to justify the representation and the symmetry of $\mathbf{N}^{(1)}$. Letting $V_l = -g_1\phi_l$ and using (5.6) we immediately find (6.13). Moreover, using $M_{12}^* = -M_{21}$ leads to

$$\begin{aligned} [\![\phi_k, M_{11}\phi_{l+4} + M_{12}V_l]\!] &= -[\![M_{11}\phi_k, \phi_{l+4}]\!] - \frac{1}{L}\int_0^L \langle M_{21}\phi_k, V_l\rangle dx \\ &= -\,[\![\mathcal{L}_I\phi_{k+4}, \phi_{l+4}]\!] - \frac{1}{L}\int_0^L \langle K_2\frac{d}{dx}V_k - D^2_{Y_2}SV_k, V_l\rangle dx\,, \end{aligned}$$

which is symmetric in k and l. ■

The next step is to find expansions of the matrices $\mathbf{N}^{(j)}(I)$ in order to express the lowest order terms of the dispersion relation $\Lambda(\lambda,\sigma,I)$. Therefore, we first note that all entries in the matrices will have an expansion in even powers of $\sqrt{2I_1}$. This is easily seen, as the factor $\sqrt{2I_1}$ is always associated with $\sin(H_1x)$ or $\cos(H_1x)$ and the scalar product cancels all terms with odd powers of the basic harmonics. By Lemma 6.2 we know that ϕ_j for $j=3,\ldots,6$ are of order $(2I_1)^{-1/2}$, while $\phi_1=\mathcal{O}(\sqrt{2I_1})$ and $\phi_2=\mathcal{O}(1)$. It is therefore easy to see that no $\mathbf{N}^{(j)}(I)$ has an entry of order I_1^{-1}. Recall that $\mathbf{N}^{(3)}(I)$ is known explicitly. For the other two matrices the above expansions of the ϕ_j's and the expansion of M_{11} in Appendix D give the following result. However, note that the coefficients H_{11}, H_{12}, H_{22}, Δ_K and ε_5 as defined above depend on I_1. For all five functions we know the value when $I_1=0$ and can avoid the calculation of the next term in the expansion by grouping the terms appropriately. In fact the following analysis relies heavily on this delicate cancellation properties, otherwise the straightforward expansion of all desired terms would lead to extremely long calculations. These problems are avoided by using the special structure of the problem.

Proposition 6.4.

$$\begin{aligned}\mathbf{N}^{(1)}(I) &= \frac{H_{22}}{\Delta_K}\begin{pmatrix}\varepsilon_5^2+\nu_o\Delta_K\,2I_1 & \varepsilon_5\varepsilon_6\\ \varepsilon_5\varepsilon_6 & \varepsilon_6^2-\frac{\Delta_K}{gb^2}\end{pmatrix}+\begin{pmatrix}\mathcal{O}((2I_1)^2) & \mathcal{O}(2I_1)\\ \mathcal{O}(2I_1) & \mathcal{O}(2I_1)\end{pmatrix}\\ \mathbf{N}^{(2)}(I) &= \frac{H_{22}}{\Delta_K}\begin{pmatrix}2\varepsilon_5 & \varepsilon_6-\varepsilon_1\varepsilon_5\\ \varepsilon_6-\varepsilon_1\varepsilon_5 & -2\varepsilon_1\varepsilon_6+2\frac{c}{gb^2}\Delta_K\end{pmatrix}+\begin{pmatrix}\mathcal{O}((2I_1)^2) & \mathcal{O}(2I_1)\\ \mathcal{O}(2I_1) & \mathcal{O}(2I_1)\end{pmatrix}\end{aligned}\tag{6.14}$$

where

$$\nu_o=\frac{t\varepsilon_5^2}{2a^2}\frac{\dot c_g}{c}=-\frac{1}{2ga^2}\left(1+\frac{4t^2(1-t^2)}{(F^2+t^2-1)^2}\right)<0\,,\quad \text{with } \dot c_g=\frac{\partial c_g}{\partial k}\,.\tag{6.15}$$

Proof. The proof involves direct calculations using ϕ_3, ϕ_4, ϕ_5, ϕ_6, V_1 and V_2 and is given in Appendix F. ■

It remains to analyze the roots of $\Lambda(\lambda,\sigma,I)=0$ for $|\lambda|+|\sigma|+|I|$ small. First we split Λ into its quartic part plus a remainder

$$\Lambda(\lambda,\sigma,I)=\Lambda_2(\lambda,\sigma,I)+\mathcal{O}(|\lambda|^5+|\sigma|^5)$$

where

$$\Lambda_2(\lambda,\sigma,I)=\det\left[\lambda^2\mathbf{N}^{(1)}(I)+i\lambda\sigma\mathbf{N}^{(2)}(I)+(i\sigma)^2\mathbf{N}^{(3)}(I)\right]\,.$$

When $I=0$ we arrive, after some simple rearrangements, at

$$\Lambda_2(\lambda,\sigma,0)=-\frac{\varepsilon_5^2}{gb^2\Delta_K}\left(\lambda-i(c-c_g)\sigma\right)^2\left(\lambda-ic(1+\frac{1}{F})\sigma\right)\left(\lambda-ic(1-\frac{1}{F})\sigma\right)\,.$$

Note that this expression is independent of ε_6, which drops out when calculating the determinant.

The equation $\Lambda_2(\lambda, \sigma, 0) = 0$ has two simple roots

$$\lambda_{3,4} = ic(1 \pm 1/F)\sigma = i(c \pm \sqrt{gh})\sigma$$

which are purely imaginary and are associated with perturbations of the mean flow. Small perturbations arising from $I \neq 0$ cannot drive them off the imaginary axis. However, at $\lambda_{1,2} = i(c - c_g)\sigma$ there is a double root, which may either split and remain on the imaginary axis or leave the imaginary axis as a complex pair with $\overline{\lambda}_1 = -\lambda_2$.

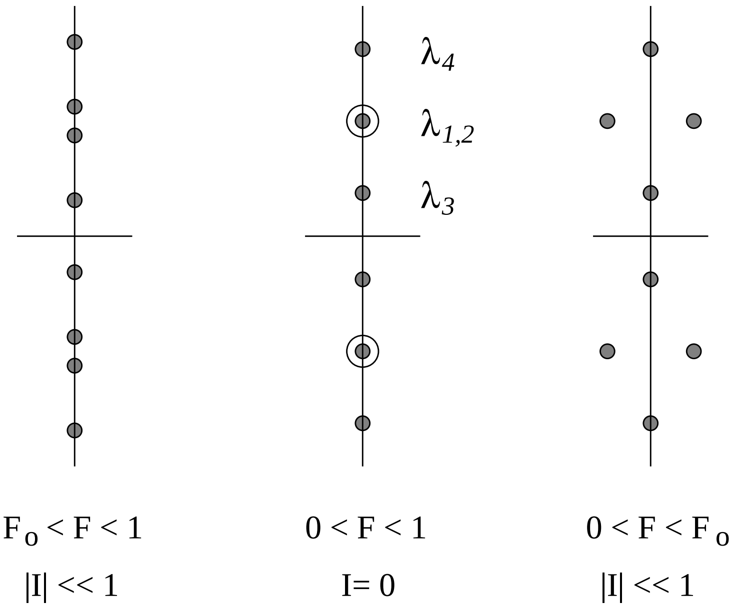

Figure 2. Position of the temporal eigenvalues $\lambda_j \in \mathbb{C}$, $j = 1, \ldots, 4$ for $|I|$ and $|\sigma|$ small.

The position of the roots of $\Lambda_2(\lambda, \sigma, 0)$, and their complex conjugates, is shown in the center picture of Figure 2. For gravity waves with $F \in (0, 1)$ we have $\sqrt{gh} > c > c_g$ resulting in the positions for the roots when $I = 0$ as shown. It is interesting to note that a triple root is not possible for gravity waves as $gh - c_g^2 > 0$; however, when surface tension is present, there exists values of the parameters at which $c_g = \sqrt{gh}$. This singularity is known as the long-wave short-wave resonance and has been analyzed for capillary-gravity waves using modulation equations by Djordjevic & Redekopp [1977]. Note also that $c = c_g$ is not possible for gravity waves but for capillary-gravity waves there exists critical parameter values at which $c = c_g$ in which case the double eigenvalue $i(c - c_g)$ will lie at the origin leading to a different type of stability.

We now verify that when $I_1 \neq 0$ the positions of the eigenvalues shift as shown in the two outside pictures of Figure 2. Let

$$\lambda = i(c - c_g)\sigma + \sqrt{2I_1}\kappa\sigma \tag{6.16}$$

where κ is I-dependent. If, for $|I|$ sufficiently small, we can show there exists a solution of $\Lambda_2(\lambda, \sigma, I) = 0$ with $\mathrm{Re}(\kappa) \neq 0$ we can conclude linear instability. Substitution of the above expression into $\Lambda_2(\lambda, \sigma, I)$ results in

$$\begin{aligned}
0 &= \Lambda_2(i(c - c_g)\sigma + \sqrt{2I_1}\kappa\sigma, \sigma, I) \\
&= \det\left[\frac{\sigma^2 H_{22}}{\Delta_K}\begin{pmatrix} 2I_1(\kappa^2\varepsilon_5^2 - \nu_o\Delta_K/\varepsilon_5^2) & -i\sqrt{2I_1}\kappa(\varepsilon_6 + \varepsilon_1\varepsilon_5) \\ -i\sqrt{2I_1}\kappa(\varepsilon_6 + \varepsilon_1\varepsilon_5) & \varepsilon_8 \end{pmatrix} + \mathcal{O}_*\right] \\
&= 2I_1\left(\frac{\sigma^2 H_{22}}{\Delta_K}\right)^2\left[\varepsilon_8(\kappa^2\varepsilon_5^2 - \nu_o\Delta_K/\varepsilon_5^2) + \kappa^2(\varepsilon_6 + \varepsilon_1\varepsilon_5)^2\right] + \mathcal{O}((2I_1)^{3/2})\sigma^4
\end{aligned}$$

where $\mathcal{O}_* = \sigma^2\begin{pmatrix} \mathcal{O}((2I_1)^{3/2}) & \mathcal{O}(2I_1) \\ \mathcal{O}(2I_1) & \mathcal{O}(\sqrt{2I_1}) \end{pmatrix}$ and

$$\varepsilon_8 = -\frac{1}{\varepsilon_5^2}\left(\varepsilon_6^2 - \frac{\Delta_K}{gb^2}\right) - \frac{1}{\varepsilon_5}\left(2\varepsilon_1\varepsilon_6 - 2\frac{c}{gb^2}\Delta_K\right) - 2\varepsilon_2 = \frac{(gh - c_g^2)}{gh}\frac{k_o^3 t}{gba^4}\nu(F)$$

where

$$\nu(F) = k_o h\left(\frac{9 - 10t^2 + 9t^4}{8t^3}\right) - \frac{gh}{gh - c_g^2}\left(1 + \frac{c_g}{c}(1 - t^2) + \frac{1}{4F^2}(1 - t^2)^2\right). \tag{6.17}$$

Setting $\Lambda_2(\lambda, \sigma, I) = 0$, under the hypothesis that $I_1 \cdot \sigma \neq 0$, results in

$$[\varepsilon_8\varepsilon_5^2 + (\varepsilon_6 + \varepsilon_1\varepsilon_5)^2]\kappa^2 = \frac{\Delta_K}{\varepsilon_5^2}\nu_o\varepsilon_8 + \mathcal{O}((2I_1)^{1/2})$$

or

$$\kappa^2 = -\left(\frac{k_o^3 tb}{gh\varepsilon_5^4 a^4}\right)\nu_o\,\nu(F) + \mathcal{O}((2I_1)^{1/2}).$$

Let $\varepsilon_7^2 = -\nu_o k_o^3 tb/(gh\varepsilon_5^4 a^4)$. Then, since $\nu_o < 0$ (cf. Proposition 6.4), it follows that $\varepsilon_7^2 > 0$. Therefore the splitting of the double eigenvalue $i(c - c_g)\sigma$ takes the following form for I_1 small

$$\lambda = i(c - c_g)\sigma \pm \varepsilon_7\sqrt{2I_1\nu(F)}\,\sigma + \mathcal{O}(I_1)\sigma.$$

The function $\nu(F)$ is precisely the well-known function associated with the instability problem (cf. Benjamin [1967, p. 68], Whitham [1967, p. 49]) where it is shown that $\nu(F)$ has a unique zero $\nu(F_o) = 0$ where $F_o \approx 0.8$ and $\nu(F) > 0$ for $0 < F < F_o$. This verifies the pictures in Figure 2. Combining the above analysis of the dispersion relation $\Lambda(\lambda, \sigma, I)$ with the Lyapunov-Schmidt splitting (6.4) proves the following.

Theorem 6.5. *For $0 < F < F_o$, where F_o is the unique zero of the function $\nu(F)$ defined in (6.17), and $|I|$ sufficiently small, there exists $(\lambda, \sigma, U) \in \mathbb{C} \times \mathbb{R} \times \mathbb{Y}^1$ satisfying (6.1) with $\mathrm{Re}(\lambda) > 0$.*

Combining Theorem 6.5 with Corollary 5.3 yields the desired instability result which was anticipated in Theorem 2 in the introduction.

Corollary 6.6 (Benjamin-Feir instability). *For $0 < F < F_o$ and $|I|$ sufficiently small the Stokes' periodic wavetrain is linearly unstable.*

Recall that the local existence of the Stokes periodic wavetrain requires the non-degeneracy conditions $F \in (0,1)$, $c \neq c_g$ and $\Delta_K \neq 0$. The first condition is imposed and the second and third conditions are satisfied for gravity waves. The stability problem introduces additional non-degeneracy conditions: $gh - c_g^2 \neq 0$ and $\dot{c}_g \neq 0$. For gravity waves the above two conditions are satisfied. In fact, for gravity waves $gh - c_g^2 > 0$ and $\dot{c}_g < 0$ (cf. Whitham [1967, p. 409]). Although, it is interesting to note that, for capillary gravity waves and interfacial waves many of the above conditions are violated at particular parameter values.

7. Concluding remarks

Theorem 4.1 and 4.3 combine to complete the proof of the Stokes Manifold Theorem stated in the introduction. Corollary 5.3 and Corollary 6.6 combine to complete the proof of the Benjamin-Feir instability.

The reduced instability theory presented here can be generalized to prove other related wave instabilities. For example, capillary-gravity waves have resonances in the linear dispersion relation (the Wilton ripple resonance for example) leading to multiple branches of periodic travelling waves whose stability is not completely understood. The reduced ODE for capillary-gravity waves, obtained through the instability theory is however six dimensional. Also, capillary-gravity waves and interfacial waves violate some the the non-degeneracy conditions (i.e. there exist parameter values where $gh = c_g^2$, $\dot{c}_g = 0$ and so forth) leading to new features in the instability problem. All the above problems can be formulated, as here, and reduced to the analysis of finite-dimensional ordinary differential equations, which, in principle, can be completely analyzed.

The reduced instability theory in this paper relied heavily on the Hamiltonian structure. For systems without a Hamiltonian structure the basic formulation carries over with some modification to the analysis. The spatial center manifold reduction and the principle of reduced linear instability carry over. However the reduced stability problem will be of the form

$$V_x = D\mathcal{F}_1(\beta(x))V - \lambda M(\beta(\cdot),\lambda,x)V \quad \text{for } V \in \mathbb{C}^n \quad \text{and} \quad \beta \in \mathcal{U} \subset \mathbb{R}^n \,.$$

In the above problem $M(\beta,\lambda,x)$ and $D\mathcal{F}_1(\beta)$ have no special structure (other than the fact that $D\mathcal{F}_1(\beta)$ is obtained from the existence problem). However the problem, when β is periodic in x for example, can be again addressed using Floquet theory and methods for ordinary differential equations (cf. Mielke [1994], Bridges & Mielke [1994]).

Appendix A. Spatial and temporal Hamiltonian structure for gravity waves

The gravity wave problem is formulated as follows. The free surface is represented by $y = \eta(x,t)$. The fluid occupies the region $-h < y < \eta(x,t)$, is inviscid, irrotational and of constant density with velocity potential $\phi(x,y,t)$. The still water depth h is determined by the uniform flow (see Appendix B). The governing equations for $(x,t) \in \mathbb{R} \times \mathbb{R}_+$ are

$$\left.\begin{aligned} \eta_t + \phi_x\eta_x - \phi_y &= 0 \\ \phi_t + \tfrac{1}{2}(\phi_x^2 + \phi_y^2) + g\eta &= \widetilde{R}(t) \end{aligned}\right\} \quad \text{at } y = \eta(x,t)\,, \tag{A.1}$$

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \qquad \text{for} \quad -h < y < \eta(x,t) \tag{A.2}$$

and impermeability at the bottom requires

$$\phi_y|_{y=-h} = 0\,. \tag{A.3}$$

The equations in (A.1) are respectively the kinematic surface condition and the Bernoulli condition with Bernoulli function $\widetilde{R}(t)$ (cf. Whitham [1974]).

Consider the change of independent variable: $t' = t$ and $x' = x - ct$. The free surface boundary conditions are transformed to

$$\left.\begin{aligned} \eta_{t'} + u\eta_{x'} - \phi_y &= 0 \\ \phi_{t'} + \tfrac{1}{2}(u^2 + \phi_y^2) + g\eta &= R(t) \end{aligned}\right\} \text{ at } y = \eta \tag{A.1a}$$

with an adjustment of the Bernoulli function $\widetilde{R}(t) = R(t) - \frac{1}{2}c^2$ and where $u = \phi_x - c$.

The above equations with (A.1) replaced by (A.1a) can be written in the following form (cf. Bridges [1994a,b,c]) (dropping the primes on x and t)

$$\mathbf{M}Z_t + \mathbf{K}(u_s)Z_x = \nabla S(Z) + c\nabla Q(Z) \tag{A.4}$$

where $\mathbf{M}$ and $\mathbf{K}(u_s)$ are skew-symmetric operators and $\nabla S(Z)$ is the gradient of a functional $S(Z)$. This is verified as follows. Let

$$Z = \begin{pmatrix} \Phi \\ \eta \\ \phi \\ u \end{pmatrix}, \quad \mathbf{M} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{K}(u_s) = \begin{pmatrix} 0 & -u_s & 0 & 0 \\ u_s & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{A.5}$$

where

$$\Phi = \phi|_{y=\eta}, \quad u = \phi_x - c \quad \text{and} \quad u_s = u|_{y=\eta}. \tag{A.6}$$

The functional $S(Z)$ is the flow force (Benjamin & Lighthill [1954], Benjamin [1984], Bridges [1992a,1994a]) for the water-wave problem in terms of the Z-coordinates

$$S(Z) = \tfrac{1}{2}\int_{-h}^{\eta} (u^2 - \phi_y^2)\, dy - \tfrac{1}{2} g\eta^2 + R(t)\eta.$$

The functional $Q(Z) = \int_{-h}^{\eta} u\, dy$ is the mass flux. For the steady problem, the flow force and the mass flux are spatial invariants, but, in the time-dependent problem the flow force is not, in general, x and/or t independent; that is, $S_x \cdot S_t \neq 0$. The gradients of $S(Z)$ and $Q(Z)$, taken with respect to the inner product,

$$\langle\!\langle U, V \rangle\!\rangle_Z = U_1V_1 + U_2V_2 + \int_{-h}^{\eta} (U_3V_3 + U_4V_4)\, dy \tag{A.7}$$

are

$$\nabla S(Z) \stackrel{\text{def}}{=} \begin{pmatrix} \delta S/\delta\Phi \\ \delta S/\delta\eta \\ \delta S/\delta\phi \\ \delta S/\delta u \end{pmatrix} = \begin{pmatrix} -\phi_y|_{y=\eta} \\ \frac{1}{2}u_s^2 + \frac{1}{2}\phi_y^2|_{y=\eta} - g\eta + R(t) \\ \phi_{yy} \\ u \end{pmatrix} \quad \text{and} \quad \nabla Q(Z) = \begin{pmatrix} 0 \\ u_s \\ 0 \\ 1 \end{pmatrix}. \tag{A.8}$$

Using the definitions of $\mathbf{M}$, $\mathbf{K}(u_s)$, $\nabla S(Z)$ and $\nabla Q(Z)$ in (A.5) and (A.8), the equation (A.4) when written out is

$$\begin{aligned} -\eta_t - u_s\eta_x &= -\phi_y|_{y=\eta} \\ \Phi_t + u_s\Phi_x &= \tfrac{1}{2}u_s^2 + \tfrac{1}{2}\phi_y^2|_{y=\eta} - g\eta + R(t) + cu_s \\ -u_x &= \phi_{yy} \\ \phi_x &= u + c\,. \end{aligned} \tag{A.9}$$

The first of (A.9) recovers the kinematic surface condition. The third and fourth equations in (A.9) combine to recover Laplace's equation and the second of (A.9) results in the Bernoulli equation by noting that $\Phi_t = \phi_t|_{y=\eta} + \eta_t\phi_y|_{y=\eta}$ and $\Phi_x = \phi_x|_{y=\eta} + \eta_x\phi_y|_{y=\eta}$. Associated with (A.4) are the conditions (A.3) and (A.6).

The formulation (A.4) corresponds to a Hamiltonian system on a multi-symplectic structure (Bridges [1994c]) in the following way. There are independent symplectic structures associated with space and time. The skew-symmetric operators $\mathbf{M}$ and $\mathbf{K}(u_s)$ define two-forms by

$$\omega_{\scriptscriptstyle Z}{}^{(1)}(U,V) = \langle\!\langle U, \mathbf{M}V \rangle\!\rangle_{\scriptscriptstyle Z}$$
$$\omega_{\scriptscriptstyle Z}{}^{(2)}(U,V) = \langle\!\langle U, \mathbf{K}(u_s)V \rangle\!\rangle_{\scriptscriptstyle Z}$$

where the inner product is as defined in (A.7) and U, V are arbitrary tangent vectors to the manifold $\mathcal{M}$ defining the phase space. A Hamiltonian system on a multi-symplectic structure is then $(\mathcal{M}, \omega^{(1)}, \omega^{(2)}, S)$; a manifold, a family of closed two forms and a Hamiltonian functional with vector field

$$\omega_{\scriptscriptstyle Z}{}^{(1)}(\zeta, Z_t) + \omega_{\scriptscriptstyle Z}{}^{(2)}(\zeta, Z_x) = \langle\!\langle \zeta, dS(Z) \rangle\!\rangle_{\scriptscriptstyle Z}$$

for any ζ in the tangent space of $\mathcal{M}$. One can verify that the above two-forms for the water-wave problem are closed (Bridges [1994c]). The precise specification of the manifold is not needed here as the analysis in the body of the paper will be carried out for the above system restricted to a moving frame of reference. It is the spatial Hamiltonian structure, obtained by restricting (A.4) to a moving frame of reference, that forms the basis for the present analysis.

Appendix B. Uniform flows, Bernoulli constant and the swallowtail

Steady solutions of the gravity wave problem relative to a moving frame of reference, moving at speed $c \in \mathbb{R}$ satisfy

$$\mathbf{K}(u_s) Z_x = \nabla S(Z) + c \nabla Q(Z) \tag{B.1}$$

where the functionals $S(Z)$, the flow force, and $Q(Z)$, the mass flux, are absolute spatial invariants. The operator $\mathbf{K}(u_s)$ and the vector-valued function Z are defined in (A.5). In this appendix the (x, y) coordinates for the system are chosen with the free surface at $y = \eta$ and the bottom at $y = 0$. The shift in origin is temporary and its intention is to illustrate how the still water depth, h, arises as a natural consequence of the chosen uniform flow.

The functionals $S(Z)$ and $Q(Z)$ take the following form (where the presence of R, the Bernoulli constant, in the functional $S(Z)$ is to be particularly noted)

$$\begin{aligned} S(Z) &= \tfrac{1}{2} \int_0^\eta (u^2 - \phi_y^2)\, dy - \tfrac{1}{2} g \eta^2 + R\eta \\ Q(Z) &= \int_0^\eta u\, dy \end{aligned} \tag{B.2}$$

from which it follows, formally, using the inner product (A.7), with integration adjusted so that $y \in (0, \eta)$, that

$$\nabla S(Z) = \begin{pmatrix} -\phi_y|_{y=\eta} \\ \frac{1}{2}(u_s^2 + \phi_y^2)|_{y=\eta} - g\eta + R \\ \phi_{yy} \\ u \end{pmatrix}, \qquad \nabla Q(Z) = \begin{pmatrix} 0 \\ u_s \\ 0 \\ 1 \end{pmatrix}.$$

An interesting feature of the form (B.1) is that stationary states satisfy

$$\nabla S(Z) + c\,\nabla Q(Z) = 0\,. \tag{B.3}$$

Therefore, uniform states of the gravity wave problem, can be classified as critical points of S on level sets of Q with R constant (given) in which case $c \in \mathbb{R}$ is a Lagrange multiplier. Equation (B.3) is then the Lagrange necessary condition for the constrained variational problem. Evaluating (B.3) results in

$$\phi = \text{constant}, \quad u = -c, \quad \eta \stackrel{\text{def}}{=} h = \frac{1}{g}(R - \tfrac{1}{2}c^2)\,. \tag{B.4}$$

For fixed $R > 0$ evaluation of S and Q on the uniform state (B.4) results in

$$Q_0 = \int_0^\eta u\,dy = -ch = -\frac{c}{g}(R - \tfrac{1}{2}c^2) \tag{B.5}$$

$$S_0 = \tfrac{1}{2}\int_0^\eta (u^2 - \phi_y^2)\,dy - \tfrac{1}{2}g\eta^2 + R\eta = hc^2 + \tfrac{1}{2}gh^2 = \frac{1}{2g}\left(R^2 + Rc^2 - \tfrac{3}{4}c^4\right). \tag{B.6}$$

For $-\sqrt{2R} < c < \sqrt{2R}$ the curve $(Q(c), S(c))$, with R and g fixed (and positive), form the boundary of a swallowtail as shown in Figure 1. Moreover $F = 1$ corresponds to the cusp point in the picture separating the upper branch (along which the flow is subcritical) from the lower branch (along which the flow is supercritical). Along the branch we have

$$Q_0{}'(c) = h(1 - F^2)$$
$$S_0{}'(c) = -ch(1 - F^2)$$

verifying that $S_0{}'(c) = Q_0{}'(c) = 0$ (the cusp point) corresponds to $F = 1$. In fact we could use $Q_0{}'(c)$ to define sub(super)-criticality:

$$\begin{aligned} Q_0{}'(c) < 0 \quad &\Rightarrow \text{ supercritical uniform flow} \\ Q_0{}'(c) > 0 \quad &\Rightarrow \text{ subcritical uniform flow}\,. \end{aligned} \tag{B.7}$$

Moreover it follows from the Lagrange multiplier theory that $c = -\partial S/\partial Q$; the slope at any point on the boundary curve in Figure 1 is determined by the value of c.

The above formulation recovers the essential features of the classical uniform flow theory for shallow water (BENJAMIN & LIGHTHILL [1954], SEWELL & PORTER [1980, Section 4]). The classical equations for shallow water theory are

$$\begin{aligned} h_t + uh_x + hu_x &= 0 \\ u_t + uu_x + gh_x &= 0 \end{aligned} \tag{B.8}$$

for a one-dimensional velocity field $u(x,t)$ and elevation $h(x,t)$. The first of (B.8) corresponds to mass conservation and the second to energy conservation. They can also be combined to obtain the momentum conservation equation:

$$\frac{\partial}{\partial t}(uh) + \frac{\partial}{\partial x}(hu^2 + \tfrac{1}{2}gh^2) = 0\,. \tag{B.9}$$

Let

$$Q = uh, \quad R = gh + \tfrac{1}{2}u^2, \quad S = hu^2 + \tfrac{1}{2}gh^2\,. \tag{B.10}$$

Then it is clear from (B.8) and (B.9) that on time-independent states (uniform flows) the functions Q, R and S are constants. The function represented by (B.10) is a map from $\mathbb{R}^2 = (u,h)$ to $\mathbb{R}^3 = (S,Q,R)$ and the surface in $\mathbb{R}^3$ is a swallowtail (cf. SEWELL & PORTER [1980]) and for fixed $R > 0$ the result is precisely as shown in Figure 1.

For the analysis of the Stokes manifold we suppose R is arbitrary but fixes the properties of the uniform flow and hence h. To obtain the governing equations for the system perturbed about the uniform flow let $\widetilde{Z} = (\widetilde{\Phi}, \widetilde{\eta}, \widetilde{\phi}, \widetilde{u})$ with

$$\begin{aligned} u &= -c + \widetilde{u} \\ \eta &= h + \widetilde{\eta} \\ S + cQ &= S_0 + cQ_0 + \widetilde{S}(\widetilde{Z}) \\ Q &= Q_0 + \widetilde{Q}(\widetilde{Z}) \end{aligned}$$

where $S_o = hc^2 + \frac{1}{2}gh^2$ and $Q_0 = -ch$. Then shifting y so that $-h < \widetilde{y} < \widetilde{\eta}$,

$$\begin{aligned} \widetilde{S}(\widetilde{Z}) &= \tfrac{1}{2}\int_{-h}^{\tilde{\eta}} (\widetilde{u}^2 - \widetilde{\phi}_{\tilde{y}}^2)\, d\tilde{y} - \tfrac{1}{2}g\widetilde{\eta}^2 \\ \widetilde{Q}(\widetilde{Z}) &= \int_{-h}^{\tilde{\eta}} \widetilde{u}\, d\tilde{y} - c\widetilde{\eta} \end{aligned} \tag{B.11}$$

and the governing equations are then

$$\mathbf{K}(\tilde{u}_s - c)\widetilde{Z}_{\tilde{x}} = \nabla \widetilde{S}(\widetilde{Z}) \tag{B.12}$$

which is the starting point for the analysis in Section 2. Note that R does not appear in the formulation (B.11)-(B.12) after subtracting off the uniform flow; it is implicit in h.

Appendix C. Expansion of the reduced Hamiltonian system

The reduction theory of Section 4 leads to the Hamiltonian differential equation (4.13) with a non-constant symplectic form. The reduced Hamiltonian structure is simplified by Darboux transformations. Additional simplification is obtained by transforming the Hamiltonian function into normal form. In principle the constructive aspects leading to the simplified Hamiltonian function with a constant symplectic form are lengthy. However once the final form of the reduced system is known the construction of the critical coefficients (in this case ε_1 and ε_2 in Theorem 4.2) can be considerably simplified by directly equating coefficients of like polynomials to zero. This simplification is suggested by Iooss & Adelmeyer [1992, p. 65] and carries over with minimum modification to the case where the reduced vector field is Hamiltonian.

In normal form the reduced Hamiltonian vector field to leading order is

$$\begin{aligned}\dot\beta_1 &= (-k_o + \varepsilon_1\beta_4 + 2\varepsilon_2 I_1 + \cdots)\beta_2\\ \dot\beta_2 &= -(-k_o + \varepsilon_1\beta_4 + 2\varepsilon_2 I_1 + \cdots)\beta_1\\ \dot\beta_3 &= \beta_4 + \varepsilon_1 I_1 + \cdots\\ \dot\beta_4 &= 0\end{aligned} \tag{C.1}$$

and the object is to determine expressions for the coefficients ε_1 and ε_2.

One expands the expression for the center manifold $\mathcal{H}(\beta)$ in terms of the *transformed* coordinates

$$Z = Z_1 + \mathcal{H}_2(\beta) + \mathcal{H}_3(\beta) + \cdots \tag{C.2}$$

where

$$\begin{aligned}Z_1 &= \beta\cdot\xi = \beta_1\xi_1 + \beta_2\xi_2 + \beta_3\xi_3 + \beta_4\xi_4\\ \mathcal{H}_2(\beta) &= \tfrac12\beta_1^2 W_1 + \beta_1\beta_2 W_2 + \tfrac12\beta_2^2 W_3 + \tfrac12\beta_4^2 W_4 + \beta_2\beta_4 W_5 + \beta_1\beta_4 W_6\\ \mathcal{H}_3(\beta) &= \tfrac13\beta_1^3 W_7 + \cdots\,;\end{aligned}$$

the terms $\mathcal{H}_j(\beta)$ are homogeneous polynomials of degree j. The full infinite dimensional system can be written in the form

$$Z_x - \mathbf{A}Z = \mathcal{N}(Z) = \tfrac12\mathcal{N}_2(Z,Z) + \tfrac16\mathcal{N}_3(Z,Z,Z) + \cdots \tag{C.3}$$

where $\mathbf{A}$ and $\mathcal{N}(Z)$ are defined in (2.19) and (2.20) respectively and

$$\mathcal{N}_2(Z,W) \stackrel{\text{def}}{=} \begin{bmatrix} \frac{g}{c^2}[Z_2W_4 + W_2Z_4]|_{y=0} - \frac1c[W_4Z_4 + W_3'Z_3']|_{y=0}\\[2ex] \frac{1}{ch}[W_2Z_3' + Z_2W_3']|_{y=0} - \frac{1}{c^2}[W_4Z_3' + Z_4W_3']|_{y=0}\\[2ex] -\frac1c(1+\frac yh)[Z_3'W_3'|_{y=0} + W_3'Z_3'|_{y=0}]\\[2ex] \frac2h(Z_2W_3'' + W_2Z_3'') - \frac1c(1+\frac yh)[Z_4'W_3'|_{y=0} + W_4'Z_3'|_{y=0}]\end{bmatrix}. \tag{C.4}$$

Substitution of the center manifold (C.2) into (C.3) leads to

$$\begin{aligned}&\big[\beta_x\cdot\xi + D\mathcal{H}_2(\beta)\beta_x + D\mathcal{H}_3(\beta)\beta_x + \cdots\big] - \mathbf{A}(Z_1+\mathcal{H}_2+\mathcal{H}_3+\cdots) = \\ &\quad = \tfrac{1}{2}\mathcal{N}_2(Z_1,Z_1) + \mathcal{N}_2(Z_1,\mathcal{H}_2(\beta)) + \ \tfrac{1}{6}\mathcal{N}_3(Z_1,Z_1,Z_1)+\cdots\,.\end{aligned} \tag{C.5}$$

The derivatives of β with respect to x are replaced by the expressions in (C.1). One then sets each polynomial in β to zero. The coefficients of the quadratic polynomials in (C.5) result in

$$\begin{aligned}\beta_2\beta_4 &: \ \ \varepsilon_1\xi_1 - k_oW_6 - \mathbf{A}W_5 = \mathcal{N}_2(\xi_2,\xi_4)\\ \beta_1\beta_4 &: \ \ -\varepsilon_1\xi_2 + k_oW_5 - \mathbf{A}W_6 = \mathcal{N}_2(\xi_1,\xi_4)\\ \beta_1^2 &: \ \ \tfrac{1}{2}\varepsilon_1\xi_3 + k_oW_2 - \tfrac{1}{2}\mathbf{A}W_1 = \tfrac{1}{2}\mathcal{N}_2(\xi_1,\xi_1)\\ \beta_1\beta_2 &: \ \ k_oW_3 - k_oW_1 - \mathbf{A}W_2 = \mathcal{N}_2(\xi_1,\xi_2)\\ \beta_2^2 &: \ \ \tfrac{1}{2}\varepsilon_1\xi_3 - k_oW_2 - \tfrac{1}{2}\mathbf{A}W_3 = \tfrac{1}{2}\mathcal{N}_2(\xi_2,\xi_2)\\ \beta_4^2 &: \ \ -\tfrac{1}{2}\mathbf{A}W_4 = \tfrac{1}{2}\mathcal{N}_2(\xi_4,\xi_4)\,.\end{aligned}$$

These expressions can be simplified by letting

$$W_1 = U_1+U_3\,,\quad W_2 = U_2\,,\qquad W_3 = U_1 - U_3 \quad \text{and} \quad \zeta = \xi_1 - i\xi_2\,.$$

In which case

$$\begin{aligned}\mathbf{A}U_1 &= \varepsilon_1\xi_3 - \tfrac{1}{2}\mathcal{N}_2(\zeta,\overline{\zeta})\\ (\mathbf{A}-2ik_o\mathbf{I})(U_2+iU_3) &= -\tfrac{1}{2}i\mathcal{N}_2(\zeta,\zeta)\\ \mathbf{A}W_4 &= -\mathcal{N}_2(\xi_4,\xi_4)\\ (\mathbf{A}-ik_o\mathbf{I})(W_5+iW_6) &= \varepsilon_1\zeta - i\mathcal{N}_2(\zeta,\xi_4)\,.\end{aligned}$$

Solvability of the equation for W_5+iW_6 requires

$$\varepsilon_1 = \tfrac{1}{2}i\langle\zeta^*,\mathcal{N}_2(\zeta,\xi_4)\rangle = -\frac{tk_o}{2bca^2}(3-t^2) \quad \text{where} \quad \zeta^* = \xi_1^* - i\xi_2^*\,. \tag{C.6}$$

This verifies the first expression in (4.8).

In order to construct an expression for ε_2 we need only U_1, U_2 and U_3. The above inhomogeneous problems can be explicitly solved to find

$$U_1 = \varepsilon_1\xi_4 + \frac{k_o^2}{2ga^2}\begin{pmatrix}0\\ -(1-t^2)\\ 0\\ 2c(1+\frac{y}{h})\dot{\mathrm{ch}}(y)\end{pmatrix} \tag{C.7a}$$

and

$$U_2+iU_3 = -\frac{i}{a^2}\begin{pmatrix}\frac{ik_o}{4c}(3t^{-2}+t^2)\\ \frac{k_o^2}{2g}(1-3t^{-2})\\ \frac{3ik_o}{2c}(t^{-1}-t)\frac{\cosh 2k_o(y+h)}{\sinh 2k_oh} + \frac{it}{c}(1+\frac{y}{h})\dot{\mathrm{ch}}(y)\\ -\frac{3k_o^2}{c}(t^{-1}-t)\frac{\cosh 2k_o(y+h)}{\sinh 2k_oh} - \frac{k_ot}{c}(1+\frac{y}{h})\dot{\mathrm{ch}}(y)\end{pmatrix}. \tag{C.7b}$$

For the coefficients of the cubic polynomials we will sketch the result as only a single solvability condition is needed which results in the coefficient ε_2. After some algebra we obtain

$$8i\varepsilon_2 = \langle \zeta^*, i\mathcal{N}_2(\overline{\zeta}, U_2 + iU_3) - 2\mathcal{N}_2(\zeta, U_1) - \tfrac{1}{2}\mathcal{N}_3(\zeta, \zeta, \overline{\zeta})\rangle \tag{$C.8$}$$

which upon substitution of ζ, ζ^*, U_1 and $U_2 + iU_3$ results in the second expression in (4.8).

Note that U_1, W_4 W_5 and W_6 are not uniquely defined as $\mathbf{A}$ and $(\mathbf{A} - ik_o\mathbf{I})$ have a nontrivial kernel in $\mathbb{X}_1$. This freedom can be used to modify the spatial symplectic structure $\mathbf{K}$ on the center manifold. However, these terms will not influence the final result, so we omit the calculations.

Appendix D. Expansion of the λ-part in the reduced stability problem

In this appendix the expansion of the leading λ-part of the reduced spectral problem is derived. The reduced temporal two-form with operator $M_{11}(\beta)$ is defined by

$$(\alpha, M_{11}(\beta)\gamma) = \langle D\mathcal{T}(\beta, 0)\alpha, \mathbf{M}\, D\mathcal{T}(\beta, 0)\gamma\rangle_{\mathbb{X}} \tag{$D.1$}$$

where $\mathcal{T}$ is defined in Theorem 5.1 and $\mathbf{M}$ is given in (A.5). Using the expansion $\mathcal{T}(\beta, 0) = \beta \cdot \xi + \mathcal{H}_2(\beta) + \cdots$ from (C.2) we obtain

$$D\mathcal{T}(\beta, 0)\,\alpha = \{[\xi_1\,\xi_2\,\xi_3\,\xi_4] + \beta_1[W_1\,W_2\,0\,W_6] + \beta_2[W_2\,W_3\,0\,W_5] + \mathcal{O}(|I|)\}\,\alpha \tag{$D.2$}$$

and $M_{11}(\beta) = M^{(0)} + \beta_1 M^{(1)} + \beta_2 M^{(2)} + \mathcal{O}(|I|)$ with $(M^{(0)})_{ij} = \langle \xi_i, \mathbf{M}\xi_j\rangle_{\mathbb{X}}$, where ξ_i for $i = 1, \ldots, 4$ are the basis elements for the center subspace (cf. Section 2). The result is

$$M^{(0)} = \frac{1}{c - c_g}\begin{pmatrix} 0 & 1 & \frac{a}{b} & 0 \\ -1 & 0 & 0 & -\frac{a}{k_o b} \\ -\frac{a}{b} & 0 & 0 & -\frac{a^2}{k_o b^2} \\ 0 & \frac{a}{k_o b} & \frac{a^2}{k_o b^2} & 0 \end{pmatrix}. \tag{$D.3$}$$

It is not necessary to construct all the entries in $M^{(1)}$ and $M^{(2)}$. Only the single combination

$$\varepsilon_3 \stackrel{\text{def}}{=} (M^{(1)})_{13} + (M^{(2)})_{23}$$

is needed in Section 6 to define the expansion of ϕ_6 in equations (6.8)-(6.10). From (D.2) and the above definition for ε_3 we find

$$\varepsilon_3 = \langle W_1, \mathbf{M}\xi_3\rangle_{\mathbb{X}} + \langle W_3, \mathbf{M}\xi_3\rangle_{\mathbb{X}} = 2\langle U_1, \mathbf{M}\xi_3\rangle_{\mathbb{X}}\,.$$

Using the definition for U_1 in (C.7a) the above expression evaluates to

$$\varepsilon_3 = \frac{2}{b}\left(\varepsilon_1 \frac{c}{gb} - \frac{k_o^2}{2ga^2}(1-t^2)\right) ,$$

which, after substitution for ε_1, reduces to the form (6.10).

Appendix E. Tangent vectors to the basic state

The basic state of the finite-dimensional reduced system on the center manifold is a four-parameter family of L-periodic states coupled to drift along the Γ-group orbit, see (4.9). There is an L-periodic tangent space given by the tangent vector to the flow and the tangent vector to the group orbit: $\phi_j = \frac{\partial}{\partial \mu_j}\widehat{\beta}$. The state also depends on the value of the Hamiltonian function and the value of the mass flux. This yields two other tangent vectors:

$$\widetilde{\phi}_3 = \frac{\partial \beta}{\partial I_1} \quad \text{and} \quad \widetilde{\phi}_4 = \frac{\partial \beta}{\partial I_2} .$$

The vectors $\widetilde{\phi}_3$ and $\widetilde{\phi}_4$ are not in general periodic but by adding in suitable multiples of ϕ_1 and ϕ_2 they can be made periodic. The explicit expressions are

$$\begin{aligned}
\phi_1 &= \frac{\partial \widehat{\beta}}{\partial \mu_1} = -\sqrt{2I_1}\begin{pmatrix} \sin(H_1 x) \\ \cos(H_1 x) \\ 0 \\ 0 \end{pmatrix} \qquad \phi_2 = \frac{\partial \widehat{\beta}}{\partial \mu_2} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} , \\
\widetilde{\phi}_3 &= \frac{\partial \widehat{\beta}}{\partial I_1} = \frac{1}{\sqrt{2I_1}}\begin{pmatrix} \cos(H_1 x) \\ -\sin(H_1 x) \\ 0 \\ 0 \end{pmatrix} + xH_{11}\phi_1 + xH_{21}\phi_2 , \\
\widetilde{\phi}_4 &= \frac{\partial \widehat{\beta}}{\partial I_2} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + xH_{12}\phi_1 + xH_{22}\phi_2 .
\end{aligned} \tag{E.1}$$

Independent L-periodic functions can be constructed from $\widetilde{\phi}_3$ and $\widetilde{\phi}_4$ by subtracting off the non-periodic part. Define

$$\begin{aligned}
\phi_3 &= x\phi_1 - c_1\widetilde{\phi}_3 - c_2\widetilde{\phi}_4 \\
\phi_4 &= x\phi_2 - c_3\widetilde{\phi}_3 - c_4\widetilde{\phi}_4 .
\end{aligned} \tag{E.2}$$

Imposing L-periodicity,

$$\begin{aligned}
\phi_3(x+L) - \phi_3(x) &= 0 \\
\phi_4(x+L) - \phi_4(x) &= 0
\end{aligned}$$

fixes values for the constants $c_1,\dots,c_4$ resulting in

$$\begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}^{-1} = \frac{1}{\Delta_K}\begin{pmatrix} H_{22} & -H_{12} \\ -H_{21} & H_{11} \end{pmatrix}$$

where $\Delta_K = H_{11}H_{22} - H_{12}H_{21}$ is the "KAM-determinant" for the basic state. Substitution of the coefficients $c_1,\dots,c_4$ into into (E.2) results in the following L-periodic functions

$$\begin{aligned} \phi_3 &= -\frac{1}{\sqrt{2I_1}}\frac{H_{22}}{\Delta_K}K_1\phi_1 + \frac{H_{12}}{\Delta_K}K_1\phi_2 \\ \phi_4 &= \frac{1}{\sqrt{2I_1}}\frac{H_{12}}{\Delta_K}K_1\phi_1 - \frac{H_{11}}{\Delta_K}K_1\phi_2\,. \end{aligned} \tag{E.3}$$

Appendix F. Proof of Proposition 6.4

The coefficients of $N^{(2)}$ are defined by $N^{(2)}_{kl} = [\![\phi_k, K_1\phi_{l+4}]\!] + [\![\phi_l, K_1\phi_{k+4}]\!]$ (cf. Lemma 6.3). We find

$$\begin{aligned} N^{(2)}_{11} &= -2[\![K_1\phi_1,\phi_5]\!] = \frac{2H_{22}}{\Delta_K}\varepsilon_5 + \mathcal{O}((2I_1)^2) \\ N^{(2)}_{12} &= N^{(2)}_{21} = -[\![K_1\phi_1,\phi_6]\!] - [\![K_1\phi_2,\phi_5]\!] = \frac{H_{22}}{\Delta_K}(\varepsilon_6 - \varepsilon_1\varepsilon_5) + \mathcal{O}(2I_1) \\ N^{(2)}_{22} &= -2[\![K_1\phi_2,\phi_6]\!] = -\frac{2H_{22}}{\Delta_K}\varepsilon_1\varepsilon_6 + \frac{2c}{gb^2} + \mathcal{O}(2I_1)\,. \end{aligned}$$

This completes the construction of $N^{(2)}$. For the construction of $N^{(1)}$ we need the functions V_1 and V_2 which are defined by $V_l = -g_1(I,x)\phi_l$ for $l=1,2$ (cf. Lemma 6.3) and we note that

$$V_1 = \sqrt{2I_1}V_1^{(0)} + \mathcal{O}(2I_1) \quad \text{and} \quad V_2 = V_2^{(0)} + \mathcal{O}(\sqrt{2I_1})\,. \tag{F.1}$$

To calculate $V_j^{(0)}$ it is sufficient to construct $g^{(0)}$ given by $g_1(I,x) = g^{(0)} + \mathcal{O}(|I|)$. Setting $\widehat{\beta}=0$ in (5.6) we find that $g^{(0)} \in \mathcal{L}(\mathbb{C}^4,\mathbb{X}_2)$ is constant and satisfies

$$\mathbf{A}_2 g^{(0)} - g^{(0)}\mathbf{A}_1 = J_2\,M_{21}(0) = (\mathbf{I}-\mathbf{P})\mathbf{J}(0)\mathbf{M}[\xi_1\,\xi_2\,\xi_3\,\xi_4]\,, \tag{F.1a}$$

where $\mathbf{A}_j \in \mathcal{L}(\mathbb{X}_j,\mathbb{X}_j)$. The solution of the inhomogeneous problem (F.1a) can be explicitly constructed and is given by

$$g^{(0)} = \begin{pmatrix} g^{(0)}_{11} & 0 & 0 & 0 \\ 0 & g^{(0)}_{22} & g^{(0)}_{23} & 0 \\ g^{(0)}_{31} & 0 & 0 & 0 \\ 0 & g^{(0)}_{42} & g^{(0)}_{43} & 0 \end{pmatrix} \tag{F.1b}$$

with

$$
\begin{aligned}
g_{11}^{(0)} &= \ell_1 + \ell_2 + \ell_3 k_o t \\
g_{31}^{(0)} &= \ell_1 + \ell_2 \mathrm{ch}(y) + \ell_3 (1 + \frac{y}{h}) \dot{\mathrm{ch}}(y) \\
g_{22}^{(0)} &= -\frac{1}{ga}(\frac{h}{b^2} - \frac{k_o h F^2}{a^2}) - \frac{k_o c}{g}(\ell_1 + \ell_2 + \ell_3 k_o t) \\
g_{42}^{(0)} &= -\frac{hF^2}{acb^2} + \frac{k_o c}{ga^3}\mathrm{ch}(y) - k_o(\ell_1 + \ell_2 \mathrm{ch}(y) + \ell_3(1 + \frac{y}{h})\dot{\mathrm{ch}}(y)) \\
g_{23}^{(0)} &= -\frac{1}{gb}\left(\frac{h}{b^2} - \frac{k_o h F^2}{a^2}\right) \\
g_{43}^{(0)} &= -\frac{1}{bc}\left(\frac{hF^2}{b^2} - \frac{k_o c^2}{ga^2}\mathrm{ch}(y)\right)
\end{aligned}
$$

and

$$
\ell_1 = -\frac{2c}{gk_o ab^2}\,, \quad \ell_2 = -\frac{chk_o}{4ga^5}(1 - F^2 - 3t^2) \quad \text{and} \quad \ell_3 = -\frac{ch}{ga^3}\,.
$$

Using this result the functions $V_j^{(0)}$ take the form

$$
V_1^{(0)} = \begin{pmatrix} g_{11}^{(0)} \\ 0 \\ g_{31}^{(0)} \\ 0 \end{pmatrix} \sin(H_1 x) + \begin{pmatrix} 0 \\ g_{22}^{(0)} \\ 0 \\ g_{42}^{(0)} \end{pmatrix} \cos(H_1 x) \quad \text{and} \quad V_2^{(0)} = -\begin{pmatrix} 0 \\ g_{23}^{(0)} \\ 0 \\ g_{43}^{(0)} \end{pmatrix}. \tag{F.2}
$$

For the off-diagonal terms in $N^{(1)}$ we find

$$
N_{12}^{(1)} = N_{21}^{(1)} = -[\![M\phi_2, \phi_5]\!] + [\![\phi_2, \mathcal{O}(\sqrt{2I_1})]\!] = \frac{1}{\Delta_K}\varepsilon_5\varepsilon_6 + \mathcal{O}(2I_1)\,.
$$

From the definition we have that

$$
\begin{aligned}
N_{22}^{(1)} &= [\![\phi_2, M_{11}(\hat{\beta})\phi_6 + M_{12}(\hat{\beta})V_2]\!] \\
&= [\![\phi_2, M_{11}(\hat{\beta})\phi_6]\!] - \frac{1}{L}\int_0^L \langle \mathbf{M}\xi_3, V_2 \rangle_{\mathbb{X}}\, dx\,.
\end{aligned}
$$

But, using the expression for V_2 in (F.1)–(F.2),

$$
\begin{aligned}
-\frac{1}{L}\int_0^L \langle \mathbf{M}\xi_3, V_2 \rangle_{\mathbb{X}}\, dx &= -\langle \mathbf{M}\xi_3, V_2^{(0)} \rangle_{\mathbb{X}} + \mathcal{O}(\sqrt{2I_1}) = \frac{1}{b} g_{23}^{(0)} + \mathcal{O}(\sqrt{2I_1}) \\
&= -\frac{1}{gb^2}\left(\frac{h}{b^2} - \frac{k_o h F^2}{a^2}\right) + \mathcal{O}(\sqrt{2I_1})
\end{aligned}
$$

and

$$
\begin{aligned}
[\![\phi_2, M_{11}(\hat{\beta})\phi_6]\!] = &- [\![M^{(0)}\phi_2, \phi_6]\!] \\
&+ \sqrt{2I_1}[\![\phi_2, (M^{(1)}\cos(H_1 x) - M^{(2)}\sin(H_1 x))\phi_6]\!] + \mathcal{O}(2I_1) \\
= &\frac{\varepsilon_6^2}{\Delta_K} - \frac{k_o c^2}{g^2 a^2 b^2}\left(1 - \frac{a^2}{k_o b^2}\right) + \mathcal{O}(2I_1)\,.
\end{aligned}
$$

Therefore

$$N_{22}^{(1)} = \frac{\varepsilon_6^2}{\Delta_K} - \frac{1}{gb^2} + \mathcal{O}(2I_1)\,.$$

The construction of the element $N_{11}^{(1)}$ is the most difficult as terms up to order $(2I_1)$ are needed. The expression for $N_{11}^{(1)}$ is

$$N_{11}^{(1)} = [\![\phi_1, M\phi_5]\!] + [\![\phi_1, M_{12}V_1]\!]\,. \tag{F.3}$$

The second term evaluates as follows:

$$\begin{aligned}
[\![\phi_1, M_{12}V_1]\!] &= \sqrt{2I_1}[\![\phi_1, M_{12}V_1^{(0)}]\!] + \mathcal{O}(I_1^2)\\
&= -\frac{\sqrt{2I_1}}{L}\int_0^L \langle M_{21}\phi_1, V_1^{(0)}\rangle_{\mathbb{X}}\,dx + \mathcal{O}(I_1^2)\\
&= \frac{2I_1}{L}\int_0^L \langle \mathbf{M}(\xi_1\sin(H_1x) + \xi_2\cos(H_1x)), V_1^{(0)}\rangle_{\mathbb{X}}\,dx + \mathcal{O}(I_1^2)\\
&= \frac{2I_1}{2a}\left(g_{22}^{(0)} - \frac{ck_o}{g}g_{11}^{(0)}\right) + \mathcal{O}(I_1^2)\,.
\end{aligned}$$

For the first term in (F.3) we find

$$[\![\phi_1, M\phi_5]\!] = -[\![M^{(0)}\phi_1, \phi_5]\!] + \mathcal{O}(I_1^{3/2}) = \frac{H_{22}}{\Delta_K}\varepsilon_5^2 - \frac{2I_1}{2ga^2}\frac{3hF^2}{b^2} + \mathcal{O}(I_1^{3/2})\,.$$

Therefore

$$N_{11}^{(1)} = \frac{H_{22}}{\Delta_K}\varepsilon_5^2 + \nu_o\,2I_1 + \mathcal{O}(I_1^{3/2})$$

where

$$\nu_o = \frac{1}{2a}\left(g_{22}^{(0)} - \frac{ck_o}{g}g_{11}^{(0)}\right) - \frac{F^2}{2ga^2}\frac{3h}{b^2}\,. \tag{F.4}$$

Using the expressions for $g_{22}^{(0)}$ and $g_{11}^{(0)}$ in (F.1b) we find

$$\begin{aligned}
g_{22}^{(0)} - \frac{ck_o}{g}g_{11}^{(0)} &= -\frac{1}{ga}\left(\frac{h}{b^2} - \frac{k_ohF^2}{a^2}\right) - \frac{2k_oc}{g}(\ell_1 + \ell_2 + \ell_3k_ot)\\
&= \frac{1}{ga}\left(-\frac{(4F^2-1)}{1-F^2} - \frac{4t^2(1-t^2)}{(F^2+t^2-1)^2}\right)
\end{aligned}$$

and therefore ν_o in (F.4) reduces to the expression for ν_o in (6.15).

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References

V.I. ARNOLD [1988] *Dynamical Systems III*. Enc. Math. Sci., Springer-Verlag.

C. BAESENS & R.S. MACKAY [1992] *Uniformly travelling water waves from a dynamical systems viewpoint: some insights into bifurcations from Stokes' family*, J. Fluid Mech. **241**, pp. 333-47

T.B. BENJAMIN [1967] *Instability of periodic wavetrains in nonlinear dispersive systems*, Proc. Roy. Soc. Lond. **A299**, pp. 59-75

T.B. BENJAMIN [1984] *Impulse, flow force and variational principles*, IMA J. Appl. Math. **32**, pp. 3-68

T.B. BENJAMIN & J.E. FEIR [1967] *The disintegration of wave trains on deep water, Part 1.*, J. Fluid Mech. **27**, pp. 417-30

T.B. BENJAMIN & M.J. LIGHTHILL [1954] *On cnoidal waves and bores*, Proc. Roy. Soc. **A224**, pp. 448-60

D.J. BENNEY & G.J. ROSKES [1969] *Wave instabilities* , Stud. in Applied Math. **48**, pp. 377-85

T.J. BRIDGES [1992a] *Spatial Hamiltonian structure, energy flux and the water-wave problem*, Proc. Royal Soc. Lond. **A439**, pp. 297-315

T.J. BRIDGES [1992b] *Hamiltonian bifurcations of the spatial structure for coupled nonlinear Schrödinger equations*, PhysicaD **57**, pp. 375-94

T.J. BRIDGES [1994a] *Hamiltonian spatial structure for 3D water-waves relative to a moving frame of reference*, J. Nonlinear Sci. **4**, pp. 221-51

T.J. BRIDGES [1994b] *An instability criterion for water waves of arbitrary amplitude on finite depth*, preprint, Uni-Stuttgart.

T.J. BRIDGES [1994c] *Multi-symplectic structures and wave propagation*, preprint, Uni-Stuttgart.

T.J. BRIDGES & A. MIELKE [1994] *Instability of spatially-periodic states for a family of semilinear PDE's on an infinite strip*, Math. Nachr., *to appear*.

T.J. BRIDGES & G. ROWLANDS [1994] *Instability of spatially quasiperiodic states of the Ginzburg-Landau equation*, Proc. Roy. Soc. Lond. **A444**, pp. 347-62

V.D. DJORDJEVIC & L.G. REDEKOPP [1977] *On two-dimensional packets of capillary-gravity waves*, J. Fluid Mech. **79**, pp. 703-14

H. HASIMOTO & H. ONO [1972] *Nonlinear modulations of gravity waves*, J. Phys. Soc. Japan **33**, pp. 805-11

J. Henrard [1992] *The adiabatic invariant in classical mechanics.* Dynamics Reported **2** (New Series), pp. 117-236

G. Iooss & M. Adelmeyer [1992] *Topics in Bifurcation Theory and Applications*, World Scientific: Singapore

G. Iooss & K. Kirchgässner [1992] *Water waves for small surface tension: an approach via normal form*, Proc. Roy. Soc. Edin. **122A**, pp. 267-99

K. Kirchgässner [1982] *Wave solutions of reversible systems and applications*, J. Diff. Eqns. **45**, pp. 113-27

P. Kuchment [1993] *Floquet Theory for Partial Differential Equations*, Birkhäuser: Basel

J.E. Marsden & T.J.R. Hughes [1983] *Mathematical Foundations of Elasticity*, Prentice-Hall: Englewood Cliffs, N. J.

J.W. McLean [1982] *Instabilities of finite-amplitude gravity waves on water of finite depth*, J. Fluid Mech. **114**, pp. 331-41

A. Mielke [1988] *Reduction of quasilinear elliptic equations in cylindrical domains with applications*, Math. Meth. Appl. Sci. **10**, pp. 51-66

A. Mielke [1991] *Hamiltonian and Lagrangian flows on center-manifolds with applications to elliptic variational problems*, Lecture Notes in Math. **1489**, Springer-Verlag: Berlin

A. Mielke [1992] *A center-manifold approach to steady bifurcations from spatially periodic patterns*, Preprint, IfAM, Universität Hannover.

A. Mielke [1994] *A new approach to sideband instabilities using the principle of reduced instability*, Preprint, IfAM, Universität Hannover.

P. Saffman & H.C. Yuen [1985] *Three dimensional waves on deep water*, Adv. in Nonlinear Waves (Ed. L. Debnath), Res. Notes in Math. **111**, Pitman Adv. Publ. Co.

M.J. Sewell & D. Porter [1980] *Constitutive surfaces in fluid mechanics*, Math. Proc. Camb. Phil. Soc. **88**, pp. 517-46

J.T. Stuart & R.C. DiPrima [1978] *The Eckhaus and Benjmain-Feir resonance mechanisms*, Proc. Roy. Soc. Lond. **A362**, pp. 27-41

G.B. Whitham [1967] *Nonlinear dispersion of water-waves*, J. Fluid Mech. **27**, pp. 399-412

G.B. Whitham [1974] *Linear and Nonlinear Waves*, Wiley-Interscience: New York

V.E. Zakharov [1968] *Stability of periodic waves of finite amplitude on the surface of* *a deep fluid*, Zh. Prikl. Mekh. Fiz. **9**, pp. 86-94 (English trans: J. Appl. Mech. Tech. Phys. **2**, p. 190)

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