Attractors for Modulation Equations on Unbounded Domains – Existence and Comparison –

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Abstract

We are interested in the long–time behavior of nonlinear parabolic PDEs defined on unbounded cylindrical domains. For dissipative systems defined on bounded domains, the long–time behavior can often be described by the dynamics in their finite–dimensional attractors. For systems defined on the infinite line, very little is known at present, since the lack of compactness prevents application of the standard existence theory for attractors. We develop here an abstract theorem based on the interaction of a uniform and a localizing (weighted) norm which allows us to define global attractors for some dissipative problems on unbounded domains such as the Swift–Hohenberg and the Ginzburg–Landau equation.

The second aim of this paper is the comparison of attractors. The so–called Ginzburg–Landau formalism allows us to approximate solutions of weakly unstable systems which exhibit modulated periodic patterns. Here we show that the attractor of the Swift–Hohenberg equation is upper semicontinuous in a particular limit to the attractor of the associated Ginzburg–Landau equation.

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1 Introduction

The phenomenon of pattern formation has attracted a lot of interest over the last few decades. In a typical situation one is concerned with a translationally invariant problem where the wavelength of the pattern is much smaller than the size of the physical domain. In such situations it is advantageous to study the system on an infinite physical domain. However, this leads to mathematical difficulties due to continuous spectra and noncompactness. Recently, new methods have been developed to uncover new mathematical structures associated with these phenomena. These topics range from convective versus absolute stability of traveling waves, sideband instabilities and diffusive stability, cf. [CE90a] and later work by the same authors. Another direction of research is concerned with the derivation and justification of so–called amplitude or modulation equations, which may be considered as a generalization of the center manifold reduction. The main idea is to describe the slow spatial and temporal modulations of a basic periodic pattern by the solutions to a simple differential equation for the amplitude function A. This formal multiple scaling method is also called the Ginzburg–Landau formalism, see [NW69, IMD89].

It is our aim to develop a theory which provides a general framework for attractors of problems defined on unbounded domains. The first part of the paper is devoted to a rigorous proof of the existence of attractors and estimates of their size. The main difficulties stem from the fact that the semiflows on unbounded domains are not compact, nor is the attractor. In the second part we consider a system close to the threshold of instability ($\varepsilon > 0$ small) which can be studied by the modulation theory. Thus, we have an attractor $\mathcal{A}^\varepsilon$ in the original system and another attractor $\mathcal{A}_\mathcal{G}$ in the associated modulation equation. The question arises as to how well the attractor $\mathcal{A}^\varepsilon$ can be described by $\mathcal{A}_\mathcal{G}$.

In this paper we start the investigation with simple model problems in one space dimension. In future work we will treat generalizations to higher dimensional and vector–valued problems. The simplest examples where the Ginzburg–Landau formalism applies are the Swift–Hohenberg equation ($\alpha = \varepsilon^2,\ \beta = 0,\ \gamma > 0$) and the weakly unstable Kuramoto–Shivashinsky equation ($\alpha = \varepsilon^2,\ \beta \neq 0,\ \gamma = 0$)

$$\partial_t u = -(1+\partial_x^2)^2 u + \alpha u + \beta u \partial_x u - \gamma u^3, \qquad x \in \mathbb{R}. \tag{1.1}$$

Our aim is to study this problem in function spaces containing all spatially bounded solutions, such as spatially (quasi–) periodic or traveling wave solutions. Our basic phase space Z will contain all functions in $C_b^2(\mathbb{R}, \mathbb{R})$.

Linearizing (1.1) with $\alpha = \varepsilon^2 \ll 1$ at $u \equiv 0$ gives solutions of the form $v(x,t) = e^{\widehat{\lambda} t + ikx}$, where $\widehat{\lambda}(k, \varepsilon^2) = -(1-k^2)^2 + \varepsilon^2$. One observes that $\widehat{\lambda}(k,\varepsilon^2)$ is positive for k close to ± 1 with height $\mathcal{O}(\varepsilon^2)$ and width $\mathcal{O}(\varepsilon)$. Therefore, it is natural to expect that for small $\varepsilon^2 > 0$ the long–time behavior of (1.1) is described by solutions u, which behave approximately as

$$u = \widetilde{\psi}_\varepsilon(A) := \varepsilon A(T,X)e^{ix} + \varepsilon\overline{A}(T,X)e^{-ix}, \tag{1.2}$$

with slow time scale $T = \varepsilon^2 t$ and large spatial scale $X = \varepsilon x$. This ansatz for u is the starting point of the Ginzburg–Landau formalism, and formally equating equal powers in ε and e^{ix} to zero shows that the complex valued amplitude $A = A(T,X) \in \mathbb{C}$ should satisfy a Ginzburg–Landau equation

$$\partial_T A = 4\partial_X^2 A + A - (\beta^2/9 + 3\gamma)|A|^2 A. \tag{1.3}$$

Recent work (see [CE90b, KSM92, vH91, Sch94a, Sch94b]) has demonstrated that the Ginzburg–Landau formalism provides a valid approximation in the sense that $\tilde{u}(t,x) = \widetilde{\psi}_\varepsilon(A)$ approximates a true solution u of (1.1) whenever A solves (1.3). Moreover, in [Eck93, Sch94c, Sch95] the attractivity property of the set of solutions in the form (1.2) was shown.

The above–mentioned results are only concerned with single solutions. Here we compare the long–time behavior of all solutions u of the original system (1.1) with the long–time behavior of the solutions A of (1.3). Assuming the existence of an attractor $\mathcal{A}^\varepsilon$ for (1.1) and an attractor $\mathcal{A}_\mathcal{G}$ for (1.3), it is natural to ask whether $\mathcal{A}^\varepsilon$ can be compared with $\widetilde{\psi}_\varepsilon(\mathcal{A}_\mathcal{G})$. Similar questions concerning (singular) limits of attractors are treated in [HR92a, HR92b]. There, PDEs defined on thin domains $\Omega \times (0,\varepsilon)$ are compared with their limit problems on Ω.

In our case of unbounded domains, the first difficulty is that the existence of an attractor is not clear at all. The problem is that the semigroup is not compact; in fact, the ω–limit sets of solutions can be empty; e.g., this is the case for a traveling pulse when translationally invariant norms are considered. Moreover, any attractor has to be translationally invariant, hence, if it is non–trivial, it cannot be compact in a Banach space with a uniform norm (i.e., translationally invariant norm).

Thus, neither the classical methods for dissipative systems nor those for damped hyperbolic systems on bounded domains (cf., [Te88, Ha88, BV92]) apply here. In [BV90] a first approach is introduced using weighted function spaces. By choosing appropriate weights the compactness is restored, at least in the weak topology, and the existence of attractors can be concluded. We build our theory on new ideas by Feireisl et al [Fei95, FLS94] which uses the strong topology in weighted spaces.

To be specific, let us consider the space $H^1_\rho(\mathbb{R})$ which contains all functions in $H^1_{\mathrm{loc}}(\mathbb{R})$ such that the weighted norm $\|u\|^2_{H^1_\rho} = \int \rho(u^2 + u'^2)\,dx$ is finite. Here ρ can be either $\rho(x) = 1/(1+x^2)$ or $\rho(x) = 1/\cosh(x)$. To introduce the basic phase space Z we use the translation operators $T_y : u \mapsto u(\cdot + y)$. Now, Z is the set

$$H^1_{l,u}(\mathbb{R}) = \{\, u \in H^1_\rho \;:\; \|u\|_{H^1_{l,u}} < \infty \text{ and } \|T_y u - u\|_{H^1_{l,u}} \to 0 \text{ for } y \to 0 \,\},$$

where the uniform norm is given by $\|u\|_{H^1_{l,u}} := \sup_{y\in I\!R} \|T_y u\|_{H^1_\rho}$. Note that this space is large enough to contain all sufficiently smooth bounded solutions, such as fronts, quasiperiodic, or periodic functions.

Denoting by $\mathcal{S}_t : H^1_{l,u} \to H^1_{l,u}$ and $\mathcal{G}_t : H^1_{l,u} \to H^1_{l,u}$ the semiflow of (1.1) and (1.3), respectively, we can show the existence of global attractors $\mathcal{A}^\varepsilon$ and $\mathcal{A}_\mathcal{G}$ for the Swift–Hohenberg equation and for the Ginzburg–Landau equation, respectively.

Theorem 1.1
There is a unique non–empty closed bounded set $\mathcal{A} \subset H^1_{l,u}$ with the following properties:
(i) $\mathcal{A}$ is compact in $H^1_\rho(I\!R)$.
(ii) $\mathcal{A}$ is time and translation invariant, i.e., $\mathcal{S}_t(\mathcal{A}) = T_y\mathcal{A} = \mathcal{A}$ for all $t \geq 0,\ y \in I\!R$.
(iii) $\mathcal{A}$ attracts bounded sets in $H^1_{l,u}$ with respect to the H^1_ρ–distance, i.e., for any bounded $B \subset H^1_{l,u}$, we have

$$\mathrm{dist}_{H^1_\rho}(\mathcal{S}_t(B), \mathcal{A}) = \sup_{b\in B} \inf_{a\in\mathcal{A}} \|\mathcal{S}_t(b) - a\|_{H^1_\rho} \to 0 \text{ for } t \to \infty.$$

This existence result is proved in Section 2 in a more abstract form. In fact, exploiting the translational invariance we are able to strengthen the attractivity to an intermediate distance, namely

$$\mathrm{dist}^*_{H^1_\rho}(\mathcal{S}_t(B), \mathcal{A}) = \sup_{y\in I\!R} \mathrm{dist}_{H^1_\rho}(T_y\mathcal{S}_t(B), T_y\mathcal{A}) \to 0 \text{ for } t \to \infty.$$

Using the linear heat equation as an example we show that attractivity of $\mathcal{A}$ is false in $H^1_{l,u}$. It is only possible to establish uniform convergence on compact subintervals $[x_1, x_2] \subset I\!R$. Note that $\mathrm{dist}^*_{H^1_\rho}(\mathcal{S}_t(u_0), \mathcal{A}) \leq \varepsilon$ implies that for each subinterval $[x_1, x_2]$ there exists an $a \in \mathcal{A}$ with $\sup\{\, |\mathcal{S}_t(u_0)(x) - a(x)| \ :\ x \in [x_1, x_2]\,\} \leq C(\rho, x_2 - x_1)\varepsilon$. Here, a generally depends on the interval $[x_1, x_2]$, but the estimate only depends only on the length $x_2 - x_1$.

In Section 3 we apply the abstract result to our problems (1.1) and (1.3). Moreover, we are able to give a–priori estimates of the diameter of the attractors in terms of the parameter of the equations. For instance, for $\gamma > 0$ we are able to show that the global attractor of (1.1) is contained in the ball

$$B_{H^1_{l,u}}(\Delta_1) = \{\, u \in H^1_{l,u} \ :\ \|u\|_{H^1_{l,u}} \leq \Delta_1 \,\} \quad \text{with } \Delta_1 = C\Delta_0 M(1 + |\alpha| M^{-4}),$$

where $\Delta_0 = (\alpha(\alpha + 1 + \beta^2/\gamma)\gamma^{-2})^{1/4}$ for $\alpha > 0$ and $\Delta_0 = 0$ for $\alpha \leq 0$ and $M = 1 + (|\beta|\Delta_0)^{2/5} + (\gamma\Delta_0^2)^{1/3}$. This result is established by weighted energy estimates using appropriate weights ρ depending on the parameters. Note that for $\alpha = \varepsilon^2$ the radius Δ_1 is of the order $\sqrt{\varepsilon}$, which is not the best possible. We believe that weighted energy estimates are not strong enough to deliver optimum results in cases where small solutions are considered (see the remark after Cor. 3.5).

As we expect that $\mathcal{A}^\varepsilon$ and $\psi_\varepsilon(\mathcal{A}_\mathcal{G})$ are close, we wish to improve the bound Δ_1 to $C\varepsilon$, and this is done by using information about the dynamics of the equation. In particular, we appeal to the Ginzburg–Landau formalism which tells us that the dynamics of all small solutions is dominated by the dynamics of the modulation equation. For this purpose we recall, without proofs, the main theorems of the modulation theory in a slightly generalized form (Section 4).

The improvement of the diameter of $\mathcal{A}^\varepsilon$ from order $\sqrt{\varepsilon}$ to order ε is given in Section 5. In fact, we prove much more: namely, for every weakly unstable modulation system with associated amplitude equation (1.3) there is a small ball with ε–independent radius which is positively invariant and is absorbed into a ball of radius $C\varepsilon$. (For example, this applies to (1.1) with $\gamma < 0$ as long as $\beta^2 + 27\gamma > 0$.) The method relies on shadowing the solutions by means of pseudo–orbits in the Ginzburg–Landau equation. As the Ginzburg–Landau equation has more structure, such as the maximum principle, we obtain much better control over the size of the solutions.

In the last section we finally compare the global attractor $\mathcal{A}_\mathcal{G}$ of (1.3) with the global attractor $\mathcal{A}^\varepsilon$ of (1.1). Since the limit attractor $\mathcal{A}_\mathcal{G}$ can be embedded via $\widetilde{\psi}_\varepsilon$ into the original system, it is natural to show $\frac{1}{\varepsilon}\mathrm{dist}^*_{H^1_\rho}(\mathcal{A}^\varepsilon, \widetilde{\psi}_\varepsilon(\mathcal{A}_\mathcal{G})) \to 0$ for $\varepsilon \to 0$. However, we prove a stronger result by rescaling the attractor $\mathcal{A}^\varepsilon$ in such a way that it can be compared with the ε–independent attractor $\mathcal{A}_\mathcal{G}$. To this end we define a mapping Φ_ε which extracts the Ginzburg–Landau modes A from any solution $u \in H^1_{l,u}$. The other modes belong to the damped Fourier modes and are controlled by a mode filter E_s.

We show the upper semicontinuity of the attractor $\mathcal{A}^\varepsilon$ for $\varepsilon \to 0$ in the following sense:

Theorem 1.2
For every $\sigma > 0$ there exist $C, \varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$ the estimates

$$\mathrm{dist}^*_{H^1_\rho}(\Phi_\varepsilon \mathcal{A}^\varepsilon, \mathcal{A}_\mathcal{G}) \le \sigma \quad \text{and} \quad \mathrm{dist}_{H^1_{l,u}}(E_s \mathcal{A}^\varepsilon, \{0\}) \le C\varepsilon^{5/4},$$

hold.

This result means that the complexity of the attractor $\mathcal{A}^\varepsilon$ cannot be much larger than the complexity of the Ginzburg–Landau attractor $\mathcal{A}_\mathcal{G}$. The question of lower semicontinuity is still open.

It should be noted that, in principle, the theories given are restricted neither to scalar problems nor to one unbounded spatial dimension. In future research we will consider generalizations of this work to hydrodynamical problems, such as the Bénard problem or the Taylor–Couette problem. See [Sch94b]) for first results in this direction. Of course it might be difficult (or impossible) to find global attractors for the Navier–Stokes equation on unbounded domains in the relevant uniform function spaces. But the existence of local attractors can be obtained by generalizing the pseudo–orbit technique to vector–valued problems.

Throughout this paper constants are denoted by C. Moreover we assume $0 < \varepsilon \ll 1$.

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2 An abstract existence theorem for attractors

We are interested in attractors for semigroups $(\mathcal{S}_t)_{t\geq 0}$ on a Banach space $(Z, \|\cdot\|)$ in cases where the semigroup is not compact. In our applications the semigroups have smoothing properties and noncompactness is due to translation to infinity. In particular, the ω–limit set of a particular solution can be empty.

Our basic assumption is that $\mathcal{S}_t(u) \in Z$ depends continuously on $(t, u) \in [0, \infty) \times Z$. A subset $\mathcal{B} \subset Z$ is called positively invariant (for $\mathcal{S}_t$) if $\mathcal{S}_t(\mathcal{B}) \subset \mathcal{B}$ for all $t > 0$, and it is called an *absorbing set* for $\mathcal{S}_t$, if it is bounded, positively invariant and every bounded set $B \subset Z$ is absorbed into $\mathcal{B}$ in finite time (i.e., there exists $t > 0$ such that $\mathcal{S}_t(B) \subset \mathcal{B}$).

As we will see it is essential to work with at least two topologies in problems on unbounded domains. So let us denote by Z_ρ the set Z, but equipped with a weaker topology induced by a norm $\|\cdot\|_\rho$ such that $\|u\|_\rho \leq \|u\|$.

Definition 2.1
*Let $\mathcal{B} \subset Z$ be positively invariant, then a subset $\mathcal{A} \subset \mathcal{B}$ is called an (Z, Z_ρ)–**attractor** for $\mathcal{S}_t$ in $\mathcal{B}$ if the following conditions hold:*
a) $\mathcal{A}$ is nonempty, closed, bounded in Z, and compact in Z_ρ.
b) $\mathcal{A}$ is invariant under $\mathcal{S}_t$, i.e., $\mathcal{S}_t(\mathcal{A}) = \mathcal{A}$ for all $t > 0$.
c) Every $B \subset \mathcal{B}$ which is bounded in Z is attracted to $\mathcal{A}$ in the distance induced by the norm of Z_ρ, i.e.,

$$\operatorname{dist}_{Z_\rho}(\mathcal{S}_t(B), \mathcal{A}) := \sup_{b\in B} \inf_{a\in\mathcal{A}} \|\mathcal{S}_t(b) - a\|_\rho \to 0 \quad \textit{for } t \to \infty.$$

If $\mathcal{A}$ is an (Z, Z_ρ)–attractor in $\mathcal{B} = Z$, then it is called the (global) attractor of $\mathcal{S}_t$.

From this definition it is standard to conclude the uniqueness of a (Z, Z_ρ)–attractor, since any second candidate $\mathcal{A}_2$ would be attracted to $\mathcal{A}$ as well as attract cA towards itself. Using the invariance and closedness we obtain

$$\operatorname{dist}_{Z_rho}(\mathcal{A}, \mathcal{A}_2) = \operatorname{dist}_{Z_rho}(\mathcal{A}_2, \mathcal{A}) = 0,$$

which implies $\mathcal{A} = \mathcal{A}_2$. The definition is a special case of that in [BV90, BV92], where Z_ρ can be an arbitrary topological space. If $Z = Z_\rho$ in the sense of topological spaces the definition coincides with the usual definition of an attractor ([Te88, Ha88]).

For the Banach space Z we have in mind the function space in which the solutions of the problems defined on unbounded domains should be in. Since this space should contain fronts, periodic and quasiperiodic solutions it should be equipped with a uniform norm, i.e $Z = L^\infty$ or $Z = H^1_{l,u}$ for instance.

As already explained such spaces have the disadvantage that the semigroup to the problem is not compact for them, and so classical theorems ([Ha88, Te88]) on the existence of attractors do not apply for this case.

As the following example shows attractivity in such norms is not true, i.e. we cannot expect the existence of a (Z, Z)–attractor.

Example 2.2 Consider the linear heat equation $u_t = u_{xx}$ with $x \in \mathbb{R}$ in the space $Z = C^0_{b,u}(\mathbb{R}, \mathbb{R})$, the space of uniformly continuous and bounded functions over the real axis with the classical sup–norm $\|\cdot\|_\infty$. The space Z_ρ is the space Z equipped with the norm $\|u\|_\rho = \|\rho u\|_\infty$, where $\rho(x) = 1/(1+x^2)$. We are interested in the attractor $\mathcal{A}$ in the positively invariant set $\mathcal{B} = \{u \in Z : \|u\|_\infty \le R\}$. The only solutions which stay in $\mathcal{B}$ for all negative times are the constants, thus $\mathcal{A} = \{u \equiv \theta : \theta \in [-R, R]\}$ is the only candidate for an attractor.

Consider now the solution $\mathcal{S}_t(u_0)$ with the initial condition $u_0 = (2/\pi)\arctan(x)$. Using the explicit solution formula $u(t,x) = \int_{\mathbb{R}} G(t, x-y) u_0(y) dy$ with the heat kernel $G(t,x) = \frac{1}{2\sqrt{\pi t}} e^{(x-y)^2/(4t)}$ is not difficult to show that for all t we have $u(t,x) \to \pm 1$ for $x \to \pm\infty$. Hence, we always have $\operatorname{dist}_Z(S_t(u_0), \mathcal{A}) = 1$, and attractivity of $\mathcal{A}$ cannot hold in the uniform norm of Z; but as we will show later, we have $\|\mathcal{S}_t(u_0) - 0\|_\rho \to 0$ for $t \to \infty$.

As this simple example shows a new theory for problems on unbounded domains has to be developed, and so in [BV90] it is proposed to work in two different topologies Z and Z_ρ. There, the space $Z = H^1_\rho$ is chosen and Z_ρ is H^1_ρ equipped with its weak topology. So there attractivity is only obtained in the weak topology, whereas our results will be in a slightly stronger sense.

Our choice of spaces is initiated by a work of [Fei95]. For our problems on unbounded domains, the loss of compactness is due to the translational invariance of the problem. This invariance can be abstractly formulated in the existence of a translation group $(T_y)_{y \in \mathbb{R}}$ acting on Z as isometries and commuting with the semigroup.

We assume that there exists a localizing norm $\|\cdot\|_\rho$ on Z with the following properties:

(A1) *The translations T_y are continuous with respect to the norm $\|\cdot\|_\rho$ and*

$$\|u\| = \sup\{\, \|T_y u\|_\rho \,:\, y \in \mathbb{R}\,\}.$$

For notational convenience we let $Z_u = (Z, \|\cdot\|)$ and $Z_\rho = (Z, \|\cdot\|_\rho)$ to indicate that Z is equipped with different norms. However, we mostly omit the subscript u to denote the topological space $Z = Z_u$. The *uniform* space Z_u is the original Banach space, whereas Z_ρ is a normed space only which is not necessarily complete. In fact, our interest lies precisely in those cases where the translations T_y are not uniformly bounded in the ρ–norm, which implies that Z_ρ is not complete.

From now on we use the abbreviation $B_Z(r, u_0)$ for the closed ball of radius r in Z with center u_0, i.e., $\{ u \in Z \, : \, \|u - u_0\| \le r \}$. Moreover we let $B_Z(r) = B_Z(r, 0)$. As a direct consequence of (A1) we obtain the following result which will be needed in Theorem 2.6.

Lemma 2.3
Let $A \subset Z$ be contained in $B_Z(r)$ for some $r > 0$, then $\overline{A}^\rho = \mathrm{closure}_{Z_\rho}(A) \subset B_Z(r)$.

Proof: Let $u_n \in B$ be a sequence with limit u in Z_ρ. Since $\|u_n\| < r$ and since T_y is continuous in Z_ρ we obtain for fixed y

$$\|T_y u\|_\rho \le \|T_y u_n\|_\rho + \|T_y u_n - T_y u\|_\rho \le r + \varepsilon_n$$

with $\varepsilon_n \to 0$ for $n \to \infty$. Therefore $\|u\| \le r$, which is the desired result. □

As an example consider Z and Z_ρ as in Example 2.2 with the translations T_y are given by $T_y u(x) = u(x + y)$. Convergence in Z_u is uniform convergence, whereas convergence in Z_ρ means uniform convergence on each compact interval. Consider $u : x \to \tanh(x)$ and let $A = \{ T_y u \, : \, y \in \mathbb{R} \}$. Then A is closed in Z_u but its closure in Z_ρ is $\overline{A}^\rho = A \cup \{v_{-1}, v_1\}$, where $v_{\pm 1} \equiv \pm 1$.

These two norms allows us to define two different distances between sets, dist_{Z_ρ} and dist_{Z_u}. Since our problem is translational invariant we introduce a third intermediate distance, $\mathrm{dist}^*_{Z_\rho}$. We define

$$\begin{aligned}
\mathrm{dist}_{Z_\rho}(b, A) &= \inf_{a \in A} \|b - a\|_\rho \\
\mathrm{dist}^*_{Z_\rho}(b, A) &= \sup_{y \in \mathbb{R}} \inf_{a \in A} \|T_y b - T_y a\|_\rho = \sup_{y \in \mathbb{R}} \mathrm{dist}_{Z_\rho}(T_y b, T_y A) \\
\mathrm{dist}_{Z_u}(b, A) &= \inf_{a \in A} \|b - a\| = \inf_{a \in A} \sup_{y \in \mathbb{R}} \|T_y b - T_y a\|_\rho
\end{aligned}$$

For all three distances we let $\mathrm{dist}(B, A) = \sup_{b \in B} \mathrm{dist}(b, A)$ for $B \subset Z$. Obviously, we have the ordering $\mathrm{dist}_{Z_\rho}(B, A) \le \mathrm{dist}^*_{Z_\rho}(B, A) \le \mathrm{dist}_{Z_u}(B, A)$. Note that all these (nonsymmetric) distance measures satisfy $\mathrm{dist}(A, B) = 0 \Leftrightarrow A \subset \overline{B} = \mathrm{closure}(B)$, $A \subset B \Rightarrow \mathrm{dist}(A, C) \le \mathrm{dist}(B, C)$, and $\mathrm{dist}(A, C) \le \mathrm{dist}(A, B) + \mathrm{dist}(B, C)$.

Example 2.4 In order to see the differences between these distances we consider the families of functions $u(t, x) = (2/\pi) \arctan(t + x)$ and $v(t, x) = (2/\pi) \arctan(x/(t + 1))$.

Let Z, Z_ρ and the set $\mathcal{A}$ be given as in Example 2.2. We obtain $\mathrm{dist}_{Z_\rho}(u(t),\mathcal{A}) \leq C/t$ for $t>0$, but $\mathrm{dist}^*_{Z_\rho}(u(t),\mathcal{A}) = \mathrm{dist}_{Z_u}(u(t),\mathcal{A}) \equiv 1$. For the second family we also have $\mathrm{dist}_{Z_u}(v(t),\mathcal{A}) \equiv 1$. However, $|v(t,x+y)-v(t,y)| \leq 2|x|/(1+t)$ implies

$$\|T_y v(t) - \theta(y)\|_\rho \leq \sup_{x\in\mathbb{R}} \frac{2|x|}{(1+x^2)(1+t)} \leq C/(1+t)$$

if θ is chosen as $\theta(y) = v(t,y)$. Thus, $\mathrm{dist}_{Z_\rho}(v(t),\mathcal{A}) \leq \mathrm{dist}^*_{Z_\rho}(v(t),\mathcal{A}) \leq C/(1+t)$.

This example shows that the three distance measures are really different for cases of interest in dynamics on unbounded domains. However, the next lemma states that $\mathrm{dist}^*_{Z_\rho}$ and dist_{Z_ρ} coincide for translationally invariant sets.

Lemma 2.5
Let $A, B \subset Z$ with $T_y A = A$, $T_y B = B$, for all $y \in \mathbb{R}$, then $\mathrm{dist}_{Z_\rho}(B,A) = \mathrm{dist}^*_{Z_\rho}(B,A)$.

Proof: We may interchange the two supremum in $\mathrm{dist}^*_{Z_\rho}(B,A)$ as follows.

$$\mathrm{dist}^*_{Z_\rho}(B,A) = \sup_{b\in B}\sup_{y\in\mathbb{R}}\inf_{a\in A}\|T_y b - T_y a\|_\rho = \sup_{y\in\mathbb{R}} \mathrm{dist}_{Z_\rho}(T_y B, T_y A) = \mathrm{dist}_{Z_\rho}(B,A).$$

In the last equality we used the translational invariance of A and B. □

With all the preparations from above we can show the existence of an (Z_u, Z_ρ)–attractor for problems on unbounded domains:

Theorem 2.6
Let Z_u, T_y, and Z_ρ be given as above such that (A1) holds. Moreover, let $\mathcal{S}_t$ be a continuous semigroup on Z_u which is translationally invariant ($T_y\mathcal{S}_t = \mathcal{S}_t T_y$) and has a nonempty, bounded, and positively invariant set $\mathcal{B} \subset Z_u$. Assume that the following additional assumptions hold:

(A2) (localized continuity) *For each $t \geq 0$ the evolution operator $\mathcal{S}_t$ is continuous from Z_ρ into itself.*

(A3) (compactness) *For all subsets $B \subset \mathcal{B}$ there is a $t_0 > 0$ such that $\mathcal{S}_{t_0}(B)$ is precompact in Z_ρ.*

Then there exists an unique (Z_u, Z_ρ)–attractor $\mathcal{A}$ for $\mathcal{S}_t$ in $\mathcal{B}$. If, additionally, $\mathcal{B}$ is an absorbing set, then $\mathcal{A}$ is the global attractor. Moreover, $\mathcal{A}$ has the following properties:
(i) $\mathcal{A}$ is shift–invariant, i.e. $T_y\mathcal{A} = \mathcal{A}$ for all $y \in \mathbb{R}$.
(ii) Every $B \subset \mathcal{B}$ which is bounded in Z is attracted to $\mathcal{A}$ with respect to the distance $\mathrm{dist}^*_{Z_\rho}$, i.e.,

$$\mathrm{dist}^*_{Z_\rho}(\mathcal{S}_t(B),\mathcal{A}) := \sup_{b\in B}\sup_{y\in\mathbb{R}}\inf_{a\in\mathcal{A}}\|T_y\mathcal{S}_t(b) - T_y a\|_\rho \to 0 \quad \text{for } t\to\infty.$$

(iii) Assume that Z_{ρ_1} and Z_{ρ_2} define equivalent norms in Z_u via (A1). Denote by $\mathcal{A}_i$ the attractor constructed using Z_{ρ_i}, then $\mathcal{A}_1 = \mathcal{A}_2$.

Proof: Without loss of generality we can assume that the set $\mathcal{B}$ is translationally invariant, i.e. $T_y\mathcal{B} = \mathcal{B}$ for all $y \in I\!R$. If not, take $B_1 = \cup_{y\in I\!R} T_y\mathcal{B}$ instead of $\mathcal{B}$. The attracting set is defined by

$$\mathcal{A} = \bigcap_{t\geq 0} A_t \quad \text{with } A_t = \overline{\mathcal{S}_t(\mathcal{B})}^{\rho}.$$

As $\mathcal{B}$ is positively invariant, the family $(A_t)_{t\geq 0}$ is a decreasing family, i.e., $A_{t_1} \subset A_{t_2}$ for $t_1 > t_2$. Hence, $\mathcal{A} \subset A_0$ and $A_0 = \overline{\mathcal{B}}^{\rho}$ is bounded by Lemma 2.3. Therefore, $\mathcal{A}$ is bounded in Z_u. Moreover, from (A3) the set A_{t_0}, and hence all A_t for $t \geq t_0$ are compact in Z_ρ. $(A_t)_{t\geq t_0}$ forms a decreasing family of compact and nonempty sets in Z_ρ. Thus, $\mathcal{A} = \cap_{t\geq t_0} A_t$ is nonempty and compact in Z_ρ, as it is the intersection of a decreasing family of nonempty compact sets. As $\mathcal{A}$ is closed in Z_ρ it is also closed in Z_u. This proves part a) in the Definition 2.1 for a (Z_u, Z_ρ)–attractor.

As $T_y\mathcal{S}_t(\mathcal{B}) = \mathcal{S}_t(T_y\mathcal{B}) = \mathcal{S}_t(\mathcal{B})$ we find by taking the closure in Z_ρ and by using the boundedness of T_y the relation $T_yA_t = A_t$. This implies (i).

The more difficult part of the proof is to show that $\mathcal{A}$ is in fact an attractor. It remains to show the time invariance and the attractivity, i.e. part b) and c) in Definition 2.1.

1) (Time invariance)

Let $v \in \mathcal{S}_t(\mathcal{A})$, i.e. $v = \mathcal{S}_t(u)$, where $u = \lim_{t_n\to\infty} \mathcal{S}_{t_n}(u_n)$ in Z_ρ with $u_n \in \mathcal{B}$. Because of (A2) (continuity of $\mathcal{S}_t$ in Z_ρ), we have $\mathcal{S}_{t_n}(\mathcal{S}_t(u_n)) = \mathcal{S}_t(\mathcal{S}_{t_n}(u_n)) \to \mathcal{S}_t(u) = v$ as $n\to\infty$ in Z_ρ. As $\mathcal{S}_t(u_n) \in \mathcal{B}$ we conclude $v \in \mathcal{A}$ and hence $\mathcal{S}_t(\mathcal{A}) \subset \mathcal{A}$.

For the opposite direction (and the attractivity discussed below) the compactness in Z_ρ plays a crucial role. Let $v \in \mathcal{A}$, then there exist $t_n \to \infty$ and $u_n \in \mathcal{B}$ with $t_n < t_{n+1}$ and $v = \lim_{t_n\to\infty} \mathcal{S}_{t_n}(u_n)$ in Z_ρ. For any $t > 0$ we wish to show $v \in \mathcal{S}_t(\mathcal{A})$. From (A3) the set $\{\, \mathcal{S}_{(t_n-t)}(u_n) \;:\; t_n - t \geq t_0(\mathcal{B}) \,\} \subset \mathcal{S}_{t_0(\mathcal{B})}(\mathcal{B})$ is precompact in Z_ρ. Therefore for a subsequence $w_i = \mathcal{S}_{(t_{n_i}-t)}(u_{n_i}) \to w$ in Z_ρ. Applying the continuous mapping $\mathcal{S}_t$ we find $v = \mathcal{S}_t(w)$. As $w_j \in A_{t_{n_i}-t}$ for $j \geq i$ and all A_t are closed, w lies in all A_t and hence in $\mathcal{A}$. Thus, $v = \mathcal{S}_t(w) \in \mathcal{S}_t(\mathcal{A})$ and $\mathcal{A} \subset \mathcal{S}_t(\mathcal{A})$ is proved.

2) (Attractivity in Z_ρ)

We use the compactness to give a proof by contradiction. Let $B \subset \mathcal{B}$ be arbitrary. Assume that B is not attracted to $\mathcal{A}$, then there exist $C > 0$, sequences $t_n \to \infty$ and $u_n \in B$ such that $\mathrm{dist}_{Z_\rho}(\mathcal{S}_{t_n}(u_n), \mathcal{A}) > C > 0$ for all $n \in I\!N$. Because of compactness there is a subsequence such that $v_i = \mathcal{S}_{t_{n_i}}(u_{n_i})$ converges in Z_ρ to w. As w lies in $\mathcal{A}$ this is a contradiction. Hence, we have shown $\mathrm{dist}_{Z_\rho}(\mathcal{S}_t(B), \mathcal{A}) \to 0$ for $t\to\infty$, for all $B \subset \mathcal{B}$.

3) (Attractivity in with respect to $\mathrm{dist}^*_{Z_\rho}$)

Let $B \subset \mathcal{B}$ be bounded and consider $\tilde{B} = \bigcup_{y\in I\!R} T_yB$. Then $\mathcal{S}_t(B) \subset \mathcal{S}_t(\tilde{B}) = T_y\mathcal{S}_t(\tilde{B})$. Hence,

$$\mathrm{dist}^*_{Z_\rho}(\mathcal{S}_t(B), \mathcal{A}) \leq \mathrm{dist}^*_{Z_\rho}(\mathcal{S}_t(\tilde{B}), \mathcal{A}) = \mathrm{dist}_{Z_\rho}(\mathcal{S}_t(\tilde{B}), \mathcal{A}).$$

according to Lemma (2.5). Hence (ii) follows from the attractivity in Z_ρ.

4) (Independence of ρ)

Suppose two different Z_{ρ_1} and Z_{ρ_2} which imply equivalent norms in Z. Then $\mathcal{A}_1$ is attracted in the Z_{ρ_2}–norm to $\mathcal{A}_2$. This yields $\mathrm{dist}_{Z_{\rho_2}}(\mathcal{A}_1, \mathcal{A}_2) = 0$. Interchanging the roles gives $\mathrm{dist}_{Z_{\rho_1}}(\mathcal{A}_2, \mathcal{A}_1) = 0$. The closedness implies $\mathcal{A}_1 \subset \mathcal{A}_2 = \overline{\mathcal{A}}_2^{\rho_2}$ and $\mathcal{A}_2 \subset \mathcal{A}_1 = \overline{\mathcal{A}}_1^{\rho_1}$, and so $\mathcal{A}_2 = \mathcal{A}_1$. □

Example 2.7 Let us return to Example 2.2 which meets all the assumptions of Theorem 2.6. Hence, we have attractivity in Z_ρ as well as in the intermediate distance $\mathrm{dist}^*_{Z_\rho}$. To substantiate the theory we consider solutions $u(t) = \mathcal{S}_t u_0$ with arbitrary initial conditions $u_0 \in \mathcal{B}$. Using the explicit solution formula we easily find $\|\partial_x u(t)\|_\infty \leq CR/\sqrt{t}$ for all $u_0 \in \mathcal{B}$. Choosing $\theta(y) = u(t, y)$, we obtain

$$\begin{array}{rcl} \|T_y u(t) - \theta(y)\|_\rho & \leq & \sup_{x\in\mathbb{R}} \frac{1}{1+x^2}|u(t, x + y) - u(t, y)| \\ & \leq & \sup_{x\in\mathbb{R}} \frac{1}{1+x^2}\frac{C}{\sqrt{t}}|x|\|u_0\|_\infty \leq C/\sqrt{t} \end{array}$$

Thus, we conclude $\mathrm{dist}_{Z_\rho}(\mathcal{S}_t(\mathcal{B}), \mathcal{A}) = \mathrm{dist}^*_{Z_\rho}(\mathcal{S}_t(\mathcal{B}), \mathcal{A}) \leq CR/\sqrt{t}$. Hence, for this linear case we obtain an explicit decay rate towards the attractor.

We conclude this section with some general remarks.

1) The whole theory can be generalized to more general translation groups T_g, $g \in G$, where G is a general group. For multidimensional unbounded domains one may choose $G = \mathbb{R}^d$. For problems which are only invariant under discrete translations (PDEs with periodic coefficients) the suitable group is $G = \mathbb{Z}^d$.

2) The choice of the norm $\|\cdot\|_\rho$ leads to a topology on Z which can be seen as the topology of uniform convergence on compact subsets of $\mathbb{R}$. Hence, the whole theory could be formulated without the use of any metric in Z_ρ by just appealing to this topology. This is also the reason why we have proved the independence of $\mathcal{A}$ from the specific choice of the weight ρ. One impotrant point is that the smallness of $\mathrm{dist}^*_{Z_\rho}(\mathcal{S}_t(u_0), \mathcal{A})$ implies that $\mathcal{S}_t(u_0)$ can be approximated on any subinterval $[x_1, x_2]$ by some element a of $\mathcal{A}$, where the quality of the approximation depends only on the length $x_2 - x_1$ but not its position in $\mathbb{R}$. However, we have to take into account that $a \in \mathcal{A}$ generally depends on $[x_1, x_2]$.

3 Existence of attractors

3.1 Uniformly local function spaces

The function spaces we intend to base our analysis on should be rich enough to contain all solutions which are bounded over the real line. As many of our results rely on energy estimates or on the Fourier transform methods, we introduce spaces based on L^2 theory. First we choose a positive weight function $\rho : \mathbb{R} \to (0, \infty)$ which is continuous, bounded, and has a finite integral $\int_{\mathbb{R}} \rho(x)\, dx$. For later purposes we also impose $\rho \in C^2(\mathbb{R}, \mathbb{R})$, such that $|\rho'(x)|, |\rho''(x)| \leq \rho(x)$ for all x. As a consequence we obtain $\rho(x+y) \leq e^{|y|}\rho(x)$ for all x, y. (We may fix ρ once and for all to $\rho(x) = 1/\cosh(x)$ or $\rho(x) = 2/(2+x^2)$.) Next we let

$$\widetilde{L}^2_{l,u}(I\!\!R) = \{\, u \in L^2_{\text{loc}}(I\!\!R) \;:\; \|u\|_{L^2_{l,u}} < \infty \,\}, \quad \text{where } \|u\|^2_{L^2_{l,u}} = \sup_{y \in I\!\!R} \int_{I\!\!R} \rho(y+x) u(x)^2 dx$$

and define the translation operator $T_y : \widetilde{L}^2_{l,u} \to \widetilde{L}^2_{l,u}; u \mapsto u(\,\cdot + y)$. Here and later on we omit the argument $I\!\!R$ as all function spaces are defined over the real line. Our final space of uniformly local L^2 functions is given as

$$L^2_{l,u} = \{\, u \in \widetilde{L}^2_{l,u} \;:\; \|T_y u - u\|_{L^2_{l,u}} \to 0 \text{ as } y \to 0 \,\}.$$

Note that different weight functions lead to the same uniform space with equivalent norms. For $s \in I\!\!N$ we define the associated Sobolev spaces $\widetilde{H}^s_{l,u}$ and $H^s_{l,u}$ by requiring that the first s distributional derivatives lie in the space as well.

Before studying the properties of these spaces we relate it to the abstract theory in the previous section. In our examples the space $H^1_{l,u}$ will play the role of Z_u. The weighted norm is simply given by $\|u\|^2_\rho = \int_{I\!\!R} \rho(u^2 + u'^2)dx$. We write H^1_ρ for the space Z_ρ. Clearly the translation operators T_y are bounded in the weighted norm by $\sup\{\, \rho(x+y)/\rho(x) \;:\; y \in I\!\!R \,\} \leq e^{|y|}$. Moreover, $\|u\|_u = \sup\{\, \|T_y u\|_\rho \;:\; y \in I\!\!R \,\}$ defines an equivalent norm in $H^1_{l,u}$ which is translationally invariant.

Lemma 3.1

a) All functions u in $\widetilde{H}^1_{l,u}$ are bounded and continuous and satisfy

$$|u(x)|^2 \leq C\|u\|_{L^2_{l,u}}\|u\|_{H^1_{l,u}}, \quad |u(x) - u(y)| \leq C\sqrt{|x-y|}\,\|u\|_{H^1_{l,u}}. \tag{3.1}$$

b) $L^2_{l,u}$ is a closed subspace of $\widetilde{L}^2_{l,u}$.
c) The spaces $\widetilde{H}^s_{l,u}$ and $H^s_{l,u}$ are dense in $L^2_{l,u}$.
d) $L^2_{l,u}$ is a proper subset of $\widetilde{L}^2_{l,u}$.

Proof: ad a) These estimates follow exactly as in the case of classical H^1 functions.

ad b) To show the closedness of $L^2_{l,u}$ let $(u_n)_{n\in I\!\!N}$ be a Cauchy sequence in $L^2_{l,u}$ with limit $u \in \widetilde{L}^2_{l,u}$. Then

$$\|T_y u - u\|_{L^2_{l,u}} \leq \|T_y u - T_y u_n\|_{L^2_{l,u}} + \|T_y u_n - u_n\|_{L^2_{l,u}} + \|u_n - u\|_{L^2_{l,u}} = s_1 + s_2 + s_3$$

Now choose n such that $s_1 = s_3 < \varepsilon/3$. Since $u_n \in L^2_{l,u}$ we find δ such that for all $y \in (-\delta, \delta)$ the relation $s_2 < \varepsilon/3$ holds.

ad c) Let $u \in \widetilde{H}^1_{l,u}$. With part a) we conclude

$$\|T_y u - u\|^2_{L^2_{l,u}} = \sup\{ \int_{I\!\!R} \rho(x)(u(x+y+z) - u(x+z))^2 dx \;:\; z \in I\!\!R \} \leq C|y|\|u\|^2_{H^1_{l,u}}.$$

Thus, $\widetilde{H}^1_{l,u} \subset L^2_{l,u}$ and whence $H^s_{l,u} \subset \widetilde{H}^s_{l,u} \subset L^2_{l,u}$ for $s \in I\!N$. To see that $H^1_{l,u}$ is dense in $L^2_{l,u}$ choose any $u \in L^2_{l,u}$. As $y \mapsto T_y u \in L^2_{l,u}$ is continuous the Riemann integral $v_h = \frac{1}{h}\int_0^h T_y u\, dy$ exists and $\|v_h - u\|_{L^2_{l,u}} \to 0$ for $h \to 0$. Moreover, $v_h \in H^1_{l,u}$ as $v_h(x) = \frac{1}{h}\int_0^h u(x+y)\, dy = \frac{1}{h}\int_x^{x+h} u(z)\, dz$. Iterating this process it follows that $H^s_{l,u}$ is dense in $L^2_{l,u}$.

ad d) It suffices to consider the example $u(x) = n$ for $x \in [n, n+1/n^2]$, $n \in I\!N$, and $u(x) = 0$ else. This u lies in $u \in \widetilde{L}^2_{l,u} \setminus L^2_{l,u}$. Hence, $L^2_{l,u}$ is a proper subspace of $\widetilde{L}^2_{l,u}$. □

Next we give the basic theorem concerning localized compactness. Working on unbounded domains compactness relies on two facts: first we need smoothing properties of the semigroup, and second, we have to localize the norm to control the behavior at infinity.

Theorem 3.2
Let $B \subset H^1_{l,u}$ be any set which is bounded in $H^n_{l,u}$ for some $n > 1$, then B is precompact in H^1_ρ.

Proof: For every ε we have to show that B admits in H^1_ρ a finite covering by balls of radius less than ε. We decompose every $u \in B$ into $u = v + w$ with $v = u\chi_\beta$ and $w = u(1-\chi_\beta)$, where the smooth cut–off function χ_β vanishes for $|x| \leq \beta$ and equals 1 for $|x| \geq \beta + 1$. Then $\|v\|_{H^1_\rho} < \varepsilon$ for β sufficiently large, as ρ decays for $|x| \to \infty$ and u varies in a bounded set. Moreover, $w \in H^n([-\beta-1, \beta+1])$ which can be embedded compactly into $H^1([-\beta-1, \beta+1])$. Since for functions with support $[-\beta-1, \beta+1]$ the norms in H^n and H^n_ρ are equivalent, there is a finite covering of this set by balls $B_{H^1_\rho}(\varepsilon, w_i)$ with $i = 1, \ldots, m < \infty$. Thus $\cup_{i=1,\ldots,m}\{B_{H^1_\rho}(2\varepsilon, w_i)\}$ is a finite covering of B, since $\|u - w_i\|_{H^1_\rho} \leq \|v\|_{H^1_\rho} + \|w - w_i\|_{H^1_\rho}$. □

An important tool in studying translationally invariant operators on the spaces $H^s_{l,u}$ is the so–called multiplier theory which uses Fourier transform methods. An operator $M : H^q_{l,u} \to L^2_{l,u}$ is called a multiplier if it is defined by multiplying the Fourier transform $\widehat{u} = \mathcal{F}u$ by a function $\widehat{m} \in L^\infty(I\!R, \mathbb{C})$ and then doing an inverse Fourier transform. Using the following lemma, which is proved in [Sch94b], allows us to study the mapping properties of $M : u \mapsto \mathcal{F}^{-1}(\widehat{m}\mathcal{F}u)$. Natural applications are convolution operators $Mu(x) = \int_{I\!R} m(x-y)u(y)\, dy$ with m integrable, where $\widehat{m} = \mathcal{F}m$.

Lemma 3.3
Let $q, s \geq 0$ and $w_{s-q}(k) = (1+k^2)^{(s-q)/2}\widehat{m}(k) \in C^2_b(I\!R, \mathbb{C})$. Then $M : H^q_{l,u} \to H^s_{l,u}; u \mapsto \mathcal{F}^{-1}(\widehat{m}\mathcal{F}u)$ is well defined with the estimate

$$\|Mu\|_{H^s_{l,u}} \leq C(q,s)\|w_{s-q}\|_{C^2_b(I\!R,\mathbb{C})}\|u\|_{H^q_{l,u}},$$

where $C(q,s)$ does not depend on $\widehat{m}$.

In [Sch94b] this lemma was established in the spaces $\widetilde{H}^s_{l,u}$. However, as all multipliers M are invariant under translations (i.e., $T_y M = M T_y$), it is obvious that they retain continuity with respect to translations.

3.2 The Ginzburg–Landau equation

As the first example we treat the real Ginzburg–Landau equation for the complex valued variable $A = A(T, X)$:

$$\partial_T A = \partial_X^2 A + \mathbf{a} A - \mathbf{b} A |A|^2 \tag{3.2}$$

with $\mathbf{a}$, $\mathbf{b} > 0$. The real variables $T \in [0, \infty)$ and $X \in \mathbb{R}$ are written in capital letters only for later reasons when this equation is interpreted as a modulation equation obtained from the Swift–Hohenberg problem via the scalings $T = \delta^2 t$ and $X = \delta x$.

This equation was studied extensively on L^∞ spaces, cf. [CE90b, Co94, TBDvHT94]. In [Sch94b] a treatment using $\widetilde{H}^1_{l,u\,\mathbb{C}} := H^1_{l,u} \oplus i H^1_{l,u}$ was started. We repeat here some of the material in order to ensure that the paper remains sufficiently self–contained. (We simply write $H^1_{l,u}$ for $H^1_{l,u\,\mathbb{C}}$.)

Theorem 3.4
The real Ginzburg–Landau equation generates a global semigroup $(\mathcal{G}_t)_{T\geq 0}$ in $H^1_{l,u}$ such that every solution satisfies

$$\limsup_{T\to\infty} \|A(T)\|_{H^1_{l,u}} \leq \widetilde{\Delta}_{\mathcal{G}} := C \left(\frac{\mathbf{a} + \mathbf{a}^4}{\mathbf{b}^2} \right)^{1/4}, \tag{3.3}$$

where C is a universal constant independent of A, $\mathbf{a}$, and $\mathbf{b}$.

Moreover, for all $t > 0$ and $n \in \mathbb{N}$ bounded sets $B \subset H^1_{l,u}$ are mapped into bounded sets in $H^n_{l,u}$, and for all $T \geq 0$ the mapping $\mathcal{G}_T : H^1_\rho \to H^1_\rho$ is continuous.

Proof: According to Lemma 3.3 the operator $\partial_X^2 : H^3_{l,u} \to H^1_{l,u}$ generates a holomorphic semigroup $G(t) = e^{t\partial_X^2}$, $t \geq 0$. Rewriting the equation as an integral equation we arrive at

$$A(t) = G(t)A(0) + \int_0^t G(t-r) F(A(r))\, dr \quad \text{where } F(A) = (\mathbf{a} - |A|^2)A.$$

The nonlinearity is an analytical mapping from $H^1_{l,u}$ into itself. Hence standard contraction arguments (cf. [He81]) prove local existence and continuous dependence on the initial data. Smoothness (even analyticity) in X and T is well known, see [TBDvHT94].

To show global existence we wish to employ weighted energy estimates using the scaled weight $\rho_b(X) = \rho(bX)$. We then have $|\rho_b'(X)| \leq b \rho_b(X)$ and

$$\|A\|^2_{L^2_{l,u}} \leq C(1+b) \sup\{ \int_{\mathbb{R}} \rho_b(X) A^2(X+y)\, dX : y \in \mathbb{R} \}, \tag{3.4}$$

with a similar estimate in H^1. Using the differential equation we obtain

$$\begin{array}{rcl}\frac{1}{2}\frac{d}{dT}\int_{\mathbb{R}}\rho_b|A|^2\,dX & \leq & \mathrm{Re}\int\rho_b\overline{A}A''\,dX+\int\rho_b(\mathbf{a}|A|^2-\mathbf{b}|A|^4)\,dX\\ & = & \int\Big(-\rho_b|A'|^2-\mathrm{Re}\Big[\rho_b'\overline{A}A'\Big]+\rho_b|A|^2(\mathbf{a}-\mathbf{b}|A|^2)\Big)\,dX\\ & \leq & \int\rho_b|A|^2(\mathbf{a}+b^2/4-\mathbf{b}|A|^2)dX\\ & \leq & \int\rho_b\frac{\mathbf{a}+b^2/4}{\mathbf{b}}\Big(\mathbf{a}+b^2/4-\mathbf{b}|A|^2\Big)\,dX.\end{array}\tag{3.5}$$

For the estimate of the derivative we use the relation

$$\int\rho_b|A'|^2\,dX=-\int(\rho_b'A'+\rho_bA'')\overline{A}\,dX\leq\int\rho_b(b|A'|+|A''|)|A|\,dX,\tag{3.6}$$

let $\widetilde{\alpha}=\mathbf{a}+b^2/4$, and proceed as follows

$$\begin{array}{rcl}\frac{1}{2}\frac{d}{dT}\rho_b|A'|^2\,dX & \leq & \mathrm{Re}\int\rho_b\overline{A'}\Big(A'''+\mathbf{a}A'-\mathbf{b}(2|A|^2A'+A^2\overline{A'})\Big)\,dX\\ & \leq & \int\rho_b(-|A''|^2+b|A'A''|+\mathbf{a}|A'|^2)\,dX\\ & & +\frac{\widetilde{\alpha}}{4}\left(\int\rho_b(b|AA'|+|AA''|)\,dx-\int\rho_b|A'|^2\,dX\right)\\ & \leq & -\widetilde{\alpha}\int\rho_b|A'|^2\,dX+\frac{9\widetilde{\alpha}}{16}(4\widetilde{\alpha}+9b^2)\int\rho_b|A|^2\,dX.\end{array}\tag{3.7}$$

Note that the factor of $\widetilde{\alpha}$ in the third line is nonnegative due to (3.6).

These two differential inequalities imply that the $H^1_{l,u}$ norm cannot blow up. Use the Gronwall estimate in (3.5) and then for (3.7). The desired uniform estimate is obtained by applying the weighted energy estimates to the translated solution T_yA for each $y\in\mathbb{R}$. Taking the limit $T\to\infty$ in these estimates we find

$$\begin{array}{rcl}\limsup\limits_{T\to\infty}\int_{\mathbb{R}}\rho_b|A(T,X)|^2\,dX & \leq & \dfrac{4\mathbf{a}+b^2}{4\mathbf{b}}\displaystyle\int\rho_b\,dX,\\ \limsup\limits_{T\to\infty}\int_{\mathbb{R}}\rho_b|A'(T,X)|^2\,dX & \leq & \dfrac{9(4\mathbf{a}+b^2)(2\mathbf{a}+5b^2)}{32\mathbf{b}}\displaystyle\int\rho_b\,dX.\end{array}$$

Using the estimate (3.4) and $\int\rho_b(X)\,dX=\frac{1}{b}\int\rho_1(X)\,dX$ we can optimize with respect to b. Using $b=\sqrt{\mathbf{a}}$ leads to the desired result.

In order to show the continuity of the semigroup in H^1_ρ, we consider a given solution $A\in H^1_{l,u}$ and the solution $A+V$ where $V(0,\cdot)$ is small in H^1_ρ but may be arbitrarily large in $H^1_{l,u}$. Our aim is to estimate $V(T)$ in terms of $V(0)$ using weighted energy estimates. We use the following pointwise estimates on the nonlinearity $f(A)=-\mathbf{b}|A|^2A$:

$$\begin{array}{l}\mathrm{Re}\,(f(A+V)-f(A))\overline{V}\leq4\mathbf{b}|A|^2|V|^2,\\ \mathrm{Re}\,[Df(A+V)(B+W)-Df(A)B]\overline{W}\leq\mathbf{b}(2|A|^2|W|^2+|B|^2|V|^2).\end{array}$$

As $\partial_TV=\partial_X^2V+\mathbf{a}V+f(A+V)-f(A)$ we obtain with the fixed weight ρ (with $|\rho'|\leq\rho$) the estimate

$$\begin{array}{l}\frac{1}{2}\frac{d}{dT}\|V\|^2_{H^1_\rho}=\frac{1}{2}\frac{d}{dT}\int\rho(|V|^2+|V'|^2)\,dX\ \leq\\ \quad\leq\int\rho\Big\{(-|V'|^2-|V''|^2+|VV'|+|V'V''|+\mathbf{a}(|V|^2+|V'|^2)+\\ \qquad\qquad\mathrm{Re}\,[(f(A+V)-f(A))\overline{V}+\{Df(A+V)(A'+V')-Df(A)A'\}\overline{V'}]\Big\}dX\\ \quad\leq\int\rho\left((1+\mathbf{a})(|V^2|+|V'|^2)+\mathbf{b}\Big[2|A|^2|V|^2+2|A|^2|V'|^2+|A'|^2|V|^2\Big]\right)\,dX\\ \quad\leq(1+\mathbf{a}+2\mathbf{b}\|A\|^2_\infty)\|V\|^2_{H^1_\rho}+\mathbf{b}\|A\|^2_{H^1_{l,u}}\|\sqrt{\rho}V\|^2_\infty\ \leq C\|V\|^2_{H^1_\rho}.\end{array}$$

Here $\|\cdot\|_\infty$ is the classical sup–norm, which can be estimated by the H^1 norm, and $A(T)$ is bounded in $H^1_{l,u}$. As $V(T) = \mathcal{G}_T(A(0)+V(0)) - \mathcal{G}_T(A(0))$ the Gronwall estimate yields $\|\mathcal{G}_T(A_0+V_0)-\mathcal{G}_T(A_0)\|_{H^1_\rho} \leq e^{CT}\|V_0\|_{H^1_\rho}$, where C depends only on A_0. This is the desired continuity in H^1_ρ and the theorem is proved. $\square$

As a general remark we may mention that all the statements concerning the limsup of the norms of solutions of the systems given in this paper are in fact uniform with respect to initial conditions in bounded sets in $H^1_{l,u}$. We omit this fact in the theorems for the sake of readability. However, this information is necessary to prove that the ball with radius $\widetilde{\Delta}_{\mathcal{G}}$ is in fact an absorbing set in the sense of Section 2.

Corollary 3.5
The Ginzburg–Landau equation with $\mathbf{b} > 0$ *has a global* $(H^1_{l,u}, H^1_\rho)$*–attractor* $\mathcal{A}_{\mathcal{G}}$ *which satisfies attractivity in* $\mathrm{dist}^*_{H^1_\rho}$*, is translationally invariant, and invariant under the rotations* $R_\phi : A \mapsto e^{i\phi}A$.

Proof: To show the existence of the attractor, it only remains to prove that there is an absorbing set for the semigroup $\mathcal{G}_T$. From the a–priori estimates in the proof of Theorem 3.4 we know that every bounded set is absorbed into a ball of radius $\widetilde{\Delta}_{\mathcal{G}} + \varepsilon$. Moreover, there is a $T_1 > 0$ such that all solutions starting in $B_{H^1_{l,u}}(2\widetilde{\Delta}_{\mathcal{G}})$ satisfy $A(T) = \mathcal{G}_T(A_0) \in B_{H^1_{l,u}}(3\widetilde{\Delta}_{\mathcal{G}}/2)$ for $T \geq T_1$. Now let $\mathcal{B} = \cup_{T\in[0,T_1]}\mathcal{G}_T(B_{H^1_{l,u}}(2\widetilde{\Delta}_{\mathcal{G}}))$. Clearly, $\mathcal{B}$ is bounded, invariant, and globally absorbing. Thus, the existence of $\mathcal{A}_{\mathcal{G}}$ is established.

The invariance under rotations follows from the uniqueness and the rotational invariance of the Ginzburg–Landau equation. $\square$

Remark: Weighted energy estimates have a weakness when small solutions are considered. This means we cannot obtain optimum bounds and decay rates when solutions are very small. For the real Ginzburg–Landau equation it is known (see also below) that all solutions in the attractor have L^∞ norm less or equal $\sqrt{\mathbf{a}/\mathbf{b}}$. From the explicit family of stationary solutions $A(X) = re^{i\sqrt{\mathbf{a}-\mathbf{b}r^2}\,x}$ we can easily find the lower bound $c\sqrt{(\mathbf{a}+\mathbf{a}^2)/\mathbf{b}}$. For small $\mathbf{a}$ our upper bound of $\mathbf{a}^{1/4}$ is too rough. The reason for this is that the weighted energy estimate involves derivatives of the weight function. Our basic estimate $|\rho_b'| \leq b\rho$ allows for weights of the form $\rho_b(x) = e^{-b|x|}$. However, it is well known that in such spaces the spectrum of the linear part is moved to the right. This additional growth rate has to be compensated for by the nonlinearity and leads to larger domains of attractions. For weights with algebraic decay the spectrum is not moved and better estimates should be expected.

Analogous a–priori estimates for the case of the complex Ginzburg–Landau equation $\partial_T A = \mathbf{d}\partial_X^2 A + \mathbf{a}A - \mathbf{b}A|A|^2$ with $\mathbf{d}, \mathbf{b} \in \mathbb{C}$ are derived in [MS94].

We provide more exact bounds on the size of the attractor and the decay rates in the real Ginzburg–Landau equation in order to employ them in Section 5. The main tool is the maximum principle.

Lemma 3.6
Let $A(T) = \mathcal{G}_T(A_0)$ be any solution of the real Ginzburg–Landau equation with $A_0 \in H^1_{l,u}$. For every fixed $\nu > 0$ and $T > 0$ we have

$$\|A(T)\|_\infty = \sup\{\, |A(T,X)| \;:\; x \in I\!R \,\} \leq \frac{2\mathbf{b}}{\nu}\left(\frac{\mathbf{a}+\nu}{3\mathbf{b}}\right)^{3/2}(1-e^{-\nu T}) + e^{-\nu T}\|A_0\|_\infty. \quad (3.8)$$

Moreover, for each $\tau \in [0,T)$ we have

$$\|A(T)\|_{H^1_{l,u}} \leq C\left(1 + \frac{1}{\sqrt{T-\tau}} + Me^{M(T-\tau)}\sqrt{T-\tau}\right)\|A(\tau)\|_\infty \quad (3.9)$$

where $M = \mathbf{a} + 3\mathbf{b}\max\{\sqrt{\mathbf{a}/\mathbf{b}}, \|A(\tau)\|_\infty\}$ and C is a universal constant independent of $\mathbf{a}$, $\mathbf{b}$, and A.

Proof: Let $A(T,X) = r(T,X)e^{i\phi(T,X)}$, then we obtain for r the equation

$$\partial_T r = \partial_X^2 r - (\partial_X\phi)^2 r + \mathbf{a}r - \mathbf{b}r^3.$$

Using the maximum principle it is possible to compare r with the solution of $a_T = \mathbf{a}a - \mathbf{b}a^3$. Hence, if $\|A_0\|_\infty = a(0)$ then $\|A(T)\|_\infty \leq a(T)$ for all $T > 0$. From $\mathbf{a}a - \mathbf{b}a^3 \leq 2\mathbf{b}((\mathbf{a}+\nu)/(3\mathbf{b}))^{3/2} - \nu a$ the desired L^∞ estimate follows. In particular, we have $\|A(T)\|_\infty \leq \max\{\sqrt{\mathbf{a}/\mathbf{b}}, \|A(0)\|_\infty\}$.

The H^1 estimate is derived by using the constant of variations formula for $V = A'$. From $\partial_T V = \mathbf{a}V + \partial_X^2 V - \mathbf{b}(2|A|^2 V + A^2\overline{V})$ we find $V(T) = G(T-\tau)\partial_X A(\tau) + \int_\tau^T G(T-r)\big(\mathbf{a}V(r) - \mathbf{b}(\ldots)\big)\,dr$ and hence

$$\|V(T)\|_{L^2_{l,u}} \leq \frac{C}{\sqrt{T-\tau}}\|A(\tau)\|_{L^2_{l,u}} + \int_\tau^T C(\mathbf{a} + 3\mathbf{b}\|A(r)\|_\infty^2)\|V(r)\|_{L^2_{l,u}}\,dr.$$

Applying a Gronwall estimate the result follows. □

Using the above lemma the estimate of the size of the attractor can be improved for small $\mathbf{a}$. According to the remark subsequent to Corollary 3.5 the following result is the best possible.

Corollary 3.7
The solutions $A(T) = \mathcal{G}_T(A_0)$ of (3.2) satisfy

$$\limsup_{T\to\infty}\|A(T)\|_\infty \leq \sqrt{\frac{\mathbf{a}}{\mathbf{b}}}, \quad \limsup_{T\to\infty}\|A(T)\|_{H^1_{l,u}} \leq \Delta_{\mathcal{G}} := C\sqrt{\frac{\mathbf{a}+\mathbf{a}^2}{\mathbf{b}}}.$$

Proof: For the L^∞ estimate we choose $\nu = 2\mathbf{a}$ in (3.8) and the first result follows. In the H^1 estimate (3.9) we optimize τ such that $T - \tau = 1/M$. Then $\|A\|_{H^1_{l,u}} \le C(1+M^{1/2})\|A(T-1/M)\|_\infty$. With $T \to \infty$ we find $\limsup_{T\to\infty} \|A(T)\|_{H^1_{l,u}} \le C(1+\mathbf{a}+(\mathbf{ab})^{1/2})(\mathbf{a}/\mathbf{b})^{1/2}$. By rescaling $A \to \widetilde{A} = \mathbf{b}^{-1/2}A$ we can reduce the analysis to the case $\mathbf{b} = 1$. Redoing the scaling shows $\Delta_{\mathcal{G}} \le C(1+\mathbf{a})(\mathbf{a}/\mathbf{b})^{1/2}$, which is better than the result in Theorem 3.4 for small $\mathbf{a}$ but worse for large $\mathbf{a}$. Taking the minimum of both bounds gives the result. $\square$

3.3 The Swift–Hohenberg and Kuramoto–Shivashinsky equation

In this section we study an equation which involves a fourth–order derivative in space.

$$u_t = -(1+\partial_x^2)^2 u + \alpha u + \beta u u' - \gamma u^3. \tag{3.10}$$

For $\beta = 0$, $\gamma > 0$, and $\alpha = \varepsilon^2$ this problem is called the Swift–Hohenberg equation. For $\gamma = 0$ and $\alpha = \beta = 1$ it is the classical Kuramoto–Shivashinsky equation.

The linear part is more difficult than in the case of a simple diffusion operator. In particular there is no maximum principle available to obtain sharp L^∞ bounds. Linear estimates in weighted norm are derived in the following lemma.

Lemma 3.8

a) Let ρ be an integrable weight function with $|\rho''(x)| \le \rho(x)$ for all $x \in \mathbb{R}$. For $b \in \mathbb{R}$ set $\rho_b(x) = \rho(bx)$. Then, for all $u \in H^4_{l,u}(\mathbb{R})$, the following estimate holds:

$$-\int_{\mathbb{R}} \rho_b u(1+\partial_x^2)^2 u\,dx \le (2b^2 + \tfrac{3}{2}b^4)\int_{\mathbb{R}} \rho_b u^2 dx. \tag{3.11}$$

b) The linear operator $L : D(L) \to L^2_{l,u};\ u \mapsto -(1+\partial_x^2)^2 u$ with $D(L) = H^4_{l,u}$ generates a bounded holomorhic semigroup e^{Lt}, $t \ge 0$, with

$$\|e^{Lt}\|_{L^2_{l,u}\to H^s_{l,u}} \le C(1+t^{-s/4}) \quad \text{for } t > 0. \tag{3.12}$$

The same is true for the restriction $\widetilde{L} : D(\widetilde{L}) = H^5_{l,u} \subset H^1_{l,u} \to H^1_{l,u}$.

c) For each $d > 0$ there is a $D > 0$ such that

$$-\int_{\mathbb{R}} \rho u(1+\partial_x^2)^2 u\,dx \le -d\int_{\mathbb{R}} \rho u'^2 dx + D\int_{\mathbb{R}} \rho u^2 dx.$$

Proof: ad a) By several partial integrations we obtain

$$-\int_{\mathbb{R}} \rho_b u(1+\partial_x^2)^2 u\,dx = -\int_{\mathbb{R}} \{\rho_b(u^2 + 2uu'' + u''^2) - \rho_b'' u'^2 + \rho_b'' u u''\}dx.$$

Using $|\rho_b''| \leq b^2\rho_b$ and

$$\begin{aligned}\int \rho_b u'^2\,dx &= -\int(\rho_b u')'u\,dx = -\int \rho_b'(\tfrac{1}{2}u^2)'\,dx - \int \rho_b u''u\,dx\\ &= \int \tfrac{1}{2}\rho_b'' u^2 dx - \int \rho_b u u'' dx \leq \tfrac{b^2}{2}\int \rho_b u^2\,dx + \int \rho_b|uu''|\,dx\end{aligned}$$

we conclude

$$-\int_{\mathbb{R}} \rho_b u(1+\partial_x^2)^2 u\,dx \leq \int_{\mathbb{R}} \rho_b\left\{(b^4/2-1)u^2 + 2(1+b^2)|uu''| - u''^2\right\}dx,$$

which yields the desired result after maximizing the integrand for each x with respect to u''.

ad b) The semigroup e^{Lt} is the multiplier associated with the function $\widehat{m}(k) = e^{-(1-k^2)^2 t}$. The desired result is thus a direct consequence of Lemma 3.3.

ad c) Applying the estimates of part a) with $b = 1$ we find

$$\int \rho\left(-u(1+\partial_x^2)^2 u + du'^2\right)dx \leq \int \rho\left(\frac{d-1}{2}u^2 + (4+d)|uu''| - u''^2\right)dx.$$

Maximizing with respect to u'' gives the result with $D = (2d-2+(4+d)^2)/4$. □

Using these linear estimates we are prepared to estimate the solutions of our equation (3.10). We proceed exactly as in the case of the Ginzburg–Landau equation. Our first aim is to construct a global semiflow and an absorbing ball for this system. For $\gamma < 0$ it is easy to see that there are spatially constant solutions which blow up in finite time. Hence, we concentrate on the case $\gamma > 0$. For the border case $\gamma = 0$ see the discussion below.

Theorem 3.9
The equation (3.10) with $\gamma > 0$ defines a global semiflow $u(t) = \mathcal{S}_t(u_0)$. Moreover, there is a universal constant C (independent of α, β, and γ) such that

$$\begin{aligned}\limsup_{t\to\infty}\|u(t)\|_{L^2_{l,u}} &\leq \Delta_0(\alpha,\beta,\gamma) := C\begin{cases}\left(\frac{\alpha(\alpha+1+\beta^2/\gamma)}{\gamma^2}\right)^{1/4} & \text{for } \alpha > 0,\\ 0 & \text{for } \alpha \leq 0;\end{cases}\\ \limsup_{t\to\infty}\|u(t)\|_{H^1_{l,u}} &\leq \Delta_1 := C\Delta_0 M(1+|\alpha|M^{-4}),\end{aligned}$$

where $M = 1 + (|\beta|\Delta_0)^{2/5} + (\gamma\Delta_0^2)^{1/3}$. Hence, $\alpha \leq 0$, $\gamma > 0$ is the stable case where all solutions decay to zero.

Moreover, for each $t > 0$ the nonlinear mapping $\mathcal{S}_t$ maps bounded maps in $H^1_{l,u}$ into bounded sets in $H^2_{l,u}$ and it is continuous from H^1_ρ into itself.

Proof: The local existence of solutions and continuous dependence on the initial condition in $H^1_{l,u}$ follows from standard argument when rewriting the problem as

$$u(t) = e^{Lt}u(0) + \int_0^t e^{L(t-r)}F(u(s))\,ds \quad \text{where } F(u) = \alpha u + \beta u u_x - \gamma u^3.$$

As F is a smooth mapping from $H^2_{l,u} = D(L^{1/4})$ into $H^1_{l,u}$ the methods of [He81] are fully available. In particular, the smoothing property is clear.

To show global existence it is sufficient to bound the $H^1_{l,u}$ norm. The L^2 estimate follows from standard weighted energy estimates using part a) of Lemma 3.8. We let $\tilde\alpha = \max\{0,\alpha\}$, then

$$\begin{aligned}
\tfrac12 \tfrac{d}{dt}\int \rho_b u^2\,dx &= \int \rho_b\{-u(1+\partial_x^2)^2u + \tilde\alpha u^2 + \beta(\tfrac13 u^3)' - \gamma u^4\}dx\\
&\le 4b^2\|u\|^2_{\rho_b} + \int\{\rho_b u^2(\tilde\alpha - \gamma u^2) - \beta{\rho_b}' u^3/3\}dx\\
&\le (4b^2+\tilde\alpha)\|u\|^2_{\rho_b} + \int \rho_b(b|\beta|\,|u|^3 - \gamma u^4)\,dx\\
&\le c_1 - c_2\|u\|^2_{\rho_b} \quad \text{with } c_1 = \tfrac{1}{2\gamma}\Big(\tfrac{b^2\beta^2}{2\gamma} + d\Big)^2\int\rho_b dx,\ c_2 = d - 4b^2 - \tilde\alpha,
\end{aligned}$$

where $b \le 1$ was used and $d \ge 0$ is arbitrary. In the last step the integrand was estimated as follows

$$\delta u^3 - \gamma u^4 \le \frac{\delta^2}{2\gamma}u^2 - \frac{\gamma}{2}u^4 \le (\delta^2/(2\gamma)+d)^2/(2\gamma) - du^2.$$

Assuming $c_2 > 0$ and applying Gronwall's estimate to the differential inequality yields $\|u(t)\|^2_{\rho_b} \le e^{-2c_2t}\|u(0)\|^2_{\rho_b} + \frac{c_1}{c_2}(1-e^{-2c_2t})$. Letting $t\to\infty$ and choosing optimal d leads to

$$\limsup_{t\to\infty}\|u(t)\|^2_{\rho_b} \le \frac{C}{\gamma}(\tilde\alpha + b^2(1+\beta^2/\gamma))\int\rho_b\,dx.$$

With $\int\rho_b dx = C/b$ and (3.4) we obtain the desired estimate when optimizing with respect to $b\in(0,1]$.

The estimate in $H^1_{l,u}$ is derived via the constant of variations formula. With Lemma 3.8b)

$$u(t) = e^{L(t-\tau)}u(\tau) + \int_\tau^t e^{L(t-r)}\{\alpha u(r) + \beta u(r)u'(r) - \gamma u(r)^3\}dr.$$

We consider this equation in $H^1_{l,u}$ and use the shorthand notation $\|u\|_s = \|u\|_{H^s_{l,u}}$ for $s=0,1,2$. The nonlinear terms can be estimated as follows:

$$\begin{aligned}
\|e^{Lr}u^3\|_1 &\le \|e^{Lr}\|_{L^2\to H^1}\|u^3\|_0 \le Cr^{-1/4}\|u\|_0\|u\|^2_{L^\infty}\\
&\le Cr^{-1/4}\|u\|_0^2\|u\|_1,\\
\|e^{Lr}uu'\|_1 &\le Cr^{-1/4}\|uu'\|_0 \le Cr^{-1/4}\|u\|_0^{1/2}\|u\|_1^{3/2},\\
\|e^{Lr}uu'\|_1 &= \|(e^{Lr}u^2/2)'\|_1 \le \|e^{Lr}u^2\|_2\\
&\le \|e^{Lr}\|_{L^2\to H^2}\|u^2\|_0 \le Cr^{-1/2}\|u\|_0^{3/2}\|u\|_1^{1/2}
\end{aligned}$$

Combining the last two estimates we find

$$\|e^{Lr}uu'\|_1 \le Cr^{-3/8}\|u\|_0\|u\|_1.$$

Abbreviating $e_s(t) = \|u(t)\|_s$ we find the integral inequality

$$\begin{aligned}
e_1(t) \le C\Big\{&(t-\tau)^{-1/4}e_0(\tau) + |\alpha|\int_\tau^t(t-r)^{-1/4}e_0(r)\,dr +\\
&|\beta|\int_\tau^t(t-r)^{-3/8}e_0(r)e_1(r)\,dr + |\gamma|\int_\tau^t(t-r)^{-1/4}e_0(r)^2e_1(r)\,dr\Big\},
\end{aligned}\tag{3.13}$$

where $\tau \in [t-1,t]$. We assume $e_0(r) \le E$ on $[t-1,t]$ and employ Young's inequality for convolutions

$$\left(\int_\tau^{\tau+\delta} \left|\int_\tau^t (t-r)^{-\kappa} e_1(r)\,dr\right|^p dt\right)^{1/p} \le \int_\tau^{\tau+\delta} |(t-r)^{-\kappa}|dt \left(\int_\tau^{\tau+\delta} |e_1(r)|^p dt\right)^{1/p}.$$

We choose $p=3$ and estimate $I_\delta = (\int_\tau^{\tau+\delta} e_1^3 dt)^{1/3}$ using (3.13):

$$I_\delta \le C\left\{E\delta^{1/12} + |\alpha|E\delta^{13/12} + |\beta|E\delta^{5/8}I_\delta + |\gamma|E^2\delta^{3/4}I_\delta\right\}$$

Thus, for sufficiently small $\delta \in (0,1]$ the integral I_δ can be estimated from above in terms of α, β, γ, E, and δ, namely for

$$\delta = \min\{1, (4C|\beta|E)^{-8/5}, (4C|\gamma|E^2)^{-4/3}\}$$

we have $I_\delta \le 2CE(1+|\alpha|\delta)\delta^{1/12}$.

Inserting this result into (3.13) we find the pointwise estimate

$$\begin{aligned} e_1(\tau+\delta) &\le C\Big\{E\delta^{-1/4} + |\alpha|E\delta^{3/4} + |\beta|E\delta^{7/24}I_\delta + |\gamma|E^2\delta^{5/12}I_\delta\Big\}. \\ &\le CE(1+|\alpha|\delta)\Big[\delta^{-1/4} + |\beta|E\delta^{3/8} + |\gamma|E^2\delta^{1/2}\Big]. \end{aligned}$$

This shows that $u(t)$ is bounded in $H^1_{l,u}$ a–priori and the bound depends only on the parameters and the L^2 bound E. Hence, solutions exist globally and for $t \to \infty$ the bound E can be replaced by Δ_0. Since $\delta \sim M^{-4}$ the result follows.

It remains to prove the continuity of $\mathcal{S}_t$ in the localized space H^1_ρ. We take a fixed solution $u \in H^1_{l,u}$ and have to estimate $v(t) = \mathcal{S}_t(u(0)+v(0)) - \mathcal{S}_t(u(0))$ in H^1_ρ in terms of $\|v(0)\|_{H^1_\rho}$. As $u(0)+v(0) \in H^1_{l,u}$ the difference v satisfies $\partial_t v = (L+\alpha)v + s(u+v) - s(u)$ where $s(u) = \beta u u' - \gamma u^3$. In order to carry through the weighted energy estimates with a fixed weight ρ we start with

$$\begin{aligned} &\int \rho v[s(u+v)-s(u)]\,dx = \int\left\{\rho\beta v(uv)' - \rho'\beta\tfrac{v^3}{3} - \rho\gamma v^2(3u^2+3uv+v^2)\right\}dx \\ &\quad\le \int \rho\left\{|\beta|(u'v^2+uvv') + \tfrac{b^2\beta^2}{2\gamma}v^2 + \tfrac{\gamma}{2}v^4 + \gamma v^2(-3u^2+\tfrac{9}{2}u^2+\tfrac{1}{2}v^2-v^2)\right\}dx \\ &\quad\le C\left\{\|u\|_{H^1_{l,u}}\|\sqrt{\rho}\,v\|_\infty\|v\|_{L^2_\rho} + \|u\|_\infty\|v\|_{L^2_\rho}\|v'\|_{L^2_\rho} + (1+\|u\|^2_\infty)\|v\|^2_{L^2_\rho}\right\}. \end{aligned}$$

$$\begin{aligned} \int \rho v'[s(u+v)'-s(u)']\,dx = \int \rho\{&\beta v'(uv)'' + \beta v'(vv')' \\ &-\gamma v'(3u^2v)' - \gamma v'(3uv^2)' - \gamma v^2v'^2\}\,dx. \end{aligned}$$

We estimate the terms as follows:

$$\begin{aligned} \int \rho v'(uv)''dx &= -\int(\rho'v'+\rho v'')(uv)'dx \le C\|v\|_{H^2_\rho}(\|u\|_\infty\|v\|_{H^1_\rho} + \|u\|_{H^1_{l,u}}\|v\|_{L^2_\rho}), \\ \int \rho v'(vv')'dx &= -\int \rho' vv'^2dx - \int \rho vv'v''dx \le \int \rho(|v|v'^2 + \varepsilon v^2v'^2 + \tfrac{1}{4\varepsilon}v''^2)dx, \\ -\int \rho v'(uv^2)'dx &= \int \rho(2uvv'^2 + u'v^2v')dx \le \int \rho(2\varepsilon v^2v'^2 + \tfrac{1}{\varepsilon}u^2v'^2 + \tfrac{1}{4\varepsilon}u'^2v^2)dx. \end{aligned}$$

Together with Lemma 3.8 we find

$$\begin{aligned} &\tfrac{1}{2}\tfrac{d}{dt}\|v\|^2_{H^1_\rho} = \tfrac{1}{2}\tfrac{d}{dt}\int \rho(v^2+v'^2)dx \\ &\quad= \int \rho\Big[v(L+\alpha)v + v'(L+\alpha)v' + v(s(u+v)-s(u)) + v'(s(u+v)'-s(u)')\Big]dx \\ &\quad\le -d(\|v'\|^2_{L^2_\rho} + \|v''\|^2_{L^2_\rho}) + (D+\alpha)(\|v\|^2_{L^2_\rho} + \|v'\|^2_{L^2_\rho}) + C(1+\|u\|_{H^1_{l,u}})^2\|v\|_{H^2_\rho}\|v\|_{H^1_\rho}. \\ &\quad\le C(1+\|u\|_{H^1_{l,u}})^4\|v\|^2_{H^1_\rho}. \end{aligned}$$

Hence we obtain $\|v(t)\|_{H^1_\rho} \le e^{C^* t}\|v(0)\|_{H^1_\rho}$ which is the desired local Lipschitz continuity of $\mathcal{S}_t$ in the H^1_ρ norm. □

To illustrate the result we treat as an example the Swift–Hohenberg case $\beta = 0$. We obtain

$$\Delta_0 = C(\alpha+\alpha^2)^{1/4}\gamma^{-1/2}, \quad M = (1+\alpha)^{1/3}, \quad \Delta_1 = C\Delta_0.$$

This result is sharp for large α, as $u \equiv \sqrt{(\alpha-1)/\gamma}$ is a constant solution. It is the task of the next section to show that the result for small α can be improved to $\Delta_j = C\sqrt{\alpha/\gamma}$.

The limit towards the Kuramoto–Shivashinsky equation is given by $0 < \gamma \to 0$ while $\alpha > 0$ and $\beta \neq 0$ are fixed. For this case we find

$$\Delta_0 = C\gamma^{-3/4}, \quad M = \gamma^{-3/10}, \quad \Delta_1 = C\gamma^{-21/20}.$$

For $\gamma = 0$ and $\alpha \ge 1$ the equation cannot admit an absorbing ball as $u(t,x) = re^{(\alpha-1)t}$ is a solution for each $r \in \mathbb{R}$.

Conjecture: *The damped Kuramoto–Shivashinsky equation with $\gamma = 0$, $\beta = 1$, and $\alpha \in (0,1)$ has an absorbing set.*

As in the previous subsection the existence of a global attractor follows immediately in the case $\gamma > 0$.

Corollary 3.10
The equation (3.10) with $\gamma > 0$ defines a global semiflow $(\mathcal{S}_t)_{t\ge 0}$ which has a global $(H^1_{l,u}, H^1_\rho)$–attractor which satisfies attractivity in $\mathrm{dist}^*_{H^1_\rho}$ and is translationally invariant.

It is shown in [Sch94c] that for all γ with $\beta^2 + 27\gamma > 0$ and sufficiently small α the (possibly local) semiflow $\mathcal{S}_t$ maps a small ball $\mathcal{B}^* = B_{H^1_{l,u}}(C\sqrt{\alpha/(\beta^2+\gamma)})$ after a time t^* into itself. Hence, the set $B = \cup_{t\in[0,t^*]}\mathcal{S}_t(\mathcal{B}^*)$ is a positively invariant set. The above theory immediately implies that there is a (local) attractor in B. Below we will improve on this result and show that the radius of $\mathcal{B}^*$ can in fact be chosen independent of α as long as α remains sufficiently small.

4 Ginzburg–Landau approximation and shadowing by pseudo–orbits

In this section we study the relation of the weakly unstable case of (3.10), namely $\alpha = \varepsilon^2$ with small positive ε,

$$\partial_t u = -(1+\partial_x^2)^2 u + \varepsilon^2 u + \beta u \partial_x u - \gamma u^3, \tag{4.1}$$

and the Ginzburg–Landau equation. From now on we consider β and γ as fixed parameters such that all constants C may now also depend on β and γ. Only the dependence on ε will always be displayed explicitly. In the following we are free to allow γ to be slightly negative, as long as $\beta^2 + 27\gamma$ is larger than 0. The reason for this relation is explained below.

From Theorem 3.9 we know that for $\gamma > 0$ the global attractor is contained in a ball of radius $C\sqrt{\varepsilon}$. The solutions in the attractor are small and are strongly dominated by the linear part $L + \varepsilon^2$, which damps out all Fourier modes e^{ikx} with k not close to $k = \pm 1$. Hence it is reasonable to expect the solutions of (4.1) to have Fourier transforms which are strongly concentrated around $k = \pm 1$. The scaling ansatz

$$u(t,x) = \widetilde{\psi}_\delta(A)(t,x) := \delta(A(T,X)e^{ix} + \overline{A}(T,X)e^{-ix}) \quad \text{with } T = \delta^2 t \text{ and } X = \delta x \quad (4.2)$$

with small δ fulfills exactly this property.

Note that we slightly generalize the scaling ansatz with respect to previous work by taking the scaling variable δ to be independent of ε. This does not lead to any conceptual differences but has the advantage that we are able to consider solutions of size $\varepsilon \leq \delta$.

The main observation of modulation theory is that the function $u = \psi_\delta(A)$ is a good approximation of a true solution of the original system (4.1) whenever δ is sufficiently small and A is a solution of the associated Ginzburg–Landau equation

$$\partial_T A = 4\partial_X^2 A + \mathbf{a}A - \mathbf{b}|A|^2 A \qquad \text{with } \mathbf{b} = 3\gamma + \beta^2/9, \quad (4.3)$$

where $\mathbf{a} = \varepsilon^2/\delta^2$. On the formal level this can be seen by inserting the ansatz (4.2) into (4.1) and comparing equal powers in δ and e^{ix}.

Above we have constructed a global semiflow $\mathcal{S}_t^\varepsilon$ for (4.1) on the space $Z = H^1_{l,u}$ and the global semiflow $\mathcal{G}_T$ (4.3) on $Y = H^1_{l,u\,\mathbb{C}}$. Here we have introduced different notations Z and Y for $H^1_{l,u}$ and $H^1_{l,u\,\mathbb{C}}$ to distinguish functions depending on x and X. Also future generalizations to hydrodynamics will be easier: for instance, (4.1) can be replaced by the Navier–Stokes equation with the space $Z = H^1_{l,u}(\mathbb{R} \times \Sigma)$ for some bounded cross–section Σ, cf. [Sch94b]. Nevertheless, the modulation equation (4.3) and the space Y will stay the same.

We call the mapping $\widetilde{\psi}_\delta : Y \to Z$ the lift from the Ginzburg–Landau problem into the original problem (4.1). We also want to go from the original problem in u to the associated modulation amplitude A and need an operator $\Phi_\delta : Z \to Y$ which extracts the critical modes close to $k = k_{\text{crit}} = 1$. To this end we define mode filters via the multiplier theory of Lemma 3.3. We choose an even cut–off function $\widehat{\phi}_0 \in C_0^\infty(\mathbb{R}, [0,1])$ with $\widehat{\phi}_0(k) = 1$ for $k \in [-1/6, 1/6]$ and $\widehat{\phi}_0(k) = 0$ for $k \notin [-1/3, 1/3]$.

To extract the modes close to $k = -1, 0, 1$ we define $\widehat{e}_j(k) = \widehat{\phi}_0(k - j)$ for $j \in \{-1, 0, 1\}$, $\widehat{e}_c = \widehat{e}_1 + \widehat{e}_{-1}$, and $\widehat{e}_s = 1 - \widehat{e}_c$. According to Lemma 3.3 we associate to $\widehat{e}_\sigma$, $\sigma \in \{s, c, 1, 0, -1\}$, an operator $E_\sigma : Z_{\mathbb{C}} \to Z_{\mathbb{C}}$ which extracts the Fourier modes belonging to wave numbers in the relevant intervals. We call E_σ a mode filter, as it is close to mode projections which would be used in cases of a discrete spectrum. Obviously, $E_{c,s} : Z \to Z$ (real); and $E_c u \in Z$ contains the critical modes of $u \in Z$ and $E_s u \in Z$ the stable modes of u.

To deal with the slow spatial scale $X = \delta x$ we need scaling operators $S_\delta : Z_{\mathbb{C}} \to Y$ defined by $(S_\delta u)(X) = u(X/\delta)$ and the multiplication operator $\theta : Z_{\mathbb{C}} \to Z_{\mathbb{C}}; u \mapsto (\theta u)(x) = e^{-ix}u(x)$ which is a translation operator in Fourier space. With this the lift $\widetilde{\psi}_\delta$ can be written

$$\widetilde{\psi}_\delta(A) = \delta(\theta^{-1}S_\delta^{-1}A + \overline{\theta^{-1}S_\delta^{-1}A}).$$

To go from $u \in Z$ we have to extract the critical mode and rescale it to find the associated A. This operation is given by

$$\Phi_\delta : Z \to Y; u \mapsto \frac{1}{\delta}S_\delta \theta E_1 u.$$

Obviously $\Phi_\delta(u)$ contains all the information of $u_c = E_c u$, since $u_c = E_1 u + E_{-1}u$ with $E_{-1}u = \overline{E_1 u}$. Thus, if we control $\Phi_\delta(u)$ and u_s we control all of u. For technical reasons we introduce a modified lift $\psi_\delta : Y \to Z$ defined as

$$\psi_\delta(A) = \delta(\theta^{-1}E_0 S_\delta^{-1}A + \overline{\theta^{-1}E_0 S_\delta^{-1}A}).$$

In this paper we only give some ideas as to why the mode splitting $u = E_c u + E_s u$ and the concentration of $u_c = E_c u$ is essential to the modulation theory which relates the original problem (4.1) and the modulation equation (4.3). For more details and the proofs of the three theorems given below we refer the reader to [Sch94a, Sch94c].

The first important fact is that the stable uncritical modes $u_s = E_s u$ are strongly damped such that they are slaved to the critical ones after the transient time. Secondly, the critical modes $u_c = E_c u$ will be concentrated around the critical wave numbers $k = \pm 1$ with a typical width of the peak of size δ. This justifies the spatial rescaling.

These two effects are purely linear and seen by considering a solution $v(t)$ of the linearized system $\partial_t v = -(1+\partial_x^2)^2 v + \varepsilon^2 v$. Using Fourier transform and multiplier theory we see that $v(t)$ is obtained from $v(0)$ via the multiplier $\widehat{g}(t,k) = e^{(-(1-k^2)^2+\varepsilon^2)t}$. Firstly, $\widehat{e}_s(k)$ vanishes for $||k|-1| \le 1/6$, hence we find $\|v_s(t)\|_Z = \|E_s v(t)\|_Z \le Ce^{-t/10}\|v(0)\|_Z$. Secondly, the critical mode $v_c = E_c v$ may grow on a slow time scale while the wave numbers k with $||k|-1| \ge 2\varepsilon$ are still damped. Defining $\widetilde{A}(T) = \Phi_\delta(v(T/\delta^2))$ we see that the Fourier transform of $\widetilde{A}(T)$ is obtained from $v(0)$ as

$$(\mathcal{F}\widetilde{A})(T,K) = \widehat{\phi}_0(\delta K)e^{[-K^2(2+\delta K)^2+\mathbf{a}]T}\widehat{v}_0(1+\delta K) = e^{[-K^2(2+\delta K)^2+\mathbf{a}]T}(\mathcal{F}\widetilde{A})(0,K),$$

where $\mathbf{a} = \varepsilon^2/\delta^2$. Since the linearized Ginzburg–Landau equation would correspond to a multiplier $e^{[-4K^2+\mathbf{a}]T}$ it is clear that Φ_δ defines the best way to extract the Ginzburg–Landau mode A from any given $u \in Z$.

The third important feature relates to the nonlinearity. Note that the original problem may have quadratic terms while the modulation equation is always cubic. This can be understood by the fact that quadratic interactions of critical modes are never critical, which can be stated mathematically as $E_c[E_c u \cdot E_c v] = 0$ for all $u, v \in Z$. This property is essential in proving the three theorems below, but will not enter further in the present work.

Above we have proved the existence of the (global) attractors $\mathcal{A}^\varepsilon \subset Z$ for $\mathcal{S}_t^\varepsilon$ and $\mathcal{A}_\mathcal{G} \subset Y$ for $\mathcal{G}_T$. (We continue to use a superscript ε for $\mathcal{S}_t^\varepsilon$ and $\mathcal{A}^\varepsilon$ to indicate the dependence on this parameter.) Our aim is to compare these attractors and the dynamics on them. It might seem natural to ask whether the distance of $\mathcal{A}^\varepsilon$ from $\psi_\varepsilon(\mathcal{A}_\mathcal{G})$ in Z tends to zero faster than ε, which is the diameter of the two sets. This will in fact be a consequence of our result below, as we prove a stronger result exhibiting the mode concentration:

$$\operatorname{dist}^*_{Y_\rho}(\Phi_\varepsilon(\mathcal{A}^\varepsilon), \mathcal{A}_\mathcal{G}) + \frac{1}{\varepsilon}\operatorname{dist}_Z(E_s\mathcal{A}^\varepsilon, \{0\}) \to 0 \qquad \text{for } \varepsilon \to 0,$$

where $Y_\rho = H^1_{\rho\,\mathbb{C}}$. The second term in this limit will be estimated using the following lemma.

Lemma 4.1
The linear operator $\Phi_\delta : Z \to Y$ satisfies for all $\delta \in (0,1]$ the estimate

$$\|E_s\psi_\delta(A)\|_Z \le C\delta^{3/2}\|A\|_Y \qquad \text{for all } A \in Y. \tag{4.4}$$

Proof: Let $v = E_s\theta^{-1}E_0S_\delta^{-1}A$ then $u = E_s\psi_\delta(A) = \delta(v + \overline{v})$ and it is sufficient to show $\|v\|_Z \le C\delta^{1/2}\|A\|_Y$. We do this similarly to Lemma 5 in [Sch94c] and introduce the spaces $Y^0 = L^2_{l,u\,\mathbb{C}}$ and $Z^0 = L^2_{l,u\,\mathbb{C}}$. As multiplier operators commute we find $v = \theta^{-1}FE_0S_\delta^{-1}A = \theta^{-1}E_0FS_\delta^{-1}A$ where $F : Z_\mathbb{C} \to Z_\mathbb{C}$ is associated with the multiplier $\widehat{f}(k) = \widehat{e}_s(k+1) = 1 - \widehat{\phi}_0(k) - \widehat{\phi}_0(k+2)$. The scaled operator $F_\delta = S_\delta FS_\delta^{-1} : Y \to Y$ is again a multiplier operator with kernel $\widehat{f}_\delta(K) = \widehat{f}(\delta K)$. We now have $v = \theta^{-1}E_0S_\delta^{-1}F_\delta A$ and estimate

$$\|v\|_Z \le \|\theta^{-1}\|_{Z_\mathbb{C}\to Z_\mathbb{C}}\|E_0\|_{Z^0\to Z_\mathbb{C}}\|S_\delta^{-1}\|_{Y^0\to Z^0}\|F_\delta\|_{Y\to Y^0}\|A\|_Y.$$

Obviously, $\|\theta^{-1}\| \le C$ and $\|E_0\|_{Z^0\to Z_\mathbb{C}} \le C$ by Lemma 3.3 as $(1+k^2)^{1/2}\widehat{\phi}_0(k) \in C^2_b(\mathbb{R},\mathbb{R})$. Moreover, we have $\|S_\delta^{-1}\|_{Y^0\to Z^0} \le C\delta^{-1/2}$ and

$$\|F_\delta\|_{Y\to Y^0} \le C\|(1+K^2)^{-1/2}\widehat{f}_\delta(K)\|_{C^2_b(\mathbb{R},\mathbb{R})} \le C\delta,$$

since $\widehat{f}_\delta(K) = 0$ for $|K| \le 1/(6\delta)$. These estimates give the desired result. □

With these connections between the underlying spaces we want to compare the dynamics of (4.1) and (4.3). This is done again by lifting up the semigroup $\mathcal{G}_T$ from Y into Z or extracting the Ginzburg–Landau mode from $\mathcal{S}_t$.

The first important fact is the attractivity of the so–called Ginzburg–Landau set, i.e., the set of all functions $u \in Z$ having the form of $\psi_\delta(A)$ for some appropriate $A \in Y$. The second result shows the so–called approximation property, which states that $\psi_\delta[\mathcal{G}_{\delta^2 t}(\Phi_\delta(u_0))] \in Z$ is a good approximation of $\mathcal{S}_t(u_0)$ and its counterpart in Y. Both theorems given below are slight generalizations of Theorem 10–12 in [Sch94c] where only the additional scaling parameter $\delta \geq \varepsilon$ was introduced.

Theorem 4.2
(Attractivity)
For each $r_0 > 0$ there exists constants $C, T_0, R_1, \delta_0 > 0$ such that for all $0 < \varepsilon \leq \delta \leq \delta_0$ the following estimates hold:

$$\begin{aligned} \mathrm{dist}_Z(\mathcal{S}^\varepsilon_{T_0/\delta^2}(B_Z(\delta r_0)), \psi_\delta(B_Y(R_1))) &\leq C\delta^{5/4} && (4.5)\\ \mathrm{dist}_Y(\Phi_\delta[\mathcal{S}^\varepsilon_{T_0/\delta^2}(B_Z(\delta r_0))], B_Y(R_1)) &\leq C\delta^{1/4}. && (4.6) \end{aligned}$$

Theorem 4.3
(Approximation)
For all $R_1, T_1, d > 0$ there exists $C, \delta_0 > 0$ such that for all $0 < \varepsilon \leq \delta \leq \delta_0$ the following holds: Let $A_0 \in B_Y(R_1)$ and $u_0 \in Z$ with $\|u_0 - \psi_\delta(A_0)\|_Z \leq d\delta^{5/4}$, then

$$\begin{aligned} \sup_{0 \leq t \leq T_1/\delta^2} \|\mathcal{S}^\varepsilon_t(u_0) - \psi_\delta(\mathcal{G}_{\delta^2 t}(A_0))\|_Z &\leq C\delta^{5/4} && (4.7)\\ \|\Phi_\delta(\mathcal{S}^\varepsilon_{T_1/\delta^2}(u_0)) - \mathcal{G}_{T_1}(A_0)\|_Y &\leq C\delta^{1/4}. && (4.8) \end{aligned}$$

As a consequence of these two results and of the existence of the globally attracting set for the Ginzburg–Landau equation (our Cor. 3.5), it is possible to show that all solutions $u(t) = \mathcal{S}^\varepsilon_t(u_0)$ can be shadowed by the lift of a pseudo–orbit in the Ginzburg–Landau equation. A (T_1, κ)–pseudo–orbit is pieced together from true orbits of time span T_1 with jumps of maximal size κ inbetween. The exact definition is as follows.

Definition 4.4
*Let $T_1 > 0$ and $\kappa > 0$. We call a function $A = A(T)$ a (T_1, κ)–**pseudo–orbit** in Y for (4.3) if for all $n \in I\!N$ the relations*

$$\begin{aligned} &A((n-1)T_1 + \tau) = \mathcal{G}_\tau(A((n-1)T)) \text{ for all } \tau \in [0, T_1),\\ &\|A(nT_1 - 0) - \mathcal{G}_{T_1}(A((n-1)T_1))\|_Y \leq \kappa \end{aligned}$$

hold, where $\mathcal{G}_T$ is the semigroup associated with (4.3) and $A(T-0) = \lim_{\tau \nearrow T} A(\tau)$.

We state here Theorem 3 of [Sch94c] in the generalization with $0 < \varepsilon \leq \delta$.

Theorem 4.5
(Shadowing by pseudo–orbits)

Let β and γ be fixed with $\beta^2 + 27\gamma > 0$. Then, for all $T_1 > 0$ there exist positive constants δ_0, C, and T_0 such that for all $\delta \in (0, \delta_0]$ the following is true:

For all $\varepsilon \in (0, \delta]$ (i.e., $\mathbf{a} \leq 1$) and all initial conditions u_0 with $\|u_0\|_Z \leq \delta$ the solution $u(t) = \mathcal{S}_t(u_0)$ exists for all time and there is a $(T_1, C\delta^{1/4})$–pseudo–orbit A for (4.3) which satisfies $\|A(0)\|_Y \leq C$ and approximates $u(t)$ as follows:

$$\|u(t) - \psi_\delta(A(\delta^2 t - T_0))\|_Z \leq C\delta^{5/4} \quad \textit{for all } t \geq T_0/\delta^2.$$

5 Improved estimates on the attractor $\mathcal{A}^\varepsilon$

To estimate the size of the solutions $u(t)$ for $t \to \infty$ we study the $\limsup_{T\to\infty} \|A(T)\|_Y$ for pseudo–orbits.

Lemma 5.1
Assume $\mathbf{b} > 0$ to be fixed. Then there exists a constant C such that for all $\mathbf{a}, \kappa \in (0, 1]$ and all $T_1 \geq 1$ the following is true: Let $A = A(T)$ be a (T_1, κ)–pseudo–orbit of (4.3). Then

$$\limsup_{T\to\infty} \|A(T)\|_Y \leq C\Pi \textit{ with } \Pi = \sqrt{\mathbf{a}} + (\kappa/T_1)^{1/3} + \kappa. \tag{5.1}$$

Proof: We first estimate the L^∞ norm as we have sharp estimates from the maximum principle. Set $g_n = \|A(nT_1)\|_\infty$, then (3.8) and the jump condition gives

$$g_{n+1} \leq \frac{2\mathbf{b}}{\nu}\left(\frac{\mathbf{a}+\nu}{3\mathbf{b}}\right)^{3/2}(1 - e^{-\nu T_1}) + e^{-\nu T_1} g_n + \kappa.$$

Hence, we find

$$\limsup_{n\to\infty} g_n \leq \frac{2\mathbf{b}}{\nu}\left(\frac{\mathbf{a}+\nu}{3\mathbf{b}}\right)^{3/2} + \frac{\kappa}{1 - e^{-\nu T_1}}.$$

Using the estimate $1/(1 - e^{-\nu T_1}) \leq 1 + 1/(\nu T_1)$ and optimizing with respect to ν yields $\limsup_{T\to\infty} \|A(T)\|_\infty \leq C\Pi$. With (3.9) (where now $M \leq C$ and $\tau = T - 1$) we conclude $\limsup_{n\to\infty} \|A(nT_1 + s)\|_Y \leq C\Pi$ for all $s \in [1, T_1)$. The missing intervals can be estimated by the uniform continuity of the semigroup $\mathcal{G}_T$ for $T \in [0, 1]$. □

The two results above give a–priori bounds on the solutions $u(t) = \mathcal{S}_t(u_0)$ in Theorem 4.5. The idea is that the above results are applied iteratively with a decreasing sequence of δ_j such that in each step the size of $u(t) = \psi_{\delta_j}(A_j(\delta_j^2 t - \tau_j)) + \mathcal{O}(\delta_j^{5/4})$ is decreased. The obtained result is optimal, as for the Swift–Hohenberg equation it is well known that stationary solutions of the form $u(x) = \varepsilon \cos(k_\varepsilon x) + \mathcal{O}(\varepsilon^2)$ with $k_\varepsilon = 1 + \mathcal{O}(\varepsilon)$ exist.

Theorem 5.2
Let β and γ be fixed with $\beta^2 + 27\gamma > 0$. Then there exist positive constants ε_0, δ_1, and C such that for all $\varepsilon \in (0, \varepsilon_0]$ the following is true:

Every solution $u(t)=\mathcal{S}_t(u_0)$ of (4.1) with $\|u_0\|_Z\le\delta_1$ satisfies

$$\limsup_{t\to\infty}\|u(t)\|_Z\le C\varepsilon.$$

Proof: We take $\varepsilon_0\le\delta_1\le\delta_0$ where δ_0, T_0, and T_1 are the same values as in Theorem 4.5. By induction over j we construct a sequence $\delta_1>\delta_j>\delta_{j+1}>\delta_J=C\varepsilon$ where $C\ge 1$.

Assume we know $\|u(t_j)\|_Z\le\delta_j$, which is true for $j=1$ with $t_1=0$.

We apply Theorem 4.5 with $\delta=\delta_j\le\delta_0$ to obtain a $(T_1,C\delta_j^{1/4})$–pseudo–orbit A_j such that

$$\|u(t-t_j)-\psi_{\delta_j}(A_j(\delta_j^2t-T_0))\|_Z\le C\delta_j^{5/4}.$$

Note that A_j satisfies (4.3) with $\mathbf{a}_j=\varepsilon^2/\delta_j^2\le 1$. Using (4.4) and the estimate on the limsup of the pseudo–orbit in (5.1) we find a time t_{j+1} with

$$\begin{aligned}\|u(t_{j+1})\|_Z &\le C\delta_j\|A_j(\delta_j^2(t_{j+1}+t_j)-T_0)\|_Y+C\delta_j^{5/4}\\ &\le C\delta_j\left(\varepsilon/\delta_j+\left(\delta_j^{1/4}\right)^{1/3}\right)+C\delta_j^{5/4}\ \le\ C_1\Big(\varepsilon+\delta_j^{13/12}\Big).\end{aligned}\tag{5.2}$$

Thus, we are finished, as we have shown that it is possible to define δ_j such that $\delta_{j+1}\ge C_1(\varepsilon+\delta_j^{13/12})$. To this end we decrease δ_1 and ε_0, if necessary, such that

$$C_1\delta_1^{1/12}\le 1/3,\quad C_1\varepsilon_0\le\min\{1,\delta_1/3\}.\tag{5.3}$$

We let $h(\varepsilon,\delta)=C_1(\varepsilon+\delta^{13/12})$ and define $\delta_{j+1}=h(\varepsilon,\delta_j)$. From (5.3) we have for all $\varepsilon\in(0,\varepsilon_0]$ the relation $\delta_2\le 2\delta_1/3$. Moreover, $\delta=h(\varepsilon,\delta)$ has a unique fixed point δ_ε^* in the interval $(0,\delta_1)$. Obviously, $\delta_\varepsilon^*\in(C_1\varepsilon,2C_1\varepsilon)$ and $\delta>h(\varepsilon,\delta)$ for $\delta\in(\delta_\varepsilon^*,\delta_1]$. Thus, the sequence δ_j decays monotonically and has the limit δ_ε^*. Hence, there is a finite J such that $\delta_J\le 2C_1\varepsilon$.

We stop at the iteration step $j=J$ and conclude with the last pseudo–orbit A_J by taking $t_{J+1}\to\infty$ in (5.2) that $\limsup_{t\to\infty}\|u(t)\|_Z\le\delta_{J+1}\le 2C_1\varepsilon$. This completes the proof. □

Remarks:

1. From the previous section we know that δ_1 in Theorem 5.2 can be taken as $C\varepsilon^{1/2}$. Using the construction in the proof we see that it is possible to assume $\delta_j=C\varepsilon^{\alpha_j}$ with $\alpha_{j+1}=\min\{\frac{13}{12}\alpha_j,1\}$. Hence, it is sufficient to do $J=10$ iteration steps.

2. Using the above methods it is possible to show that the overall time needed for a solution starting in $B_Z(\delta)$ to reach the ball $B_Z(C\varepsilon)$ is of order $1/\varepsilon^2$.

A simple consequence of the previous results is the following.

Corollary 5.3

Let β and γ be fixed such that $\beta^2+27\gamma>0$. Then there are constants C, ε_0, and $\delta_1>0$ such that for all $\varepsilon\in(0,\varepsilon_0]$ the following is true:

For $\gamma>0$ let $\mathcal{B}=Z$ and for $\gamma\le 0$ let $\mathcal{B}=\cup_{t\ge 0}\mathcal{S}_t(B_Z(\delta_1))$. Then (4.1) has an attractor in $\mathcal{B}$ which is contained in the ball $B_Z(C\varepsilon)$.

6 Comparison of attractors

In the last chapter we discuss the question of how well the attractor of the original problem (4.1) can be described by the attractor of the Ginzburg–Landau equation. As in the previous section it is clear that the attractor $\mathcal{A}^\varepsilon$ is contained in a ball of radius $r_0\varepsilon$ in Z, where the natural scaling is now $\delta = \varepsilon$. The main result is the following.

Theorem 6.1
For every $\sigma > 0$ there exist $C, \varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$ the estimates

$$\mathrm{dist}^*_{Y_\rho}(\Phi_\varepsilon \mathcal{A}^\varepsilon, \mathcal{A}_\mathcal{G}) \le \sigma \quad \textit{and} \quad \mathrm{dist}_Z(E_s \mathcal{A}^\varepsilon, \{0\}) \le C\varepsilon^{5/4},$$

hold.

Proof: From Theorem 5.2 we know that $\mathcal{A}^\varepsilon$ is contained in $B_Z(\varepsilon r_0)$ for some $r_0 > 0$. Let $v \in \mathcal{A}^\varepsilon$. Since $\mathcal{A}^\varepsilon$ is invariant under the flow $\mathcal{S}^\varepsilon_t$, there is a $u_0 \in \mathcal{A}^\varepsilon$ such that $v = \mathcal{S}_{T_0/\varepsilon^2}(u_0)$ where T_0 is chosen according to Theorem 4.2. Hence,

$$\|E_s v\|_Z = \|E_s \mathcal{S}^\varepsilon_{T_0/\varepsilon^2}(u_0)\|_Z \le \|E_s(\mathcal{S}^\varepsilon_{T_0/\varepsilon^2}(u_0) - \psi_\varepsilon(A_0))\|_Z + \|E_s \psi_\varepsilon(A_0)\|_Z \le C\varepsilon^{5/4},$$

where (4.5) and (4.4) was used. This shows the second result as $v \in \mathcal{A}^\varepsilon$ was arbitrary.

From the attractivity in (4.6) we find $R_1 > 0$ such that $\mathrm{dist}_Y(\Phi_\varepsilon \mathcal{A}^\varepsilon, B_Y(R_1)) \le C\varepsilon^{1/4}$. Since $\mathcal{A}_\mathcal{G}$ is an attractor, there exists, for given σ, a time $T_2 > 0$ such that $\mathrm{dist}^*_{Y_\rho}(\mathcal{G}_{T_2}(B_Y(R_1)), \mathcal{A}_\mathcal{G}) \le \sigma/2$. Now let $v \in \mathcal{A}^\varepsilon$ be arbitrary. By invariance there is a $u_0 \in \mathcal{A}^\varepsilon$ with $v = \mathcal{S}^\varepsilon_{T_2/\varepsilon^2}(u_0)$. Applying the approximation result (4.8) with $T_1 = T_2$ we find

$$\mathrm{dist}_Y(\Phi_\varepsilon v, \mathcal{G}_{T_2}(B_Y(R_1))) = \mathrm{dist}_Y(\Phi_\varepsilon \mathcal{S}^\varepsilon_{T_2/\varepsilon^2}(u_0), \mathcal{G}_{T_2}(B_Y(R_1))) \le C\varepsilon^{1/4}.$$

Choosing ε_0 so small that $C\varepsilon^{1/4} < \sigma/2$, then we complete the proof by

$$\mathrm{dist}^*_{Y_\rho}(\Phi_\varepsilon \mathcal{A}^\varepsilon, \mathcal{A}_\mathcal{G}) \le \mathrm{dist}_Y(\Phi_\varepsilon \mathcal{A}^\varepsilon, \mathcal{G}_{T_2}(B_Y(R_1))) + \mathrm{dist}^*_{Y_\rho}(\mathcal{G}_{T_2}(B_Y(R_1)), \mathcal{A}_\mathcal{G}) \le \sigma.$$

$\square$

Remark: As mentioned above it is also natural to compare the lift $\psi_\varepsilon(\mathcal{A}_\mathcal{G})$ of the Ginzburg–Landau attractor with the attractor $\mathcal{A}^\varepsilon \subset Z$ of the original problem. This could also be done by generalizing Theorems 4.2 and 4.3 so that they are valid in more regular spaces, such as $H^2_{l,u}$. Then one can conclude that for each $\sigma > 0$ there is a ε_0 such that for all $\varepsilon \in (0, \varepsilon_0)$ the estimate

$$\mathrm{dist}^*_{Z_\rho}(\mathcal{A}^\varepsilon, \psi_\varepsilon(\mathcal{A}_\mathcal{G})) \le \varepsilon\sigma$$

holds. Note that the diameter of $\mathcal{A}^\varepsilon$ is of order ε so that the distance is relatively small.

The result of Theorem 6.1 means that $\mathcal{A}^\varepsilon$ is upper semicontinuous towards the attractor $\mathcal{A}_\mathcal{G}$ of the Ginzburg–Landau equation in the sense that

$$\mathrm{dist}^*_{Y_\rho}(\Phi_\varepsilon\mathcal{A}^\varepsilon, \mathcal{A}_\mathcal{G}) + \frac{1}{\varepsilon}\mathrm{dist}_Z(E_s\mathcal{A}^\varepsilon, \{0\}) \to 0 \quad \text{for } \varepsilon \to 0.$$

Thus, the solutions u in the attractor $\mathcal{A}^\varepsilon$ have relatively small stable parts $u_s = E_s u$ and the critical part u_c is given approximately by a Ginzburg–Landau mode $\psi_\varepsilon(A)$ where A is in the limit attractor $\mathcal{A}_\mathcal{G}$. In this way we have obtained an upper bound on the complexity of the attractor $\mathcal{A}^\varepsilon$.

It is an unsolved problem to show the opposite direction, namely that $\mathcal{A}^\varepsilon$ is also as rich as the attractor $\mathcal{A}_\mathcal{G}$, which means a lower bound on the complexity of the attractor. In mathematical terms this means lower semicontinuity in the sense $\mathrm{dist}^*_{Y_\rho}(\mathcal{A}_\mathcal{G}, \Phi_\varepsilon\mathcal{A}^\varepsilon) \to 0$ for $\varepsilon \to 0$. For such statements one usually needs very detailed information on the flow in the limit attractor $\mathcal{A}_\mathcal{G}$ such as structural stability, cf. [HR90]. In general it may happen that the sets $\Phi_\varepsilon(\mathcal{A}^\varepsilon)$ approach only a very small subset of $\mathcal{A}_\mathcal{G}$ and that the dynamics in that subset is rather trivial while that in $\mathcal{A}_\mathcal{G}$ is not.

However, by simply restricting our view to subspaces of periodic functions we argue that the attractors $\mathcal{A}^\varepsilon$ and $\mathcal{A}_\mathcal{G}$ cannot be trivial. Fixing a spatial period $\ell > 0$, we may define the closed subspace $Z_\ell \subset Z$ of functions of this period. Clearly our semigroups leave these subspaces invariant and have (compact) attractors $\mathcal{A}^\varepsilon_\ell \subset Z_\ell$ and $\mathcal{A}_{\mathcal{G}\,\ell} \subset Y_\ell$, respectively. For the weakly unstable Kuramoto–Shivashinsky equation there is a two–parameter family of stationary spatially periodic solutions with amplitude and translation as continuous parameters. The periods are $2\pi + \mathcal{O}(\varepsilon)$ and hence these solutions are in Z_ℓ for suitable ℓ. Many of these solutions are unstable having nontrivial unstable manifolds (in the spatially periodic setting), cf. [He81]. All these stationary solutions and their unstable manifolds are contained in the attractor $\mathcal{A}^\varepsilon_\ell$ and hence in $\mathcal{A}^\varepsilon$, which is therefore nontrivial. Moreover,

$$\cup_{\ell>0}\mathcal{A}^\varepsilon_\ell \subset \mathrm{closure}_{H^1_{l,u}}(\cup_{\ell>0}\mathcal{A}^\varepsilon_\ell) \subset \mathrm{closure}_{H^1_\rho}(\cup_{\ell>0}\mathcal{A}^\varepsilon_\ell) \subset \mathcal{A}^\varepsilon.$$

It is an interesting question whether any of these inclusions is in fact an equality.

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