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Analysis of Energetic Models for Rate–Independent Materials

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Abstract

We consider rate–independent models which are defined via two functionals: the time–dependent energy–storage functional $\mathcal{I} : [0,T] \times X \to [0,\infty]$ and the dissipation distance $\mathcal{D} : X \times X \to [0,\infty]$. A function $z : [0,T] \to X$ is called a solution of the energetic model, if for all $0 \le s < t \le T$ we have

$$\text{stability:} \qquad \mathcal{I}(t,z(t)) \le \mathcal{I}(t,\widetilde{z}) + \mathcal{D}(z(t),\widetilde{z}) \text{ for all } \widetilde{z} \in X;$$
$$\text{energy inequality:} \quad \mathcal{I}(t,z(t)) + \mathrm{Diss}_{\mathcal{D}}(z,[s,t]) \le \mathcal{I}(s,z(s)) + \int_s^t \partial_\tau \mathcal{I}(\tau,z(\tau))\,\mathrm{d}\tau.$$

We provide an abstract framework for finding solutions of this problem. It involves time discretization where each incremental problem is a global minimization problem. We give applications in material modeling where $z \in \mathcal{Z} \subset X$ denotes the internal state of a body. The first application treats shape–memory alloys where z indicates the different crystallographic phases. The second application describes the delamination of bodies glued together where z is the proportion of still active glue along the contact zones. The third application treats finite–strain plasticity where $z(t,x)$ lies in a Lie group.

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1 Introduction

Many evolution equations can be written in the abstract form

$$0 \in \partial\Psi(\dot z(t)) + \mathrm{D}\mathcal{I}(t, z(t)), \tag{1.1}$$

where $z \in X$ is the state variable, $\mathcal{I}$ is the energy–storage functional, $\Psi : X \to [0,\infty]$ is a convex dissipation functional, and $\partial\Psi$ means the set–valued subdifferential (see [2] for this doubly nonlinear form). Rate–independency is realized by assuming that Ψ is homogeneous of degree 1.

We replace the above differential inclusion by a weaker energetic formulation, which is also more general since it allows for z–dependent dissipation functionals. For given $\mathcal{I} : [0,T] \times X \to [0,\infty]$ and a given dissipation distance $\mathcal{D} : X \times X \to [0,\infty]$ satisfying the triangle inequality, we impose the energetic conditions of *global stability* (S) and the *energy inequality* (E) instead of (1.1). A function $z : [0,T] \to X$ is called a *solution of the energetic model*, if for all $0 \le s < t \le T$ we have

(S) $\quad \mathcal{I}(t, z(t)) \le \mathcal{I}(t, \widetilde z) + \mathcal{D}(z(t), \widetilde z)$ for all $\widetilde z \in X$;
(E) $\quad \mathcal{I}(t, z(t)) + \mathrm{Diss}_{\mathcal{D}}(z, [s,t]) \le \mathcal{I}(s, z(s)) + \int_s^t \partial_\tau \mathcal{I}(\tau, z(\tau))\,\mathrm{d}\tau$.

Here, $\mathrm{Diss}_{\mathcal{D}}(z, [s,t])$ is called the dissipation of z on the interval $[s,t]$ and is defined as the supremum of $\sum_{j=1}^N \mathcal{D}(z(t_{j-1}), z(t_j))$ over all $N \in \mathbb{N}$ and all discretizations $s = t_0 < t_1 < \ldots < t_N = t$.

Assuming $\mathcal{D}(z_0, z_1) = \Psi(z_1 - z_0)$, convexity of $\mathcal{I}(t,\cdot)$ and further technical assumptions, this energetic formulation is equivalent to (1.1), see [16]. However, the latter form is more general as it applies to nonconvex problems and it doesn't need differentiability of $t \mapsto z(t)$ nor of $z \mapsto \mathcal{I}(t,z)$. A related energetic approach to equations of the type (1.1) is presented in [20], however, it remains unclear whether that method applies to the rate–independent case.

In Section 2 we discuss the abstract setting in more detail and in Section 3 we provide existence results for solutions for given initial values $z(0) = z_0$. The existence theory is based on time–incremental minimization problems of the form

$$z_k \in \mathrm{argmin}\{\, \mathcal{I}(t_k, z) + \mathcal{D}(z_{k-1}, z) \mid z \in X \,\}$$

and the BV bound for $z : [0,T] \to X$ obtained via the dissipation functional satisfying $\mathcal{D}(z_0, z_1) \ge c_D \, \|z_0 - z_1\|$. However, one needs additional compactness properties, if X is infinite dimensional. Here we propose a version where $\mathcal{I}$ satisfies coercivity with respect to an embedded Banach space Y, i.e., $\mathcal{I}(t,z) \ge -C_1 + c_1 \|z\|_Y^\alpha$ with $c_1, C_1, \alpha > 0$, where Y is compactly embedded in X.

For the case of $\mathcal{D}$ having the form $\mathcal{D}(z_0, z_1) = \Psi(z_1 - z_0)$ this theory was developed in [16]. The case of general $\mathcal{D}$ can be found in [10].

The flexibility of the energetic formulation allows for applications in continuum mechanics, where $z : \Omega \to Z$ plays the rôle of internal variables in the material occupying the body $\Omega \subset \mathbb{R}^d$. Note that Z may be a manifold containing the internal variables like phase indicators, plastic or phase transformations, damage, polarization or magnetization. By $\mathcal{Z}$ we denote the set of all admissible internal states. The elastic deformation is $\varphi : \Omega \to \mathbb{R}^d$ and $\mathcal{F}$ denotes the set of admissible deformations φ.

Energy storage is characterized via the functional $\mathcal{E} : [0,T] \times \mathcal{F} \times \mathcal{Z} \to \mathbb{R}$, where $t \in [0,T]$ is the (quasi–static) process time, which drives the system via changing loads. In typical material models, $\mathcal{E}$ has the form

$$\mathcal{E}(t,\varphi,z) = \int_\Omega W(x, \mathrm{D}\varphi(x), z(x))\,\mathrm{d}x - \langle \ell_{\mathrm{ext}}(t), \varphi\rangle,$$

where W is the stored–energy density and $\ell_{\mathrm{ext}}(t)$ denotes the external loadings.

Dissipation is characterized by an infinitesimal Finsler metric $\Delta : \Omega \times \mathrm{T}Z \to [0,\infty]$, such that the curve $z : [t_0,t_1] \to \mathcal{Z}$ dissipates the energy

$$\mathrm{Diss}\,(z,[t_0,t_1]) = \int_{t_0}^{t_1} \int_\Omega \Delta(x, z(t,x), \dot z(t,x))\,\mathrm{d}x\,\mathrm{d}t.$$

The global dissipation distance $\mathcal{D}(z_0,z_1)$ is then the infimum over all curves connecting z_0 with z_1. The relation to the abstract theory above is obtained by eliminating the elastic deformation via

$$\mathcal{I}(t,z) = \inf\{\, \mathcal{E}(t,\varphi,z) \mid \varphi \in \mathcal{F}\,\} \text{ for } z \in \mathcal{Z} \text{ and } \mathcal{I}(t,z) = +\infty \text{ else.}$$

Obviously, the functional $\mathcal{I}$ is now fairly complicated and it is important to have rather general conditions in the abstract theory.

In Section 4 we illustrate the usefulness of the abstract approach by discussing three quite different applications; however, the theory is used in other areas as well, e.g., in fracture mechanics [4, 3] and in micro–magnetics [8, 19].

Our first model describes phase transformations in shape–memory alloys as discussed in [15, 17, 18, 5]. Here $z : \Omega \to Z$ indicates either the microscopic distribution of the phases or a mesoscopic average of the microscopic distribution. In the first case we choose $Z = Z_p = \{e_1,\dots,e_p\} \subset \mathbb{R}^p$, where e_j denotes the j–th unit vector in $\mathbb{R}^p$ and in the second case we choose $Z = \mathrm{conv}\, Z_p$. In both cases the dissipation distance is given by a volume integral measuring the amount of volume which is transformed into another phase: $\mathcal{D}(z_0,z_1) = \int_\Omega \Delta(z_1(x) - z_0(x))\,\mathrm{d}x$, where $\Delta : \mathbb{R}^p \to [0,\infty[$ is convex and homogeneous of degree 1. This leads naturally to the basic space $X = \mathrm{L}^1(\Omega,\mathbb{R}^p)$ and $\mathcal{Z} = \{\, z \in X \mid z(x) \in Z \text{ a.e.}\,\}$.

Including in $\mathcal{E}$ an interfacial energy proportional to the area of the interfaces between regions of different phases provides a reduced energy $\mathcal{I}$ which is coercive in $Y = \mathrm{BV}(\Omega,\mathbb{R}^p)$, see [9]. For an existence result in the case without interfacial energy we refer to [17].

The second application describes the delamination of a body Ω which is glued together along n hypersurfaces Γ_j, $j = 1,\dots,n$. The internal state $z : \Gamma = \cup_1^n \Gamma_j \to [0,1]$ denotes the percentage of glue along Γ which remains in effect. The dissipation is given by a material constant $c_{\mathcal{D}}$ times the destroyed glue, i.e., $\mathcal{D}(z_0,z_1) = c_{\mathcal{D}} \int_\Gamma z_0(x) - z_1(x)\ \mathrm{d}a(x)$ for $z_1 \le z_0$ and $\mathcal{D}(z_0,z_1) = +\infty$ else. The basic underlying space is $\mathrm{L}^1(\Gamma)$ and now compactness arises via the trace operator $\mathrm{H}^1(\Omega) \to \mathrm{L}^2(\Gamma)$ which makes the reduced energy functional $\mathcal{I}$ weakly continuous.

The final application is devoted to the modeling of elasto–plasticity with finite strains. There the internal variable $z = (P,p)$ consists of the plastic transformation $P \in \mathrm{SL}(d)$ and hardening parameters $p \in \mathbb{R}^k$. Invariance under previous plastic deformations leads to dissipation metrics which are left–invariant, i.e.,

$\Delta((P,p),(\dot P,\dot p)) = \Delta((I,p),(P^{-1}\dot P,\dot p))$. This geometric nonlinearity clearly shows that we need general dissipation distances $\mathcal{D}$ avoiding any linear structure. In single–crystal plasticity Δ is piecewise linear in $P^{-1}\dot P \in \mathrm{sl}(d)$ which leads to Banach manifolds and the dissipation metric is then a left–invariant Finsler metric. For applications in this context see [1, 13, 12].

2 Abstract Setup of the Problem

We start with a Banach space X which is not assumed to be reflexive, since our applications in continuum mechanics (cf. Section 4) naturally lead to spaces of the form $\mathrm{L}^1(\Omega,\mathbb{R}^k)$. The first ingredient of the energetic formulation is the *dissipation distance* $\mathcal{D}: X\times X\to[0,\infty]$ satisfying the triangle inequality:

$$\mathcal{D}(z_1,z_3)\le \mathcal{D}(z_1,z_2)+\mathcal{D}(z_2,z_3)\quad \text{for all } z_1,z_2,z_3\in X.$$

We don't enforce symmetry, i.e., we allow for $\mathcal{D}(z_0,z_1)\neq\mathcal{D}(z_1,z_0)$ as in Section 4.2. We assume that there is a constant $c_{\mathcal{D}}>0$ such that $\mathcal{D}(z_0,z_1)\ge c_{\mathcal{D}}\|z_1-z_0\|_X$ for all $z_0,z_1\in X$. The latter condition is in fact the one which determines the appropriate function space X for a specific application. Moreover, $\mathcal{D}$ is assumed to be s–weakly lower semicontinuous. (We continue to use the abbreviation s–weak for "sequentially weak".) We call $\mathcal{D}(z_0,z_1)$ the dissipation distance from z_0 to z_1.

For a given curve $z:[0,T]\to X$ we define the total dissipation on $[s,t]$ via

$$\mathrm{Diss}_{\mathcal{D}}(z;[s,t]) = \sup\{\textstyle\sum_1^N \mathcal{D}(z(\tau_{j-1}),z(\tau_j)) \mid N\in\mathbb{N}, s{=}\tau_0{<}\tau_1{<}\cdots{<}\tau_N{=}t\,\}. \quad (2.1)$$

The second ingredient is the energy–storage functional $\mathcal{I}:[0,T]\times X\to[0,\infty]$, which is assumed to be bounded from below and then normalized such that it takes only nonnegative values. Here $t\in[0,T]$ plays the rôle of a (very slow) process time which changes the underlying system via changing loading conditions. For fixed time t, the map $\mathcal{I}(t,\cdot):X\to[0,\infty]$ is assumed to be s–weakly lower semicontinuous, i.e., $z_j\rightharpoonup z$ implies $\mathcal{I}(t,z)\le\liminf_{j\to\infty}\mathcal{I}(t,z_j)$. Moreover, we assume that for all z with $\mathcal{I}(t,z)<\infty$ the function $t\mapsto\mathcal{I}(t,z)$ is Lipschitz continuous with $|\partial_t\mathcal{I}(t,z)|\le C_{\mathcal{I}}$.

Definition 2.1 *A curve $z:[0,T]\to X$ is called a* ***solution*** *of the rate–independent model $(\mathcal{D},\mathcal{I})$, if* ***global stability (S)*** *and* ***energy inequality (E)*** *holds:*

(S) *For all $t\in[0,T]$ and all $\widehat z\in X$ we have $\mathcal{I}(t,z(t))\le I(t,\widehat z)+\mathcal{D}(z(t),\widehat z)$.*
(E) *For all t_0,t_1 with $0\le t_0<t_1\le T$ we have*
$$\mathcal{I}(t_1,z(t_1))+\mathrm{Diss}_{\mathcal{D}}(z;[t_0,t_1])\le\mathcal{I}(t_0,z(t_0))+\textstyle\int_{t_0}^{t_1}\partial_t\mathcal{I}(t,z(t))\,\mathrm{d}t.$$

The definition of solutions of (S)&(E) is such that it implies the two natural requirements for evolutionary problems, namely that *restrictions* and *concatenations* of solutions remain solutions. To be more precise, for any solution $z:[0,T]\to E$ and any subinterval $[s,t]\subset[0,T]$, the restriction $z|_{[s,t]}$ solves (S)&(E) with initial datum $z(s)$. Moreover, if $z_1:[0,t]\to E$ and $z_2:[t,T]\to E$ solve (S)&(E) on the respective intervals and if $z_1(t)=z_2(t)$, then the concatenation $z:[0,T]\to E$ solves

(S)&(E) as well. Under a few additional assumptions, it is shown in [16] that (S) and (E) together imply that, in fact, the energy inequality is in an equality, i.e., for $0 \le t_0 < t_1 \le T$ we have

$$\mathcal{I}(t_1, z(t_1)) + \mathrm{Diss}_{\mathcal{D}}(z; [t_0, t_1]) = \mathcal{I}(t_0, z(t_0)) + \int_{t_0}^{t_1} \partial_t \mathcal{I}(t, z(t)) \, \mathrm{d}t. \tag{2.2}$$

Rate–independency manifests itself by the fact that the problem has no intrinsic time scale. It is easy to show that z is a solution for $(\mathcal{D}, \mathcal{I})$ if and only if the reparametrized curve $\widetilde{z} : t \mapsto z(\alpha(t))$, with $\dot{\alpha} > 0$, is a solution for $(\mathcal{D}, \widetilde{\mathcal{I}})$, where $\widetilde{\mathcal{I}}(t, z) = \mathcal{I}(\alpha(t), z)$. In particular, the stability (S) is a static concept and the energy estimate (E) is rate–independent, since the dissipation defined via (2.1) is scale invariant like the length of a curve.

The major importance of the energetic formulation is that neither the given functionals $\mathcal{D}$ and $\mathcal{I}(t, \cdot)$ nor the solutions $z : [0, T] \to X$ need to be differentiable. In particular, applications in continuum mechanics often have low smoothness. Of course, under additional smoothness assumptions on $\mathcal{D}$ and $\mathcal{I}$ the weak energetic form (S)&(E) can be replaced by local formulations in the form of differential inclusions like (1.1) ([2, 20]) or variational inequalities. See [16] for a discussion of the implications between these different formulations.

3 Time Discretization and Existence

The major task is now to develop an existence theory for the initial value problem, i.e., to find a solution in the above sense which additionally satisfies $z(0) = z_0$. In general, we should not expect uniqueness without imposing further conditions like smoothness and uniform convexity of $\mathcal{I}(t, \cdot)$ and $\mathcal{D}$, see [16].

The stability condition (S) can be rephrased by defining the stable sets

$$\mathcal{S}(t) := \{\, z \in X \mid \mathcal{I}(t, z) \le \mathcal{I}(t, \widehat{z}) + \mathcal{D}(z, \widehat{z}) \text{ for all } \widehat{z} \in X \,\}.$$

Then, (S) simply means $z(t) \in \mathcal{S}(t)$ for all $t \in [0, T]$. The properties of the stable sets turn out to be crucial for deriving existence results.

One of the standard methods to obtain solutions of nonlinear evolution equations is that of approximation by time discretizations. To this end we choose discrete times $0 = t_0 < t_1 < \ldots < t_N = T$ and seek z_k which approximates the solution z at t_k, i.e., $z_k \approx z(t_k)$. Our energetic approach has the major advantage that the values z_k can be found incrementally via minimization problems. Since the methods of the calculus of variations are especially suited for applications in material modeling this will allow for a rich field of applications.

To motivate the following incremental variational problem consider the nonlinear parabolic problem $h(\partial_t u) = \mathrm{div}(A \, \mathrm{D}u) + g$, where we assume $h'(v) \ge 0$. The associated fully implicit incremental problem reads

$$h\big(\tfrac{1}{t_k - t_{k-1}}(u_k - u_{k-1})\big) = \mathrm{div}(A \, \mathrm{D}u_k) + g(t_k).$$

With $H(v) = \int_0^v h(w) \, \mathrm{d}w$ we see that u_k must be a minimizer of the functional

$$\mathcal{J}_k(u_{k-1}; \cdot) : u \mapsto \int_\Omega (t_k - t_{k-1}) H\big(\tfrac{1}{t_k - t_{k-1}}(u - u_{k-1})\big) + \tfrac{1}{2} \langle A \, \mathrm{D}u, \mathrm{D}u \rangle - g(t_k) u \, \mathrm{d}x.$$

In the simplest rate–independent case the function h is given by the signum function which implies $H(v) = |v|$. Hence, the length $t_k - t_{k-1}$ of the k–th time step disappears in the functional $\mathcal{J}_k$. In our more general setting the incremental problem takes the following form:

(IP) *For $z_0 \in X$ with $\mathcal{I}(0, z_0) < \infty$ find $z_1, \dots, z_N \in X$ such that*

$$z_k \in \operatorname{argmin}\{\, \mathcal{I}(t_k, z) + \mathcal{D}(z_{k-1}, z) \mid z \in X \,\} \quad \text{for } k = 1, \dots, N. \tag{3.1}$$

Here "argmin" denotes the set of all minimizers. Using the s–weak lower semi–continuity of $\mathcal{D}$ and $\mathcal{I}$ and the coercivity $\mathcal{I}(t, z) + \mathcal{D}(z_{k-1}, z) \ge c_{\mathcal{D}} \|z - z_{k-1}\|$ we obtain the following result.

Theorem 3.1 *The incremental problem (3.1) always has a solution. Each solution satisfies, for $k = 1, \dots, N$, the following properties:*

(i) z_k *is stable for time* t_k*, i.e.,* $z_k \in \mathcal{S}(t_k)$*;*

(ii) $\int_{[t_{k-1}, t_k]} \partial_s \mathcal{I}(s, z_k) \, \mathrm{d}s \le \mathcal{I}(t_k, z_k) - \mathcal{I}(t_{k-1}, z_{k-1}) + \mathcal{D}(z_{k-1}, z_k) \le \int_{[t_{k-1}, t_k]} \partial_s \mathcal{I}(s, z_{k-1}) \, \mathrm{d}s$*;*

(iii) $\mathcal{I}(t_k, z_k) + \sum_{j=1}^{k} \mathcal{D}(z_{j-1}, z_j) \le \mathcal{I}(0, z_0) + C_{\mathcal{I}} T$*;*

(iv) $\|z_k\| \le \|z_0\| + (\mathcal{I}(0, z_0) + C_{\mathcal{I}} T)/c_{\mathcal{D}}$*.*

The assertions (i) and (ii) are the best replacements for the conditions (S) and (E) in the time–continuous case.

For each discretization $P = \{0, t_1, \dots, t_{N-1}, T\}$ of the interval $[0, T]$ and each incremental solution $(z_k)_{k=1,\dots,N}$ of (IP) we define two piecewise constant functions which attain the values z_k at t_k and are constant in–between: Z^P is continuous from the left and $\widehat{Z}^P$ is continuous from the right. Summing the estimates (ii) in Theorem 3.1 over $k = j, \dots, m$ we find the following two–sided energy estimate.

Corollary 3.2 *Let P be any discretization of $[0, T]$ and $(z_k)_{k=0,\dots,N}$ a solution of (IP), then for $0 \le j < m \le N$ we have the two–sided energy inequality*

$$\begin{aligned} \mathcal{I}(t_j, Z^P(t_j)) + \textstyle\int_{t_j}^{t_m} \partial_s \mathcal{I}(s, Z^P(s)) \, \mathrm{d}s &\le \mathcal{I}(t_m, Z_P(t_m)) + \operatorname{Diss}_{\mathcal{D}}(Z_P, [t_j, t_m]) \\ &\le \mathcal{I}(t_j, Z^P(t_j)) + \textstyle\int_{t_j}^{t_m} \partial_s \mathcal{I}(s, \widehat{Z}_P(s)) \, \mathrm{d}s. \end{aligned}$$

The existence of solutions can now be established by taking a sequence $(P(l))_{l \in \mathbb{N}}$ of discretizations whose fineness $\delta^{(l)} = \max\{\, t_j^{(l)} - t_{j-1}^{(l)} \mid j = 1, \dots, N^{(l)} \,\}$ tends to 0. Moreover we assume that the sequence is hierarchical with $P(l) \subset P(l+1)$. The associated solutions of $(\mathrm{IP})^{(l)}$ define $z^{(l)} := Z^{P(l)}$. The construction of a solution of (S)&(E) consists now of two parts.

First we use the dissipation bound (iii) of Theorem 3.1 to obtain an a priori bound in $\mathrm{BV}([0, T], X)$:

$$c_{\mathcal{D}} \textstyle\int_{[0,T]} \|\mathrm{d}z^{(l)}\|_X \le \operatorname{Diss}_{\mathcal{D}}(z^{(l)}, [0, T]) \le \mathcal{I}(0, z_0) + C_{\mathcal{I}} T.$$

Then, Part (iv) in Theorem 3.1 and the following additional compactness condition (3.2) allows us to apply a selection principle in the spirit of Helly.

$$\begin{aligned}&\text{For all } R>0 \text{ and all } t\in[0,T] \text{ the sets}\\ &\mathcal{R}_R^t := \{\, z\in X \mid \mathcal{D}(z_0,z)\le R,\ \mathcal{I}(t,z)\le R\,\} \text{ are s–weakly compact.}\end{aligned} \tag{3.2}$$

Thus, we can extract a subsequence $(l_n)_{n\in\mathbb{N}}$ such that for all $t\in[0,T]$ the sequence $z^{(l_n)}(t), n\in\mathbb{N}$, converges weakly to a limit $z^{(\infty)}(t)$ with $\mathrm{Diss}_{\mathcal{D}}(z^{(\infty)},[0,T])\le \liminf_{n\to\infty}\mathrm{Diss}_{\mathcal{D}}(z^{(l_n)},[0,T])$.

Second we need to show that $z^{(\infty)}$ is a solution of (S)&(E). Using Corollary 3.2 it is easy to give conditions which guarantee that $z^{(\infty)}$ satisfies (E) for $t_0=0$ and $t_1=T$, and by (2.2) this is sufficient. To obtain stability of $z^{(\infty)}$ there are essentially two different ways. If additional compactness properties allow us to conclude that the convergence of $z^{(l_n)}(t)$ to $z^{(\infty)}$ also happens in the strong topology, then we are in the good case. Then it suffices to know that the set

$$\mathcal{S}_{[0,T]}=\{\,(t,z)\in[0,T]\times X \mid z\in\mathcal{S}(t)\,\}=\cup_{t\in[0,T]}(t,\mathcal{S}(t))$$

is closed in the strong topology. If strong convergence cannot be deduced, one needs to show that $\mathcal{S}_{[0,T]}$ is s–weakly closed. This property is quite hard to obtain, since even under nice convexity assumptions on $\mathcal{I}(t,\cdot)$ the sets $\mathcal{S}(t)$ are generally not convex.

The following theorem provides two alternative sets of assumptions which enables us to turn the above construction into a rigorous existence proof.

Theorem 3.3 *Let $\mathcal{D}$ and $\mathcal{I}$ be given as above and satisfy (3.2). If one of the conditions (a) or (b) is satisfied, then for each $z_0\in X$ with $\mathcal{I}(0,z_0)<\infty$ there is at least one solution $z\in \mathrm{BV}([0,T],X)$ of (S)&(E) with $z(0)=z_0$.*

(a) The set $\mathcal{S}_{[0,T]}$ is s–weakly closed and $z\mapsto\partial_t\mathcal{I}(t,z)$ is s–weakly continuous.

(b) The sets $\mathcal{R}_R^t$ in (3.2) are compact, the set $\mathcal{S}_{[0,T]}$ is closed, and $z\mapsto\partial_t\mathcal{I}(t,z)$ is continuous (all in the norm topology of X).

Simple nontrivial applications of this theorem with either condition (a) or (b) are as follows: Let $X=\mathrm{L}^1(\Omega)$ with $\Omega\subset\mathbb{R}^d$ bounded and choose the dissipation distance $\mathcal{D}(z_0,z_1)=c_{\mathcal{D}}\|z_1-z_0\|_X=c_{\mathcal{D}}\int_\Omega|z_1(x)-z_0(x)|\,\mathrm{d}x$. As a first case consider

$$\mathcal{I}_1(t,z)=\textstyle\int_\Omega \alpha(x)|z(x)|^\beta-g(t,x)z(x)\,\mathrm{d}x+\gamma,$$

where $\alpha(x)\ge\alpha_0>0$, $\beta>1$, and $g\in\mathrm{C}^1([0,T],\mathrm{L}^\infty(\Omega))$. The sets $\mathcal{R}_R^t$ are closed convex sets which lie in the intersection of an L^1–ball and an L^β–ball. Hence, we obtain the s–weak compactness condition (3.2). Yet, $\mathcal{R}_R^t$ is not strongly compact in $\mathrm{L}^1(\Omega)$. The stable sets for $\mathcal{I}_1$ are given by

$$\mathcal{S}_1(t)=\{\,z\in\mathrm{L}^1(\Omega)\mid |z(x)|^{\beta-2}z(x)\in[\tfrac{g(t,x)-c_{\mathcal{D}}}{\alpha(x)\beta},\tfrac{g(t,x)+c_{\mathcal{D}}}{\alpha(x)\beta}] \text{ for a.a. } x\in\Omega\,\},$$

which shows that they are s–weakly closed since they are convex and closed. Hence, condition (a) is satisfied.

As a second case consider the nonconvex energy functional

$$\mathcal{I}_2(t,z) = \int_\Omega \tfrac{1}{2}|\mathrm{D}z(x)|^2 + f(t,x,z(x))\,\mathrm{d}x \text{ for } z \in \mathrm{H}^1(\Omega) \quad \text{and } +\infty \text{ else},$$

where $f : [0,T] \times \Omega \times \mathbb{R} \to \mathbb{R}$ and $\partial_t f$ are continuous and bounded. Now, $\mathcal{R}_R^t$ is already compact in $\mathrm{L}^1(\Omega)$ since it is closed and contained in an H^1–ball. With these properties, it can be shown that condition (b) of Theorem 3.3 holds.

4 Applications in Continuum Mechanics

The flexibility of the energetic formulation allows for applications in continuum mechanics. We consider an elastic body which is given through a bounded domain $\Omega \subset \mathbb{R}^d$ with sufficiently smooth boundary. The elastic deformation is given by the mapping $\varphi : \Omega \to \mathbb{R}^d$, and the set of all admissible deformations is denoted by $\mathcal{F}$, which implements the displacement boundary conditions.

The variable $z \in Z$ includes all the internal variables like phase indicators, plastic or phase transformations, damage, polarization or magnetization. A function $z : \Omega \to Z$ gives the internal state of the material, and $\mathcal{Z}$ denotes the set of all admissible internal states. Note that Z may be a manifold with (nonsmooth) boundary. In plasticity we have $Z = \mathrm{SL}(d) \times \mathbb{R}^k$, in phase transformations we let $Z = \{\, z \in [0,1]^k \mid \sum_1^p z^{(j)} = 1 \,\}$, and in micro–magnetism z is the magnetization satisfying $|z(t,x)| = m_0 > 0$. Moreover, below we will also consider an application where z is not defined on all of Ω but at certain parts of the boundary.

Energy storage is characterized via the functional $\mathcal{E} : [0,T] \times \mathcal{F} \times \mathcal{Z} \to \mathbb{R}$ which is the sum of the total elastic energy and the potential energies due to exterior loadings (Gibbs' energy):

$$\mathcal{E}(t,\phi,z) = \int_\Omega W(x, \mathrm{D}\phi(x), z(x))\,\mathrm{d}x - \langle \ell_{\mathrm{ext}}(t), \phi \rangle.$$

Here $t \in [0,T]$ is the (quasi–static) process time which drives the system and the external loads are $\langle \ell_{\mathrm{ext}}(t), \phi \rangle = \int_\Omega f_{\mathrm{ext}}(t,x) \cdot \phi(x)\,\mathrm{d}x + \int_{\Gamma_{\mathrm{tract}}} g_{\mathrm{ext}}(t,x) \cdot \phi(x)\,\mathrm{d}a(x)$.

Dissipation is characterized via the metric $\Delta : \Omega \times \mathrm{T}Z \to [0,\infty]$ such that the curve $z : [0,T] \to \mathcal{Z}$ dissipates the energy

$$\mathrm{Diss}\,(z,[t_0,t_1]) = \int_{t_0}^{t_1} \int_\Omega \Delta(x, z(t,x), \dot z(t,x))\,\mathrm{d}x\,\mathrm{d}t \quad \text{ on } [t_1,t_2].$$

For each material point $x \in \Omega$, the infinitesimal metric $\Delta(x,\cdot,\cdot) : \mathrm{T}Z \to [0,\infty]$ defines a global distance function $D(x,\cdot,\cdot) : Z \times Z \to [0,\infty]$ and on $\mathcal{Z}$ we obtain the global dissipation distance

$$\begin{aligned} &\mathcal{D}(z_0,z_1) = \int_\Omega D(x, z_0(x), z_1(x))\,\mathrm{d}x \\ &= \inf\{\, \mathrm{Diss}\,(z,[t_0,t_1]) \mid z \in \mathrm{C}^{\mathrm{Lip}}([0,1],\mathcal{Z}), z(0) = z_0, z(1) = z_1 \,\}. \end{aligned}$$

The rate–independent problem for this material model is defined as in the above abstract part, but now the elastic deformation appears as an additional variable, which, however, does not generate any dissipation.

Definition 4.1 *A pair $(\phi, z) : [0,T] \to \mathcal{F} \times \mathcal{Z}$ is called a* ***solution*** *of the rate–independent problem associated with $\mathcal{D}$ and $\mathcal{E}$ if the* ***global stability (S)*** *and the* ***energy inequality (E)*** *hold:*

(S) *For all $t \in [0,T]$ and all $(\widehat{\phi}, \widehat{z}) \in \mathcal{F} \times \mathcal{Z}$ we have*
$$\mathcal{E}(t, \phi(t), z(t)) \le \mathcal{E}(t, \widehat{\phi}, \widehat{z}) + \mathcal{D}(z(t), \widehat{z}).$$
(E) *For all t_0, t_1 with $0 \le t_0 < t_1 \le T$ we have*
$$\mathcal{E}(t_1, \phi(t_1), z(t_1)) + \mathrm{Diss}_{\mathcal{D}}(z; [t_0, t_1]) \le \mathcal{E}(t_0, \phi(t_0), z(t_0)) + \int_{t_0}^{t_1} \partial_t \mathcal{E}(t, \phi(t), z(t)) \, \mathrm{d}t.$$

The connection with the above abstract theory is obtained by minimization with respect to the deformations $\phi \in \mathcal{F}$, since the stability condition implies that $\phi(t)$ must be a minimizer of $\mathcal{E}(t, \cdot, z(t))$. We define the associated $\mathcal{I}$ via

$$\mathcal{I}(t,z) = \inf\{\, \mathcal{E}(t, \varphi, z) \mid \varphi \in \mathcal{F} \,\} \text{ for } z \in \mathcal{Z} \quad \text{and } +\infty \text{ else.}$$

While this elimination is suitable for an abstract treatment, the practical approximation of solutions via the incremental approach is better done by keeping the deformation and eliminating the internal variable in each incremental step. In fact, in (IP) we now have to find

$$(\phi_k, z_k) \in \operatorname{argmin}\{\, \mathcal{E}(t_k, \widehat{\phi}, \widehat{z}) + \mathcal{D}(z_{k-1}, \widehat{z}) \mid (\widehat{\phi}, \widehat{z}) \in \mathcal{F} \times \mathcal{Z} \,\}. \tag{4.1}$$

In this minimization problem the internal variable occurs only locally under the integral over Ω and hence can be eliminated pointwise. Defining the local reduced constitutive functions

$$\begin{aligned} &\Psi^{\mathrm{red}}(z_{\mathrm{old}}; x, F) := \min\{\, W(x,F,z) + D(x, z_{\mathrm{old}}, z) \mid z \in Z \,\}, \\ &Z_{\mathrm{new}}(z_{\mathrm{old}}; x, F) \in \operatorname{argmin}\{\, W(x,F,z) + D(x, z_{\mathrm{old}}, z) \mid z \in Z \,\}, \end{aligned} \tag{4.2}$$

and the reduced functional $\mathcal{E}^{\mathrm{red}}(z_{\mathrm{old}}; t, \phi) = \int_\Omega \Psi^{\mathrm{red}}(z_{\mathrm{old}}; \mathrm{D}\phi) \, \mathrm{d}x - \langle \ell_{\mathrm{ext}}(t), \phi \rangle$ the solution of (4.1) is equivalent to finding $\phi \in \operatorname{argmin}\{\, \mathcal{E}^{\mathrm{red}}(z_{k-1}; t_k, \widehat{\phi}) \mid \widehat{\phi} \in \mathcal{F} \,\}$ and then letting $z_k = Z_{\mathrm{new}}(z_{k-1}; \mathrm{D}\phi_k)$. For more details we refer to [12].

4.1 Phase transformations in shape–memory alloys

We assume that, in each microscopic point y, an elastic material is free to choose one of p crystallographic phases and that the elastic energy density W is then given by $W_j(\mathrm{D}\phi)$. If the model is made on the mesoscopic level, then the internal variables are phase portions $z^{(j)} \in [0,1]$ for the j–th phase. We set $Z = \{\, z \in [0,1]^p \subset \mathbb{R}^p \mid \sum_1^p z^{(j)} = 1 \,\}$ and $X = \mathrm{L}^1(\Omega, \mathbb{R}^p)$. The material properties are described by a mixture function $W : \mathbb{R}^{d \times d} \times Z \to [0, \infty]$, see [11, 17, 5]. The dissipation can be shown to have the form $D(z_0, z_1) = \psi(z_1 - z_0)$ with $\psi(v) = \max\{\, \sigma_m \cdot v \mid m = 1, \dots, M \,\} \ge C_\psi |v|$, where $\sigma_m \in \mathbb{R}^p$ are thermodynamically conjugated threshold values.

So far we are unable to prove existence results for this model in its full generality. However, the case with only two phases ($p = 2$) has been treated in [17] under the

additional assumption that the elastic behavior is linear and both phases have the same elastic tensor. In that case, one sets $z = (\theta, 1-\theta)$ with $\theta \in [0,1]$. It can be shown that $\mathcal{I}$ is a quadratic functional in $\theta \in \mathrm{L}^1(\Omega,[0,1]) \subset \mathrm{L}^2(\Omega)$. It then follows that the compactness condition (3.2) holds and condition (a) in Theorem 3.3 can be verified using the H–measure to handle the weak convergence of the nonconvex terms.

A microscopic model is treated in [9]. There no phase mixtures are allowed, i.e., we assume $z \in Z_p := \{e_1, e_2, \dots, e_p\} \subset \mathbb{R}^p$, where e_j is the j–th unit vector. Thus, the functions $z \in \mathcal{Z}$ are like characteristic functions which indicate exactly one phase at each material point. The dissipation is assumed as above, but now the elastic energy contains an additional term measuring the surface area of the interfaces between the different regions:

$$\mathcal{E}(t,\phi,z) = \textstyle\int_\Omega W(\mathrm{D}\phi, z)\,\mathrm{d}x + \sigma \int_\Omega |\mathrm{D}z| - \langle \ell_{\mathrm{ext}}(t), \phi\rangle,$$

where σ is a positive constant and $\int_\Omega |\mathrm{D}z|$ is $\sqrt{2}$ times the area of all interfaces. Here $\mathcal{Z} = \{\, z : \Omega \to Z_p \mid \int_\Omega |\mathrm{D}z| < \infty \,\}$ and we set $\mathcal{E}(t,\phi,z) = +\infty$ for $z \notin \mathcal{Z}$.

Hence, after minimization with respect to ϕ we still have $\mathcal{I}(t,z) \ge \gamma + \sigma \int_\Omega |\mathrm{D}z|$. This term provides for $\mathcal{R}^t_R$ (cf. (3.2)) an a priori bound in $\mathrm{BV}(\Omega,\mathbb{R}^p)$ and hence we conclude compactness in $X = \mathrm{L}^1(\Omega,\mathbb{R}^p)$. Under the usual additional conditions for the elastic stored–energy densities W_j we obtain for each $z_0 \in \mathcal{Z}$ a solution (ϕ, z) with $\phi \in \mathrm{L}^\infty_{\mathrm{w}}(]0,T[\,, \mathrm{W}^{1,2}(\Omega,\mathbb{R}^d))$ and $z \in \mathrm{BV}([0,T], \mathrm{L}^1(\Omega,\mathbb{R}^p)) \cap \mathrm{L}^\infty_{\mathrm{w}*}(]0,T[\,, \mathrm{BV}(\Omega,\mathbb{R}^p))$ with $z(t) \in \mathcal{Z}$ for all $t \in [0,T]$, see [9].

4.2 A delamination problem

Here we give a simple model for rate–independent delamination and refer to [7] for a better model and the detailed analysis.

Consider a body $\Omega \subset \mathbb{R}^d$ which is given by an open, bounded, and path–connected domain. Assume that the interior of the closure of Ω differs from Ω by a finite set of sufficiently smooth hypersurfaces Γ_j, $j = 1, \dots, n$. This means that with $\Gamma := \bigcup_{j=1}^n \Gamma_j$ we have $\mathrm{int}(\mathrm{cl}(\Omega)) = \Omega \cup \Gamma$. We assume that the two sides of the body are glued together along these surfaces and that the glue is softer than the material itself. Upon loading, some parts of the glue may break and thus lose its effectiveness. The remaining fraction of the glue which is still effective is denoted by the internal state function $z : \Gamma \to [0,1]$.

We let $\mathcal{Z} = \{\, z : \Gamma \to [0,1] \mid z \text{ measurable} \,\} \subset X = \mathrm{L}^1(\Gamma)$. The dissipation distance $\mathcal{D}(z_0, z_1)$ is proportional to the amount of glue that is broken from state z_0 to state z_1:

$$\mathcal{D}(z_0,z_1) = c_{\mathcal{D}} \textstyle\int_\Gamma z_0(y) - z_1(y)\,\mathrm{d}a(y) \text{ for } z_0 \ge z_1 \quad \text{and} \quad +\infty \text{ else.}$$

Here we explicitly forbid the healing of the glue by setting $\mathcal{D}$ equal ∞, if $z_0 \not\ge z_1$.

The energy is given by the elastic energy in the body, the elastic energy in the glue, and the potential of the external loadings:

$$\mathcal{E}(t,\phi,z) = \textstyle\int_\Omega W(\mathrm{D}\phi)\,\mathrm{d}x + \int_\Gamma z(y) Q(y, [\![\phi]\!]_\Gamma(y))\,\mathrm{d}a(y) - \langle \ell_{\mathrm{ext}}(t), \phi\rangle,$$

where for $y \in \Gamma$ the vector $[\![\phi]\!]_\Gamma(y)$ denotes the jump of the deformation ϕ across the interface Γ and $Q(y,\cdot)$ is the potential for the elastic properties of the glue.

For simplicity we assume further that W provides linearized elasticity and Q is quadratic is well, then there is a unique minimizer $\phi = \Phi(t,z) \in \mathrm{H}^1(\Omega,\mathbb{R}^d)$ of $\mathcal{E}(t,\cdot,z)$. It can be shown that the mapping $\Phi(t,\cdot) : \mathcal{Z} \subset \mathrm{L}^1(\Gamma) \to \mathrm{H}^1(\Omega,\mathbb{R}^d)$ is compact, which implies that the functional $\mathcal{I}(t,\cdot) : \mathcal{Z} \to [0,\infty[$ is s–weakly continuous with respect to the L^1–topology on $\mathcal{Z}$. For the latter argument it is essential that z appears only linearly in the definition of $\mathcal{E}(t,\phi,z)$. Theorem (3.3) with condition (a) provides the existence of solutions.

4.3 Elasto–plasticity

The above theory can be applied to linearized elasto–plasticity, see [1, 13]. Here we want to report on recent results concerning elasto–plasticity with finite strain. However, for this application the abstract existence theory is not yet available.

Elasto–plasticity with finite strains is based on the multiplicative decomposition of the deformation gradient $F = \mathrm{D}\phi$ in the form $\mathrm{D}\phi = F_{\mathrm{elast}}P^{-1}$ where the plastic transformation P lies in the Lie group $\mathrm{SL}(d) = \{\, P \in \mathbb{R}^{d\times d} \mid \det P = 1 \,\}$. The internal variable has the form $z = (P,p) \in Z$ where $p \in \mathbb{R}^k$ denotes the hardening parameters. We refer to [1, 13, 12] for mechanical motivations and mathematical details. For simplicity, we mention here only the case without hardening where $z = P \in \mathrm{SL}(d) =: Z$ and refer to [6, 12] for more general cases.

The important point in finite–strain elasto–plasticity is that the dissipation distance must be invariant under previous plastic deformations, i.e., $D(QP_0,QP_1) = D(P_0,P_1)$ for all $Q \in \mathrm{SL}(d)$. Equivalently, the infinitesimal metric $\Delta : \mathrm{T}Z \to [0,\infty]$ is left–invariant, i.e., $\Delta(P,\dot P) = \Delta(I,P^{-1}\dot P)$. This implies that the dissipation distance is characterized by a norm $\Delta(I,\cdot)$ on $\mathrm{sl}(d) = \mathrm{T}_I\mathrm{SL}(d)$ and that $D(P_0,P_1)$ behaves logarithmically in $P_0^{-1}P_1$ which introduces strong geometric nonconvexities. So far, even the solution of the incremental problem (IP) is not understood completely. Even in simple cases one has to expect non–attainment in (IP), which leads to the formation of microstructure. The easiest way to see the problems is to study the reduced energy density Ψ^{red} in (4.2). If this density is not quasi–convex, then there are loadings such that (IP) has no solution and relaxation techniques have to be employed, cf. [14].

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