Numerical Approximation Techniques for Rate-Independent Enelasticity*

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1 Introduction

Incremental minimization techniques play a crucial role in the modeling of many inelastic effects. In particular, for rate-independent material models they are closely link to the solutions of the so-called energetic formulation, see [24–26]. We present recent advances in the field of space-time discretizations of such models. Using techniques from Γ-convergence of functionals, which were established in [21], we are able to establish numerical convergence results for quite general systems, including models with evolution of microstructure in terms of Young measures. Here we present the results of [15, 20] in a form that shows it easy applicability in many applications to rate-independent inelastic or hysteretic material behavior.

As an easy application of the theory we show that we obtain a simple proof of the result in [12], which states that the space-time discretization for linearized elastoplasticity with hardening converge to the solution of the space-time continuous problem. This paper seems to be the first one addressing the subtle issue of the proving this convergence without assuming any additional temporal or spatial smoothness of the solutions, as is commonly done, see e.g. [1, 11] and the references therein.

Our work is in the same spirit in using a very weak solution concept and in obtaining under general conditions. as In fact, we are dealing with the rather general concept of *energetic solutions*, which allows solutions to have jumps with respect to time and whose spatial regularity is only determined by the fact that they have finite

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energy. As is common in the nonconvex rate-independent setting, we cannot expect uniqueness of solutions and as a consequence we will only be able to show that suitable subsequences of the numerical approximations converge. Moreover, we are not able to derive convergence rates in terms of the discretization parameters.

In the last section we will address some computational results in a damage model introduced in [9] and further developed in [3,6,16,22]. A similar numerical approach to a model for shape-memory alloys is discussed in [2, 20].

2 Energetic Rate-Independent Systems

Fully rate independent models for processes describing material models occur as limits when the loading rate slows down to 0. This makes the model simpler by omitting all effects due to interior relaxation processes. However, the resulting rate-independent mathematical models are somehow degenerate. In particular, in many cases solutions for a given initial datum are no longer unique and may have jumps in time. Nevertheless, as a subclass of the generalized standard materials [7, 10], such models are widely used in engineering, in particular in the isothermal case. Mathematical analysis of such processes, based on the notion of energetic solutions introduced in [18, 23], and there is now a variety of applications in finite-strain elastoplasticity, shape-memory alloys, ferroelectric and ferromagnetic materials, in delamination, and damage, see [14] and the references therein. In fracture and crack propagation the same concept is used but often called *irreversible quasistatic evolution*, see [4, 5, 8].

Here we remain mostly in the abstract setting and refer to [15] for more elaborations on numerical approximations and to [13] for a numerical convergence result involving gradient Young measures.

We consider situations where the state of the body $\Omega \subset \mathbb{R}^d$ can be described by the displacement $u : \Omega \to \mathbb{R}^d$ and an internal variable $z : \Omega \to Z \subset \mathbb{R}^m$. Here z may be a collection of internal variables, either scalars (like in damage), vectors (like magnetization or polarization) or tensors (like the plastic deformation). The pair $q = (u, z)$ is called the state of the system and it is assumed to lie in the Banach space $\mathcal{Q} = \mathcal{U} \times \mathcal{Z}$, where $\mathcal{U}$ is the set of admissible displacements which is specified via Dirichlet boundary conditions on $\Gamma_{\mathrm{Dir}} \subset \partial\Omega$.

The properties of the body are described via an energy storage functional $\mathcal{E}(t, q) \in \mathbb{R}_\infty :=]-\infty, \infty]$ and a dissipation potential $\mathcal{R}(z, \dot{z}) \in [0, \infty]$. In most cases one can assume that these functional are given via integration over the body as

$$\mathcal{E}(t,u,z)=\int_\Omega W(x,e(u)(x),z(x))+\kappa|\nabla z(x)|^r\,\mathrm{d}x-\langle \ell(t),u\rangle$$
$$\text{where } \langle \ell(t),u\rangle=\textstyle\int_\Omega f_{\mathrm{vol}}(t,x)\cdot u(x)\,\mathrm{d}x+\int_{\partial\Omega\setminus\Gamma_{\mathrm{Dir}}} f_{\mathrm{surf}}(t,x)\cdot u(x)\,\mathrm{d}a,$$
$$\mathcal{R}(z,v)=\int_\Omega R(x,z(x),v(x))\,\mathrm{d}x,$$

where the linearized strain is $e(u)=\frac{1}{2}(\nabla u+\nabla u^{\mathsf{T}})$.

The rate-independent evolution can be written as the system given via the elastic equilibrium and the balance of the internal forces, also called Biot's law, flow rule, or switching condition:

$$\begin{array}{ll}\text{elastic equilibrium} & 0=\mathrm{D}_u\mathcal{E}(t,u(t),z(t)),\\ \text{flow rule} & 0\in\partial_{\dot z}\mathcal{R}(z(t),\dot z(t))+\mathrm{D}_z\mathcal{E}(t,u(t),z(t)),\end{array}\tag{1}$$

where $\partial\mathcal{R}(z,v)$ denotes the set-valued subdifferential of the convex and 1-homogeneous function $v\mapsto\mathcal{R}(z,v)$.

However, in many situations it is not possible to show that (1) has solutions. Hence, we will use the energetic solutions introduced in [14, 18, 23]. For this we need the dissipation distance $\mathcal{D}(z_0,z_1)\in[0,\infty]$ which denotes minimal energy that is dissipated along a smooth path when changing the internal state from z_0 into z_1:

$$\mathcal{D}(z_0,z_1):=\inf\{\,\mathrm{Diss}_{\mathcal{D}}(\widetilde z,[0,1])\mid \widetilde z(0)=z_0,\ \widetilde z(1)=z_1\,\},\tag{2}$$

where $\mathrm{Diss}_{\mathcal{D}}(\widetilde z,[t_0,t_1])=\int_{t_0}^{t_1}\mathcal{R}(\widetilde z(s),\dot{\widetilde z}(s))\,\mathrm{d}s$. Note that $\mathcal{R}$ has the physical dimension of a power whereas $\mathcal{D}$ has the dimension of energy. We will call the triple $(\mathcal{Q},\mathcal{E},\mathcal{D})$ an *energetic system*.

Definition 1. *The process $q:[0,T]\to\mathcal{Q}$ is called an* energetic solution *of the energetic systems $(\mathcal{Q},\mathcal{E},\mathcal{D})$, if for all $t\in[0,T]$, we have* stability (S) *and* energy balance (E)*:*

$$\text{(S)}\qquad \mathcal{E}\big(t,q(t)\big)\le\mathcal{E}(t,\widetilde q)+\mathcal{D}(q(t),\widetilde q)\ \textit{for all}\ \widetilde q\in\mathcal{Q}.$$
$$\text{(E)}\qquad \mathcal{E}(t,q(t))+\mathrm{Diss}_{\mathcal{D}}(q,[0,t])=\mathcal{E}(0,q(0))+\int_0^t\partial_s\mathcal{E}(s,q(s))\,\mathrm{d}s.$$

We continue to write $\mathcal{D}(q,\widetilde q)$ for $\mathcal{D}(z,\widetilde z)$ and $\mathrm{Diss}_{\mathcal{D}}(q,[0,t])$ for $\mathrm{Diss}_{\mathcal{D}}(z,[0,t])$, whenever it is clear that $q=(u,z)$ and $\widetilde q=(\widetilde u,\widetilde z)$.

It is interesting to note that the subdifferential form (1) and the energetic form (S) & (E) are in fact extremal principle in the sense of [25,26], particularly the definition (2) of the dissipation distance.

It is discussed in [14, 19] under which conditions on $\mathcal{E}$ and $\mathcal{D}$ the notion of energetic solutions is equivalent to the solutions of (1). The point is that for general, nonconvex functionals $\mathcal{E}(t,\cdot)$ one cannot expect to find solutions of (1) while there exist energetic solutions under quite general situations. The typical assumptions for an existence theory are the following. Assume that $\mathcal{Q}=\mathcal{U}\times\mathcal{Z}$ is a reflexive Banach space, e.g. $\mathcal{U}$ is a closed subspace of $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^d)$ and $\mathcal{Z}=\mathrm{W}^{1,r}(\Omega;\mathbb{R}^m)$. We introduce the sets $\mathcal{S}(t)$ of stable states at time t via

$$\mathcal{S}(t) = \{\, q \in \mathcal{Q} \mid \mathcal{E}(t,q) < \infty \text{ and } \mathcal{E}(t,q) \leq \mathcal{E}(t,\widetilde{q}) + \mathcal{D}(q,\widetilde{q}) \text{ for all } \widetilde{q} \in \mathcal{Q} \,\}.$$

If $\mathcal{E}$ and $\mathcal{D}$ satisfy the following conditions (3)–(7), then for each initial condition $q_0 \in \mathcal{S}(0)$ an energetic solution exists, see [14] for a survey.

For these conditions we introduce the sets $\mathcal{S}(t)$ of stable states

$$\begin{array}{l}\text{for all } z_1, z_2, z_3 \in \mathcal{Z} \text{ we have}\\ \text{positivity:} \qquad \mathcal{D}(z_1,z_2) = 0 \iff z_1 = z_2,\\ \text{triangle inequality: } \mathcal{D}(z_1,z_3) \leq \mathcal{D}(z_1,z_2) + \mathcal{D}(z_2,z_3);\end{array} \tag{3}$$

$$\mathcal{D} : \mathcal{Z} \times \mathcal{Z} \to [0,\infty] \text{ is weakly lower semi-continuous;} \tag{4}$$

$$\mathcal{E} : [0,T] \times \mathcal{Q} \text{ is weakly lower semi-continuous and coercive;} \tag{5}$$

$$\begin{array}{l}\text{there exist constants } c_0^{\mathcal{E}}, c_1^{\mathcal{E}} \text{ such that}\\ \mathcal{E}(0,q) < \infty \text{ implies } \mathcal{E}(\cdot,q) \in \mathrm{C}^1([0,T]) \text{ with}\\ |\partial_t \mathcal{E}(t,q)| \leq c_1^{\mathcal{E}}\big(\mathcal{E}(t,q) + c_0^{\mathcal{E}}\big);\end{array} \tag{6}$$

$$\begin{array}{l}\text{for each sequence } (t_n,q_n)_{n\in\mathbb{N}} \text{ with } (t_n,q_n) \rightharpoonup (t_*,q_*),\\ \qquad\qquad q_n \in \mathcal{S}(t_n), \text{ and } \sup_{n\in\mathbb{N}} \mathcal{E}(t_n,q_n) < \infty \text{ we have}\\ \text{(a) } q_* \in \mathcal{S}(t_*) \quad \text{and (b) } \partial_t \mathcal{E}(t_*,q_n) \to \partial_t \mathcal{E}(t_*,q_*).\end{array} \tag{7}$$

Here conditions (3) and (4) are standard assumptions on the dissipation distance; the triangle inequality follows easily from definition (2). Conditions (5) and (6) relate only to the energy functional. The first one is the standard condition in the calculus of variations, while the second one is called an *energetic control* of the power of the external forces. This condition is crucial to obtain uniform a priori bounds.

Condition (7) may be called a *compatibility condition* as it relates, via the stable sets $\mathcal{S}(t_j)$, the properties of $\mathcal{E}$ and $\mathcal{D}$ in an intrinsic manner. While part (b) is often easy to establish (time $t = t_*$ is fixed in the power $\partial_t \mathcal{E}(t,q)$), part (a) is the most delicate point. One way to establish this condition is the *joint recovery condition*, namely

$$\begin{array}{l}\text{for all } q_*, \widetilde{q} \in \mathcal{Q},\ (t_n,q_n)_{n\in\mathbb{N}} \text{ with } (t_n,q_n) \rightharpoonup (t_*,q_*),\ q_n \in \mathcal{S}(t_n),\\ \quad \text{and } \sup_{n\in\mathbb{N}} \mathcal{E}(t_n,q_n) < \infty \text{ there exists } (\widetilde{q}_n)_{n\in\mathbb{N}} \text{ with } \widetilde{q}_n \rightharpoonup \widetilde{q} \text{ such that}\\ \limsup_{n\to\infty} \mathcal{E}(t_n,\widetilde{q}_n) + \mathcal{D}(q_n,\widetilde{q}_n) - \mathcal{E}(t_n,q_n) \leq \mathcal{E}(t_*,\widetilde{q}) + \mathcal{D}(q_*,\widetilde{q}) - \mathcal{E}(t_*,q_*).\end{array} \tag{8}$$

Proposition 1. *Conditions* (5) *and* (8) *imply* (7a).

Proof. By (5) we have $\mathcal{E}(t_*,q_*) \leq \liminf_{n\to\infty} \mathcal{E}(t_n,q_n) \leq \sup_{n\in\mathbb{N}} \mathcal{E}(t_n,q_n) < \infty$, where the last inequality is assumed in (7). Next, for $\widetilde{q} \in \mathcal{Q}$ arbitrary, choose $\widetilde{q}_n \in \mathcal{Q}$ as in (8). By definition $q_n \in \mathcal{S}(t_n)$ says that $\mathcal{E}(t_n,\widetilde{q}_n) + \mathcal{D}(q_n,\widetilde{q}_n) - \mathcal{E}(t_n,q_n) \geq 0$. Taking the limsup and using (8) gives

$$0 \leq \limsup_{n\to\infty} \mathcal{E}(t_n, \widetilde{q}_n)+\mathcal{D}(q_n, \widetilde{q}_n)-\mathcal{E}(t_n, q_n) \leq \mathcal{E}(t_*, \widetilde{q})+\mathcal{D}(q_*, \widetilde{q})-\mathcal{E}(t_*, q_*),$$

which is the desired stability of q_*, since $\widetilde{q}$ was arbitrary. □

3 Space-Time Discretization

We consider two positive parameters τ and h, were $\tau > 0$ represents the fineness of a time discretization by a partition (not necessarily equidistant) of the time interval $[0, T]$. We assume that partitions $\Pi^\tau = \{0 = t_0^\tau < t_1^\tau < \cdot < t_{N_\tau-1}^\tau < t_{N_\tau}^\tau = T\}$ are given such that

$$\text{fineness}(\Pi^\tau) := \max\{\, t_j^\tau - t_{j-1}^\tau \mid j = 1, \dots, N^\tau \,\} \leq \tau. \tag{9}$$

The parameter $h > 0$ denotes a discretization of the state space $\mathcal{Q}$ by subsets $\mathcal{Q}_h$ again having the structure $\mathcal{Q}_h := \mathcal{U}_h \times \mathcal{Z}_h$. We assume that each $\mathcal{Q}_h$ is closed and the family $(\mathcal{Q}_h)_{h>0}$ is dense, namely

$$\begin{gathered}\text{for each } (t, q) \in [0, T] \times \mathcal{Q} \text{ there exist } (q_h)_{h>0} \text{ such that}\\ q_h \in \mathcal{Q}_h,\ q_h \to q,\ \text{ and } \mathcal{E}(t, q_h) \to \mathcal{E}(t, q).\end{gathered} \tag{10}$$

Hence each space-time discretization is denoted by a pair (τ, h) and we now define the approximation via an incremental minimization problem for the partition Π^τ in the discrete space $\mathcal{Q}_h$ as follows. For a given initial value $q_0^h \in \mathcal{Q}_h$ we define $(q_j^{\tau,h})_{j=0,1,\dots,N_\tau}$ via

$$q_j^{\tau,h} \in \operatorname*{Argmin}_{\widetilde{q}\in\mathcal{Q}_h} \mathcal{E}(t_j^\tau, \widetilde{q}) - \mathcal{E}(t_{j-1}^\tau, q_{j-1}^{\tau,h}) + \mathcal{D}(q_{j-1}^{\tau,h}, \widetilde{q}). \tag{11}$$

Existence of these minimizers follows easily if we assume (4) and (5).

Using these time-discrete approximations in $\mathcal{Q}_h$ we define piecewise constant interpolants $\overline{q}_{\tau,h} : [0, T] \to \mathcal{Q}_h$ via

$$\overline{q}_{\tau,h}(t) = q_k^{\tau,h} \text{ for } t \in \big[t_k, t_{k+1}\big[\text{ and } k = 0, \dots, N^\tau-1 \text{ and } \overline{q}_{\tau,h}(T) = q_{N^\tau}^{\tau,h}.$$

The first result we give may be considered as a weak analog of *stability of a numerical scheme*. In fact, it provides uniform a priori estimates.

Theorem 1. *Let* (3)–(6) *hold, then the approximations* $\overline{q}_{\tau,h} : [0, T] \to \mathcal{Q}_h$ *exist and satisfy the following conditions:*

discrete stability

$\overline{q}_{\tau,h}(t) \in \mathcal{S}_h(t)$ *for all* $t \in \Pi^\tau = \{\, t_j^\tau \mid j = 0, 1, ..., N^\tau \,\}$; (12)

upper energy estimate (for $0 \le j < k \le N^\tau$*)*

$$\mathcal{E}(t_k^\tau, \overline{q}_{\tau,h}(t_k^\tau)) + \mathrm{Diss}_{\mathcal{D}}(\overline{q}_{\tau,h}, [t_j^\tau, t_k^\tau]) \le \mathcal{E}(t_j^\tau, \overline{q}_{\tau,h}(t_j^\tau)) + \int_{t_j^\tau}^{t_k^\tau} \partial_s \mathcal{E}(s, \overline{q}_{\tau,h}(s))\,\mathrm{d}s; \tag{13}$$

a priori estimates for all $t \in [0, T]$

$$\mathcal{E}(t, \overline{q}_{\tau,h}(t)) \le \exp(c_1^{\mathcal{E}} t)\big(\mathcal{E}(0, q_0^h) + c_0^{\mathcal{E}}\big) - c_0^{\mathcal{E}} \text{ and} \tag{14}$$

$$\mathrm{Diss}_{\mathcal{D}}(\overline{q}_{\tau,h}, [0, t]) \le \exp(c_1^{\mathcal{E}} t)\big(\mathcal{E}(0, q_0^h) + c_0^{\mathcal{E}}\big). \tag{15}$$

Here the stable sets $\mathcal{S}_h(t)$ are defined in the obvious way

$$\mathcal{S}_h(t) := \{\, q_h \in \mathcal{Q}_h \mid \mathcal{E}(t, q_h) < \infty,\ \mathcal{E}(t, q_h) \le \mathcal{E}(t, \widetilde{q}_h) + \mathcal{D}(q_h \widetilde{q}_h) \text{ for } \widetilde{q}_h \in \mathcal{Q}_h \,\}.$$

Note that these stable sets may be substantially larger than $\mathcal{S}(t) \cap \mathcal{Q}_h$.

To formulate the main convergence result, we need to adjust the compatibility condition (7) to sequences of spatial approximations:

$$\begin{array}{l} \text{for each sequence } (h_n, t_n, q_n)_{n\in\mathbb{N}} \text{ with } (h_n, t_n, q_n) \rightharpoonup (0, t_*, q_*), \\ \qquad q_n \in \mathcal{S}_{h_n}(t_n), \text{ and } \sup_{n\in\mathbb{N}} \mathcal{E}(t_n, q_n) < \infty \text{ we have} \\ \text{(a) } q_* \in \mathcal{S}(t_*) \quad \text{and (b) } \partial_t \mathcal{E}(t_*, q_n) \to \partial_t \mathcal{E}(t_*, q_*). \end{array} \tag{16}$$

As given in Proposition 1 the crucial part (a) can be derived via the correspondingly adjusted *joint recovery condition*, namely

$$\begin{array}{l} \text{for all } q_*, \widetilde{q} \in \mathcal{Q},\ (h_n, t_n, q_n)_{n\in\mathbb{N}} \\ \text{with } (h_n, t_n, q_n) \rightharpoonup (0, t_*, q_*),\ q_n \in \mathcal{S}_{h_n}(t_n), \text{ and } \sup_{n\in\mathbb{N}} \mathcal{E}(t_n, q_n) < \infty, \\ \text{there exists } (\widetilde{q}_n)_{n\in\mathbb{N}} \text{ with } \mathcal{Q}_{h_n} \ni \widetilde{q}_n \rightharpoonup \widetilde{q} \text{ such that} \\ \limsup_{n\to\infty} \mathcal{E}(t_n, \widetilde{q}_n) + \mathcal{D}(q_n, \widetilde{q}_n) - \mathcal{E}(t_n, q_n) \le \mathcal{E}(t_*, \widetilde{q}) + \mathcal{D}(q_*, \widetilde{q}) - \mathcal{E}(t_*, q_*). \end{array} \tag{17}$$

Our main result provides the convergence of space-time discretizations. Because of the implicit nature of the incremental minimization problem (11) there is no "stability restriction" on the size of τ in relation to h. Of course, we cannot expect convergence of the full sequence of approximations, since in general the energetic systems $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$ may have several solutions for a given initial value $q_0 \in \mathcal{S}(0)$, and subsequences may converge to different solutions. Nevertheless, any accumulation point of the approximations is an energetic solution for $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$. Thus, there are no spurious solutions and we may call this property *consistency of the numerical scheme*.

Theorem 2. *Assume that* $\mathcal{E}$ *and* $\mathcal{D}$ *satisfy* (3)–(6). *Let* $(\Pi^\tau)_\tau$ *and* $(\mathcal{Q}_h)_h$ *be given such that* (9), (10), *and* (16) *hold. Let* $q_0 \in \mathcal{S}(0)$ *be given and choose* $q_0^h \in \mathcal{Q}_h$ *with*

$q_0^h \rightharpoonup q_0$ *and* $\mathcal{E}(0, q_0^h) \to \mathcal{E}(0, q_0)$*, and construct approximate solutions* $\overline{q}_{\tau,h} : [0, T] \to \mathcal{Q}_h$*. Then, there exists a subsequence* $(\tau_n, h_n)_{n\in\mathbb{N}}$ *with* $(\tau_n, h_n) \to (0, 0)$ *for* $n \to \infty$ *and an energetic solution* $q : [0, T] \to \mathcal{Q}$ *of* $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$ *with* $q(0) = q_0$ *such that, with the shorthand* $\overline{q}_n = (\overline{u}_n, \overline{z}_n) := \overline{q}_{\tau_n,h_n}$*, for all* $t \in [0, T]$ *the following holds:*

$$\mathcal{E}(t, \overline{q}_n(t)) \to \mathcal{E}(t, q(t)) \text{ for } n \to \infty; \tag{18a}$$

$$\mathrm{Diss}_{\mathcal{D}}(\overline{q}_n; [0, t]) \to \mathrm{Diss}_{\mathcal{D}}(q, [0, t]) \text{ for } n \to \infty; \tag{18b}$$

$$\overline{z}_n(t) \rightharpoonup z(t) \text{ in } \mathcal{Z} \text{ for } n \to \infty, \tag{18c}$$

there exists a subsequence $(n_l^t)_{l\in\mathbb{N}}$ *such that*

$$\overline{u}_{n_l^t}(t) \rightharpoonup u(t) \text{ in } \mathcal{U} \text{ for } l \to \infty; \tag{18d}$$

$$\partial_t\mathcal{E}(\cdot, \overline{q}_n(\cdot)) \to \partial_t\mathcal{E}(\cdot, q(\cdot)) \text{ in } \mathrm{L}^1(0, T) \text{ for } n \to \infty. \tag{18e}$$

If additionally, $\mathcal{E}(t, \cdot, z) : \mathcal{U} \to \mathbb{R}_\infty$ *is strictly convex, then* (18d) *can be strengthened into* $\overline{u}_n(t) \to u(t)$ *in* $\mathcal{U}$ *(without further subsequences).*

If there is only one energetic solution q *for* $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$ *with* $q(0) = q_0$*, then the whole sequence converges, i.e.,* $\overline{q}_{\tau,h}(t) \rightharpoonup q(t)$ *in* $\mathcal{Q}$ *for* $(\tau, h) \to (0, 0)$.

For a proof of this and even much more general results we refer to [15, 20]. In fact, the proof is an adaptation of the proof of theorem 3.4 in [21] which is based on general ideas of Γ-convergence for sequences of rate-independent systems $(\mathcal{Q}, \mathcal{E}_n, \mathcal{D}_n)_{n\in\mathbb{N}}$.

Since the energetic solutions are not unique in general, one may ask the opposite question. Is it possible to obtain each energetic solution of $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$ as limit of a subsequence? It is shown in [17] that this cannot be expected. However, if one uses approximate minimizers in (11), then this is true. Here approximate minimizers means that $q_j^{\tau,h}$ must be such that the functional under "Argmin" is minimized up to an error δ. In [21] it is shown that the above convergence of subsequences still holds if $\delta = o(\tau)$ for $\tau \to 0$.

4 Linearized Plasticity with Hardening

To start with, we want to demonstrate the applicability of our theory in a simple situation, namely in rate-independent linearized elastoplasticity with hardening. In fact, we are thus able to recover that result in [12], where convergence (without rates) of space-time discretization was shown for the first time under *conditions of minimal regularity*, viz. thus that are known from the classical existence theory.

Here $\mathcal{Q} = \mathcal{U} \times \mathcal{Z}$ with Hilbert spaces $\mathcal{U} = \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega; \mathbb{R}^d)$ and $\mathcal{Z} = \mathrm{L}^2(\Omega; Z)$, where Z is $\mathbb{R}^{d\times d}_{0,\mathrm{sym}} = \{ A \in \mathbb{R}^{d\times d} \mid A = A^\mathsf{T}, \mathrm{tr}A = 0 \}$ for kinematic hardening and $Z = \mathbb{R}^{d\times d}_{0,\mathrm{sym}} \times \mathbb{R}$ for isotropic hardening. The energy functional is quadratic and takes the form

$$\mathcal{E}(t,u,z) = \frac{1}{2}\langle\!\langle \mathcal{A}\binom{u}{z}, \binom{u}{z}\rangle\!\rangle - \langle \ell(t), u\rangle$$

with a bounded, symmetric and positive definite operator $\mathcal{A} : \mathcal{Q} \to \mathcal{Q}^*$. The dissipation distance reads $\mathcal{D}(q_0, q_1) = \Psi(z_1 - z_0)$ with $\Psi^* = \chi_{\mathcal{K}}$, where the closed convex cone $\mathcal{K} \subset \mathcal{Z}^*$ is called the elastic domain.

It is easy to see that $\mathcal{E}$ and $\mathcal{D}$ satisfy the assumptions (3)–(6). Moreover, part (b) in the compatibility condition (16) is also valid, as the power $\partial_t \mathcal{E}(t,q) = -\langle \dot{\ell}(t), u\rangle$ is linear and, hence, weakly continuous.

It remains to establish part (a) of (16) by using the joint recovery condition (17). For this assume that there exist interpolation operators $B_h : \mathcal{Q} \to \mathcal{Q}_h$ such that

$$B_h q \to q \text{ strongly in } \mathcal{Q} \text{ and } \Psi(B_h q) \to \Psi(q). \tag{19}$$

While the first case is the usual interpolation condition, the second condition states that this has to be consistent with the dissipation potential Ψ. Since in general Ψ is not strongly continuous, this is nontrivial. However, as $\Psi(v) = \int_\Omega \psi(x, v(x))\,\mathrm{d}x$ and $\psi(x,\cdot)$ is convex, it is sufficient to choose B_h such that z_h in $(u_h, z_h) = B_h q$ is piecewise constant taking the average value over the polyhedra in the spatial discretization. We choose $\widetilde{q}_n \in \mathcal{Q}_{h_n}$ in (17) via

$$\widetilde{q}_n = q_n + B_{h_n}(\widetilde{q} - q_*) \text{ giving } \begin{cases} \widetilde{q}_n \rightharpoonup \widetilde{q}, \\ \widetilde{q}_n - q_n \to \widetilde{q} - q_*. \end{cases} \tag{20}$$

Clearly, we have $\mathcal{D}(q_n, \widetilde{q}_n) = \Psi(B_{h_n}(\widetilde{q} - q_*)) \to \Psi(\widetilde{q} - q_*) = \mathcal{D}(q_*, \widetilde{q})$. Moreover, in the energy can use the quadratic nature to profit from cancellation effects:

$$\begin{aligned} &\mathcal{E}(t_n, \widetilde{q}_n) - \mathcal{E}(t_n, q_n) = \langle\!\langle \tfrac{1}{2}\mathcal{A}(\widetilde{q}_n + q_n) - \tbinom{\ell(t_n)}{0}, \widetilde{q}_n - q_n \rangle\!\rangle \\ &\to \langle\!\langle \tfrac{1}{2}\mathcal{A}(\widetilde{q} + q_*) - \tbinom{\ell(t_*)}{0}, \widetilde{q} - q_* \rangle\!\rangle = \mathcal{E}(t_*, \widetilde{q}) - \mathcal{E}(t_*, q_*). \end{aligned}$$

Here (20) guarantees that the first term in $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ converges weakly and the second strongly. It follows that (17) and whence (16) hold, and Theorem 2 provides convergence of the *whole discretization sequence*, since that continuous problem $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$ has a unique solution for each $q(0) \in \mathcal{S}(0)$.

5 A Damage Model

Finally we consider a damage model introduced in [6, 9] and analyzed in [16, 22] using the energetic approach. While the displacement $u \in \mathcal{U} = \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega; \mathbb{R}^d)$ is as above, the internal variable is now a scalar damage variable with $z(t,x) \in Z := [0,1]$, where $z = 1$ denotes an undamaged material whereas $z = 0$ means that all breakable pieces are broken. However, depending on the model, $z = 0$ may still have some remaining elasticity. As space of internal states we let

$$\mathcal{Z} := \{\, z \in \mathrm{W}^{1,r}(\Omega) \mid z(x) \in [0,1] \,\} \Subset \mathrm{C}^0(\overline{\Omega}),$$

where we assume $r > d$ to have the indicated embedding. The dissipation distance is chosen in such a way that increase of damage (decrease of z) cost proportional to the increase and the damaged volume. To forbid healing we set the dissipation ∞ for increasing z:

$$\mathcal{D}(z_0, z_1) = \int_\Omega \psi(x, z_1(x) - z_0(x))\,\mathrm{d}x \text{ with } \psi(x, v) = \begin{cases} \delta(x)|v| & \text{for } v \le 0, \\ \infty & \text{for } v > 0. \end{cases}$$

For simplicity we assume that the linearized elasticity can be assumed giving a quadratic energy functional $\mathcal{E}$:

$$\mathcal{E}(t, u, z) = \int_\Omega \frac{1}{2}(e(u) + e_{\mathrm{D}}(t)) : \boldsymbol{C}(z) : (e(u) + e_{\mathrm{D}}(t)) + \kappa |\nabla z|^r \,\mathrm{d}x,$$

where $e_{\mathrm{D}}(t) = e(u_{\mathrm{D}}(t))$ and $u_{\mathrm{D}} \in \mathrm{C}^1([0, T], \mathrm{H}^1(\Omega; \mathbb{R}^d)$ is given. The elasticity tensor is monotone in z, i.e., $\boldsymbol{C}'(z) \ge \boldsymbol{0}$ in the sense of symmetric operators. Moreover, the coercivity

$$e{:}\boldsymbol{C}(z){:}e \ge (\alpha_0 + \alpha_1 z^\gamma)|e|^2 \text{ for all } z \in [0, 1] \text{ and } e \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$$

will be basic, where $\alpha_1, \gamma > 0$ and $\alpha_0 \ge 0$. The case $\alpha_0 > 0$ corresponds to incomplete damage like in [8, 16], and α_0 allows for complete damage as studied in [3, 22]. To treat the latter case it is necessary to eliminate the displacement u, since it may not be well-defined because of missing coercivity. This is done via introducing the quadratic functional

$$\mathcal{V}(z, e_{\mathrm{D}}) = \liminf_{z_n \rightharpoonup z} \min_{u \in \mathcal{U}} \int_\Omega \frac{1}{2}(e(u) + e_{\mathrm{D}}) : \boldsymbol{C}(z_n) : (e(u) + e_{\mathrm{D}})\,\mathrm{d}x,$$

which even allows to control the equilibrium stresses via $\mathrm{D}_{e_{\mathrm{D}}}\mathcal{V}$.

Again it is easy to check the assumption (3)–(6); as usual the main difficulty lies in part (a) of (16). Assume again that $B_h : \mathcal{Q} \to \mathcal{Q}_h$ is an interpolant to piecewise affine functions on a triangulation $\mathcal{T}_h$ of Ω such that $B_h q = (B_h^{\mathcal{U}} u, B_h^{\mathcal{Z}} z) \to q$ in $\mathcal{Q}$ strongly. To employ the joint recovery condition (17) we choose for given $\widetilde{q} \in \mathcal{Q}$ and $q_n \in \mathcal{Q}_{h_n}$ the joint recovery sequence

$$\widehat{q}_n = (B_{h_n}^{\mathcal{U}} \widetilde{u}, \max\{0, B_{h_n}^{\mathcal{Z}} \widetilde{z} - \rho_n\}) \in \mathcal{Q}_{h_n} \text{ with } \rho_n = \| \max\{0, B_{h_n}^{\mathcal{Z}} \widetilde{z} - z_n\} \|_{\mathrm{L}^\infty(\Omega)}.$$

Since we only need to check condition (16a) for $\mathcal{D}(z, \widetilde{z}) < \infty$ we may assume $\widetilde{z} \le z_*$. Now using the embedding $\mathcal{Z} \Subset \mathrm{C}^0(\overline{\Omega})$ we find $z_n \to z_*$ in L^∞ and similarly $B_{h_n}^{\mathcal{Z}} \to \widetilde{z} \le z_*$. Thus, we have $\rho_n \to 0$ and conclude $\widetilde{q}_n \to \widetilde{q}$ in $\mathcal{Q}$ strongly. This in turn implies $\mathcal{E}(t_n, q_n) \to \mathcal{E}(t_*, q_*)$ and $\mathcal{D}(z_n, \widetilde{z}_n) \to \mathcal{D}(z_*, \widetilde{z})$. Using the lower semicontinuity of $\mathcal{E}$, condition (16a) is established.

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