The Ginzburg–Landau equation in its role as a modulation equation*

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Abstract

The Ginzburg–Landau equation (GLe)

$$\partial_t A = \operatorname{div}\left[(I+\mathrm{i}S)\nabla A\right] + \rho\, A - \widehat{c}|A|^2 A,$$

with $A(t,x) \in \mathbb{C}$, $t \geq 0$, $x \in \mathbb{R}^d$, appears in many different contexts, e.g. nonlinear optics with dissipation or the theory of superconductivity. In addition, it plays an important role as modulation equation and it serves as a simple mathematical model for studying the transition from regular to turbulent behavior when, for $\delta > 0$ fixed, the dispersion parameters $(S, \operatorname{Im}\widehat{c}\,) = s(I, -\delta)$ with $s \gg 1$ are considered. The purpose of this work is to review the latter two aspects of the GLe.

In a variety of pattern–forming systems the modulations of the basic periodic pattern can be described by the solutions of the GLe. We discuss this for three classical hydrodynamical situations: the Rayleigh–Bénard convection, Poiseuille and the Taylor–Couette problem. Indeed, the GLe should be seen as a normal form or the lowest order expansion of a bifurcation equation in the context of a weakly unstable system when continuous spectrum moves over the imaginary axis upon changing an external parameter. In particular, this theory gives a natural and simple approach to the theory of sideband instabilites.

We explain the classical derivation of the GLe as a modulation equation of an original partial differential equation $\partial_t u = Lu + N(u)$ and give the abstract setting which allows for the application of the Ginzburg–Landau formalism which is based on the ansatz

$$u(t,x) \approx U_A(t,x) = \varepsilon A(\varepsilon^2 t, \varepsilon(x - c_g t))\mathrm{e}^{\mathrm{i}(\omega t + \langle k, x\rangle)}\Phi + \text{c.c.},$$

where ε^2 is the distance of the external parameter from its critical value. We then study the mathematical properties of the GLe which are relevant for the justification of the formalism. This includes a variety of special solution classes, a global semigroup theory in function spaces containing $\mathrm{L}^\infty(\mathbb{R}^d)$ as well as the construction of the global attractor.

Finally we review the results which prove that the solutions A of the GLe inserted into the above ansatz provide good approximations for solutions u of the original system. In particular, the direct implications to hydrodynamical problems are discussed.

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Contents

1 Introduction

The purpose of this work is to review recent developments in the theory of the Ginzburg–Landau equation (GLe) as a modulation equation. While the theory of modulation equations is a standard tool in physics and mechanics, its dissemination in mathematics is very restricted. The main goal here is to explain the physical and mechanical background of this theory in mathematical terms to enable researchers in the fields of dynamical systems and partial differential equations to familiarize with the main questions and results.

Cearly, we can neither give all the relevant results here nor is it possible to give all the mathematical details which are needed for the full understanding of the phenomena. In this introductory work we can only present the main ideas, motivations and techniques to display the general methodology and the simplest versions of the results associated to the field. Hopefully this survey is a stimulating guide to the recent literature, where more details and results can be found.

The mathematical results in this field are roughly divided into two areas, (i) the analysis of the GLe in itself and (ii) the relation between the solutions of the GLe and the original PDE. Since the number of mathematical papers on the GLe published per year exceeds 100, we have to restrict our view, in area (i), to developments which are related to the questions in area (ii). In particular, this relates to questions on the solution behavior on unbounded domains.

In Section 2 we give the general setup for the theory of modulation equations which was developed to describe modulations in space and time of a basic periodic pattern which appears in a system when certain control parameters become critical. If we consider the physical domain $\Omega = \mathbb{R}^d \times \Sigma$, where Σ is a bounded cross–section, with $x \in \mathbb{R}^d$ and $y \in \Sigma$, then the basic pattern takes the form

$$\mathbf{p}(t,x,y) = \mathbf{E}(t,x)\Phi_{0,k_0}(y) \quad \text{where} \quad \mathbf{E}(t,x) = \mathrm{e}^{\mathrm{i}(\omega t + \langle k_0, x\rangle)}. \tag{1.1}$$

Here ω and $k_0 \in \mathbb{R}^d$ denote the temporal frequency and the spatial wave vector of the basic pattern, respectively. A modulated pattern is a function of the form

$$u(t,x,y) = B(\tau,\xi)\mathbf{p}(t,x) + \text{c.c.} \tag{1.2}$$

where the amplitude function $B(\tau,\xi) \in \mathbb{C}$ depends on a slow time variable τ and a large spatial variable ξ. In our context we will have

$$\tau = \varepsilon^2 t \quad \text{and} \quad \xi = \varepsilon(x - c_g t),$$

where c_g is the group velocity of the basic pattern. The amplitude $|B(\tau,\xi)|$ gives the intensity of the basic pattern while phase changes of B give rise to changes of the pattern in the temporal frequency and in the spatial wave vector.

In the theory of modulation equations one starts with a partial differential equation

$$\partial_t u = L(\mu, \partial_x, \partial_y)u + N(\mu, \partial_x, \partial_y, u) \tag{1.3}$$

which is posed on $\Omega = \mathbb{R}^d \times \Sigma$ together with suitable boundary conditions on $\partial\Omega$. This systems is supposed to have $u \equiv 0$ as the trivial solution for all values of the small parameter μ. Moreover, $\mu = 0$ is taken to be the threshold of instability, that means that $u \equiv 0$ is asymptotically stable for $\mu < 0$ and unstable for $\mu > 0$. At criticality $\mu = 0$

the linear operator admits only the basic pattern $\mathbf{p}$ as a neutral mode which is neither decaying nor growing.

The aim of modulation theory is to understand the evolution of modulated patterns in the form (1.2). This is done by deriving a partial differential equation for the modulation function B. More precisely, one inserts $\mu = \rho\varepsilon^2$ and the multiple scaling ansatz

$$u(t,x,y) = U_A(t,x,y) + \mathcal{O}(\varepsilon^2) \quad \text{with } U_A(t,x) = \varepsilon A(\varepsilon^2 t, \varepsilon(x - c_g t))\mathbf{p}(t,x,y) + \text{c.c.} \tag{1.4}$$

in (1.3) and equates equal powers of the terms $\varepsilon^j \mathbf{E}^n$ to 0. The desired partial differential equation, called modulation equation, is then found as a solvability condition for the equation associated to $\varepsilon^3 \mathbf{E}$. Our interest lies in those systems for which the associated modulation equation is the GLe

$$\partial_\tau A = \operatorname{div}_\xi \Big[\Lambda_0 \nabla_\xi A\Big] + \rho\lambda_{0,1} A + \widehat{c}|A|^2 A. \tag{1.5}$$

Here $\lambda_{0,1}, \widehat{c} \in \mathbb{C}$, and $\Lambda_0 \in \mathbb{C}^{d\times d}$ is a symmetric matrix whose real part is positive definite. The precise assumptions of this theory and the formal calculations in the general case are given in Section 2 together with a first simple example. There we also discuss questions of different scalings and of symmetries of the GLe.

We will call the procedure described above the *Ginzburg–Landau formalism*, since it is a formal procedure which derives from the original problem (1.3), via the multiple scaling ansatz (1.4), the associated modulation equation. In our cases this is the GLe, and we refer to Subsection 2.6 for a few other modulation equations. The general philosophy of modulation theory can be found in the books [Man90, BJ*98] or in the review [NPL93].

If A is a solution of the GLe, the function U_A is called the Ginzburg–Landau approximation for the solution u of (1.3) which has the same initial data $u(0,\cdot) = U_A(0,\cdot)$. The major question in the justification of the GLe as a modulation equation is whether $U_A(t,\cdot)$ remains close to the true solution $u(t,\cdot)$ over a sufficiently long time scale. Certainly this time scale has to be of order $\mathcal{O}(1/\varepsilon^2)$ since only then the evolution of $A(\tau,\cdot)$ with $\tau = \varepsilon^2 t$ has any impact on $U_A(t,\cdot)$. This justification question will be treated in Section 6.

In Section 3 it is shown how the Ginzburg–Landau formalism can be applied to the three prototype problems in hydrodynamics, namely the Rayleigh–Bénard problem, the Couette–Taylor problem and the Poiseuille problem. In all three problems we have an open flow region with either one or two unbounded directions. At a critical Reynolds number the basic flow becomes unstable and experiments clearly show modulated periodic patterns. Historically these problems mark the origin of the GLe as a modulation equation. In fact, it was the need to explain large scale effects in these hydrodynamical problems which led to the development of the modulation theory in the late 60s and beginning 70s, see [NeW69, Seg69, DES71, NeW71] for the Rayleigh–Bénard problem and [StS71, HoS72, DHS74, StD78] for the Couette–Taylor and the Poiseuille problem.

A mathematical theory for strictly periodic patterns was developed in the late 50s and 60s by using the Liapunov–Schmid reduction, see e.g. [Kir66]. However, this theory could not explain the pattern selection which was observed in experiments, since many periodic patterns are stable with respect to perturbations of the same spatial period but unstable with respect to perturbations with slightly different spatial period. First explanations of this phenomenon of sideband instability were given in [Eck65, KiS69]. This led already to a linear theory of modulated patterns. In order to understand also the nonlinear evolution of modulated patterns the Ginzburg–Landau formalism was invented. As a by–product the theory of sideband instabilities obtained a simple form, see Subsections 3.4 and 4.1. More general large scale effects which are of importance in the Couette–Taylor problem are discussed in [ChI94, Sect. VII].

In Section 4 we study the GLe in its own right. The huge popularity of the GLe arises from the two facts that it is rather simple and explicit and that it appears in a many different applications, like for instance in superconductivity [GiL65, DGP92, CHO92, BBH94], in laser optics [Kap98, MHK98] and in reaction–diffusion systems [Hag82, Kir84, IoM92, Schn98a, Sche99]. In all these fields special solution classes as well as special properties of the GLe are of importance. There is a great distinction between the real GLe (rGL), where $\lambda_{0,1}, \widehat{c} \in \mathbb{R}$ and $\Lambda_0 = I$, and the general complex GLe (cGL). In Subsection 4.1 we shortly discuss the rotating plane waves $A(\tau,\xi) = r\,\mathrm{e}^{\mathrm{i}(\omega+\langle\kappa,\xi\rangle)}$ together with their stability. This gives the relation between the classical field of sideband instabilities and the modulation theory.

In the one-dimensional case the rGL has rather special nodal properties which can be used to study the solution behavior intensively. Moreover, for the GLe in one space dimension there are numerous results on standing, rotating and traveling pulses and fronts. For the two–dimensional GLe it is possible to construct rotating waves which reduce to the Ginzburg–Landau vortices in the case of rGL. At the end of Section 4 we develop a general semigroup theory for the GLe for space dimension $d = 1, 2, 3$ for bounded as well as unbounded domains. For the case $\mathrm{Re}\,\widehat{c} > 0$ we show that the semiflow on $\mathrm{L}^\infty(\mathbb{R}^d)$ is well defined (under the usual conditions on the dispersion parameters if $d = 3$).

Section 5 is independent of the theory of the GLe although the theory and methods were developed mainly for this equation. The aim here is to show that the semiflow on $\mathrm{L}^\infty(\mathbb{R}^d)$ has a global attractor when compactness and attractivity is taken with respect to a weighted norm. The reason for considering the GLe on unbounded domains is motivated by the modulational theory. On the full space it is easy to accommodate the multiple scaling ansatz (1.4) for any $\varepsilon > 0$, since we do not have to fulfill boundary conditions. If one wants to work on bounded domains $\Omega_\ell = (-\ell,\ell)^d \times \Sigma$ for the full problem (1.3) then the length ℓ has to be chosen at least of order $1/\varepsilon$ otherwise the modulations due to $A(\varepsilon^2 t, \varepsilon x)$ would not be effective in Ω_ℓ.

Using weighted norms we develop a theory which allows us to compare the global attractors $\mathcal{A}^{\Omega_n}$ for different domains Ω_n tending to the full space $\mathbb{R}^d$ with the attractor $\mathcal{A}^{\mathbb{R}^d}$ associated to the semiflow on $\mathrm{L}^\infty(\mathbb{R}^d)$. We are able to show upper semicontinuity to this limit but lower semicontinuity is false in general. If the limit $\Omega_n \to \mathbb{R}^d$ and the limit $\tau \to \infty$ in the definition of the attractor are done simultaneously we obtain a generalized limit attractor which actually coincides with $\mathcal{A}^{\mathbb{R}^d}$.

Section 6 treats the question of the mathematical justification of the Ginzburg–Landau formalism. This theory was developed only within the last 10 years starting from [CoE90, vHa91, KSM92]. Somewhat earlier a series of papers ([IoMD89, IoM91, IoM92, AfM95]) used the Kirchgässner reduction (cf. [Kir82]), also called spatial center–manifold reduction, to derive the steady one–dimensional Ginzburg–Landau equation for some hydrodynamical problems, see Subsection 6.1.

The following subsections treat the full time–dependent problem. By now it is not possible to prove that the full problem (1.3) can be reduced to the GLe (1.5) in the sense of dynamical systems. This would mean that there is an invariant manifold in the phase space Y of (1.3) and the reduced flow on this manifold is given by (1.5), perhaps after adding higher order corrections. The main difficulty in proving such a result arises from the fact that the continuous spectrum of the linearization at $u \equiv 0$ does not have a spectral gap which would allow us to separate the critical modes from the stable modes. Certain attempts to avoid this splitting are given in [Mie92, Schn95a, Mel98].

Thus, the best one can hope for is to prove approximation of the solutions $u(t,\cdot)$ by $U_A(t,\cdot)$ over a time interval $[0, \tau_1/\varepsilon^2]$ for $\tau_1 > 0$ fixed. Under suitable assumptions such an *approximation result* can be shown. Additionally it is possible to show that all small initial data u_0 will evolve such that after time $t = 1/\varepsilon^2$ the solution $u(1/\varepsilon^2, \cdot) = \mathcal{S}_{1/\varepsilon^2}(u_0)$ is a modulated pattern, i.e. there is a function A_0 such that $u(1/\varepsilon^2, \cdot) = U_{A_0}(\cdot)$. Thus after this transient time, the solution can be described well by a solution of the GLe according to the above approximation result. This fact is called the *attractivity of the set of modulated patterns*, [Eck93, Schn94c, Schn95a].

Iterating the above–mentioned approximation and attractivity property it is clear that the true solution $u(t,\cdot)$ can be approximated by pieces of Ginzburg–Landau approximations $U_{A_n}(t,\cdot)$, $t \in [n/\varepsilon^2, (n+1)/\varepsilon^2)$, where we have to allow for jumps at the times $t_n = n/\varepsilon^2$. A delicate analysis controls these jumps showing that they are in fact relatively small. Hence, we say that the true orbit is *shadowed by a pseudo–orbit* of the GLe. As such pseudo–orbits are attracted into a small neighborhood of the global attractor $\mathcal{A}_{\mathrm{GL}}^{\mathbb{R}^d}$ (with respect to the weighted norm), it can be shown that the global attractor $\mathcal{A}^{\mathbb{R}^d \times \Sigma}$ of the full problem (1.3) has a relatively small distance to the image of $\mathcal{A}_{\mathrm{GL}}^{\mathbb{R}^d}$ under the mapping $A \mapsto U_A$. For the one–dimensional Swift–Hohenberg equation on the interval $\Omega_\varepsilon = (-\ell/\varepsilon, \ell/\varepsilon)$ with periodic boundary conditions an even better convergence statement can be shown for the comparison of the flows on the inertial manifolds of (1.3) and (1.5), see Subsection 6.6.

The major problem with the present status of the justification of the Ginzburg–Landau formalism is that it only gives information over long but not infinite times. So it is not possible to use standard methods of dynamical–systems theory to transfer results from the 'reduced system' (the GLe (1.5)) to the full system (1.3). For instance, the existence of pulse or spiral waves in cGL does not give automatically the existence of the corresponding solutions in the full problem. A classical counter example is the existence of the downstream pulses in the two–dimensional Poiseuille flow, see the discussion at the end of Section 3.3. Another example of different long–time behavior is indicated in the discussion at the end of Subsection 6.6. Similarly the question of stability arises; for instance, if we consider the rotating plane waves, then it is clear that they correspond to space–time periodic patterns of the full problem. The stability of these waves in the GLe is easily investigated as shown in Section 4.1, however, there is no direct way to infer the same stability information for the associated pattern in the full problem. One has to repeat the analogous calculations for the spectral problem associated with the linearization of the full problem at this pattern, see [Mie95, Mi97b, Mi97c, Sca99] for such an approach. Similar problems arise, if we try to transfer dynamic phenomena like phase slips, diffusive stability or diffusive mixing from the GLe to the full problem, see [EGW95, Schn98b].

In this situation, the best we can do is to learn from the analysis of the GLe to be able to make good guesses for the full problem. Then we have to transfer the analytical methods used to study the GLe to the according methods for the full problem. Hopefully we are then able to prove the desired result for the full problem directly.

2 The Ginzburg–Landau formalism

In this section we describe the general formalism how to derive the modulation equation from a general PDE. We give precise assumptions on the linear part and discuss certain scaling and symmetry effects.

We consider general PDEs on unbounded domains Ω in the form $\Omega = \mathbb{R}^d \times \Sigma$, where $\Sigma \subset \mathbb{R}^n$ is a bounded domain with Lipschitz boundary. We write $z = (x, y)$ for spatial points in $\Omega = \mathbb{R}^d \times \Sigma$. The underlying PDE is written abstractly in the form

$$\partial_t u = L(\mu, \partial_x) u + N(\mu, \partial_x, u), \tag{2.1}$$

where $u : \Omega \to \mathbb{R}^m$ denotes the unknown function and μ is a real parameter. The linear operator $L(\mu, \partial_x)$ is a partial differential operator on Ω including boundary conditions on $\partial\Omega$. It involves the derivatives ∂_x as well as ∂_y, but we only display the former ones as x plays a distinguished role as unbounded variable. The nonlinear terms $N(\mu, \partial_x, u)$ are assumed to satisfy $N(\mu, \partial_x, \varepsilon\, u) = \mathcal{O}(\varepsilon^2)$ for sufficiently smooth $u : \Omega \to \mathbb{R}^m$.

2.1 Spectral assumptions

In such a situation $u(t, x) \equiv 0$ is the trivial homogeneous solution of (2.1) and we are interested in the onset of instability. Throughout we assume that the system has $u \equiv 0$ as an asymptotically stable state for $\mu < 0$ and $u \equiv 0$ is unstable for $\mu > 0$. The question that is answered by modulation theory is which patterns are seen when the parameter μ is close to the instability threshold $\mu = 0$.

As in finite-dimensional dynamical systems this is determined by the eigendirections associated to those eigenvalues passing the imaginary axis. In our situation of a PDE with $x \in \mathbb{R}^d$ the spectral problem is solved by using Fourier transform in the x-direction. We insert the ansatz $u(t, x, y) = \mathrm{e}^{\lambda t}\, \mathrm{e}^{\mathrm{i}\langle x, k\rangle} \Phi(y)$ into the linear part of (2.1) to obtain the spectral problem

$$\lambda \Phi = L(\mu, \mathrm{i}k)\Phi \tag{2.2}$$

which is now posed on the bounded cross-section Σ. Thus, for typical problems $L(\mu, \mathrm{i}k)$ has a discrete sprectrum $\operatorname{spec} L(\mu, k)$. Note that $\operatorname{spec} L(\mu, -k) = \overline{\operatorname{spec} L(\mu, k)}$ since $L(\mu, \partial_x)$ is a real operator. In fact, for the Ginzburg–Landau formalism we only need the eigenvalue family $\lambda(\mu, k)$ with largest real part to be isolated for $\mu \approx 0$. More precisely, the general situation leading to the GLe as a modulation equation can be characterized as follows.

Basic spectral assumptions
There exist $\delta, \gamma > 0$, $k_0 \in \mathbb{R}^d$ and eigenvalues $\lambda : (-\delta, \delta) \times B_{k_0}(\delta) \to \mathbb{C}$ such that for all $\mu \in (-\delta, \delta)$ the spectrum of $L(\mu, k)$ satisfies the following:

(1) For $k \notin B_{k_0}(\delta) \cup B_{-k_0}(\delta)$ we have $\operatorname{spec}(L(\mu, k)) \subset \{\operatorname{Re}\sigma \le \gamma\}$.

(2) For $k \in B_{k_0(\delta)}$ we have

$$\operatorname{spec} L(\mu, k) = \{\lambda(\mu, k)\} \cup S_{\mathrm{s}}(\mu, k) \quad \textit{with} \quad S_{\mathrm{s}}(\mu, k) \subset \{\operatorname{Re}\sigma \le \gamma\},$$

where $\lambda(\mu, k)$ is an algebraically simple eigenvalue with the expansion

$$\lambda(\mu, k) = \lambda_0(\mu) + \langle \lambda_1(\mu), k - k_0\rangle - \langle \Lambda(\mu)(k - k_0), k - k_0\rangle + \mathcal{O}(|k - k_0|^3). \tag{2.3}$$

(3) *We impose* $\operatorname{Re}\lambda_0(0) = 0$, $(\mathrm{d}/\mathrm{d}\mu)\operatorname{Re}\lambda_0(0) > 0$, $\operatorname{Re}\lambda_1(0) = 0$ *and* $\operatorname{Re}\Lambda(0)$ *is a positive definite matrix.*

Here $S_{\mathrm{s}}(\mu, k)$ constitutes the stable part of the spectrum of $L(\mu, \mathrm{i}k)$. The conditions on $\lambda(\mu, k)$ guarantee $\operatorname{Re}\lambda(0, k) \le 0$ with equality only for $k = k_0$. Moreover, $\operatorname{Re}\lambda(\mu, k_0)$ has the same sign as μ, which exactly corresponds to our definition of the threshold of instability. Further on we use the abbreviations

$$\lambda_{0,1} = \frac{\mathrm{d}}{\mathrm{d}\mu}\lambda_0(0) \in \mathbb{C}, \quad c_g = -\mathrm{i}\lambda_1(0) \in \mathbb{R}^d \quad \text{and} \quad \Lambda_0 = \Lambda(0),$$

where c_g is called the group velocity of the basic pattern.

These assumptions lead to three different generic cases which either lead to the real or the complex GLe. There might be other cases, but we only treat these ones as they associate to generic cases without degeneracies.

Case 1: $k_0 \neq 0$ **and** $\lambda(\mu, k) \in \mathbb{R}$**.**

This situation typically appears if the system has an additional reflection symmetry $x \to -x$ and hence $\overline{\lambda(\mu, k)} = \lambda(\mu, -k) = \lambda(\mu, k)$. The diffusion part of GLe will be real and further symmetry properties determine whether the nonlinear term is real or complex.

Case 2: $k_0 \neq 0$ **and** $\lambda(0, k_0) = \mathrm{i}\omega$ **with** $\omega > 0$**.**

This is the general situation for which we obtain the anisotropic cGL.

Case 3: $k_0 = 0$ **and** $\lambda(0, k_0) = \mathrm{i}\omega$ **with** $\omega > 0$**.**

In this situation system (2.1) has a spatially homogeneous Hopf bifurcation when μ passes the origin. If (2.1) is rotationally invariant we have additionally $\lambda(\mu, k) = \lambda(\mu, |k|)$ and we obtain an isotropic cGL.

We will refer to these cases regularly in the subsequent discussions.

2.2 Formal derivation of modulation equations

We now want to show how the GLe

$$\partial_\tau A = \operatorname{div}_\xi(\Lambda_0 \nabla_\xi A) + \rho\lambda_{0,1} A + \widehat{c}|A|^2 A \tag{2.4}$$

is obtained from the original system (2.1) by a multiple scale expansion. At this stage the procedure is formal since it assumes that the original problem has solutions which are modulations of the basic periodic pattern over sufficiently long time intervals. The mathematical justification of this assumption is studied in Section 6.

To simplify the notations we omit the cross-sectional variable y, however, the approach is fully general.

2.2.1 The linear part

As $\lambda(\mu,k)$ is an eigenvalue of $L(\mu,\mathrm{i}k)$ with the eigenfunction $\Phi_{\mu,k}$ we see that $v(t,x)=\mathrm{e}^{\lambda(\mu,k)t+\mathrm{i}\langle k,x\rangle}\Phi_{\mu,k}$ solves

$$\partial_t v = L(\mu,\partial_x)v, \tag{2.5}$$

which is the linearization of (2.1). For $(\mu,k)\approx(0,k_0)$ we use the parameter scaling

$$\mu=\varepsilon^2\rho \quad \text{and} \quad k=k_0+\varepsilon\kappa \quad \text{with} \quad 0<\varepsilon\ll 1.$$

In most cases one uses only $\rho=1$, however, for consistency and for a possible treatment of subcritical bifurcations ($\mu<0$) we have introduced the fixed constant $\rho\in[-1,1]$.

Via (2.3) this scaling leads to $\lambda(\varepsilon^2\rho,k_0{+}\varepsilon\kappa)=\mathrm{i}\omega+\varepsilon^2\rho\lambda_{0,1}-\varepsilon\mathrm{i}\langle c_g,\kappa\rangle-\varepsilon^2\langle\Lambda_0\kappa,\kappa\rangle+\mathcal{O}(\varepsilon^3)$. The above solution v of (2.5) can be written as

$$v(t,x)=\mathrm{e}^{(\rho\lambda_{0,1}-\langle\Lambda_0\kappa,\kappa\rangle)\varepsilon^2 t+\mathrm{i}\langle\varepsilon(x-c_g t),\kappa\rangle+\mathcal{O}(\varepsilon^3)}\,\mathrm{e}^{\mathrm{i}(\omega t+\langle x,k_0\rangle)}\Phi_{\varepsilon^2\rho,k_0+\varepsilon\kappa}. \tag{2.6}$$

This solution is a product whose second factor is approximately the basic periodic pattern $\mathbf{p}(t,x)=\mathrm{e}^{\mathrm{i}(\omega t+\langle x,k_0\rangle)}\Phi_{0,k_0}$ underlying the whole modulation theory. For $\mu=0$ this pattern and its complex conjugate are the only patterns which are not damped. The first factor in (2.6) depends only on the slow time variable and the large space variable

$$\tau=\varepsilon^2 \quad \text{and} \quad \xi=\varepsilon(x{-}c_g t). \tag{2.7}$$

In fact, if we define the linear parabolic equation

$$\partial_\tau B=\mathrm{div}_\xi\Big(\Lambda_0\nabla_\xi B\Big)+\rho\lambda_{0,1}B \tag{2.8}$$

for $B(\tau,\xi)\in\mathbb{C}$, then it has the general solution

$$B(\tau,\xi)=\int_{\kappa\in\mathbb{R}^d}\mathrm{e}^{\mathrm{i}\langle\kappa,\xi\rangle}\mathrm{e}^{(\rho\lambda_{0,1}-\langle\Lambda_0\kappa,\kappa\rangle)t}f(\kappa)\mathrm{d}m(\kappa)$$

where m is a complex-valued Radon measure on $\mathbb{R}^d$. Thus, we expect that the modulation ansatz

$$v(t,x)\approx U_B(t,x)=B(\varepsilon^2 t,\varepsilon(x{-}c_g t))\mathbf{p}(t,x)+\text{c.c.}, \text{ with } \mathbf{p}(t,x)=\mathrm{e}^{\mathrm{i}(\omega t+\langle x,k_0\rangle)}\Phi_{0,k_0}, \tag{2.9}$$

approximates solutions of (2.5) whenever B satisfies (2.8). We say that the amplitude function B modulates the basic periodic pattern $\mathbf{p}$. In particular, $|B|$ modulates the amplitude of $\mathbf{p}$ and $\arg B$ modulates its phase.

Unfortunately, the ansatz (2.9) is not a sufficiently good approximation in the case when $\Phi_{\mu,k}$ really depends on k. In this situation it is necessary to control things in Fourier space. We introduce the Fourier transform $\mathcal{F}_x[u]$ and $\mathcal{F}_\xi[B]$, then we find

$$\mathcal{F}_x[U_B](t,k_0+\varepsilon\kappa)=\mathrm{e}^{\mathrm{i}(\omega-\langle c_g,\varepsilon\kappa\rangle)t}\varepsilon^{-d}(\mathcal{F}_\xi[B(\varepsilon^2 t,\cdot)])(\kappa)\Phi_{0,k_0}+\text{c.c.}$$

However, using the operator-valued exponential mapping $\mathrm{e}^{tL(\mu,k)}$ we find the general solution of (2.5) with $v(0,x)=v_0(x)$ in the form $\mathcal{F}[v(t,\cdot)](k)=\mathrm{e}^{tL(\mu,k)}\mathcal{F}[v_0](k)$. In particular, by our spectral assumptions we have

$$\mathcal{F}[v(t,\cdot)](k)=\mathrm{e}^{\lambda(\mu,k)t}\widehat{B}(k)\Phi_{\mu,k}+\mathcal{O}(\mathrm{e}^{-\gamma t}),$$

for $t \to \infty$. If we compare with the above formula, we see that the expansion of the eigenvalue is matched properly for solutions B of (2.8), however, the difference $\Phi_{\mu,k} - \Phi_{0,k_0}$ may introduce errors of order $|k - k_0| = \varepsilon|\kappa|$. Thus, for exact results the approximation V_B in (2.9) should be replaced by

$$v(t,x) \approx V_B(t,x) = \mathcal{F}_k^{-1}\Big[\chi_{\pm k_0}(k)\{\mathcal{F}_\xi[B(\varepsilon^2 t,\cdot)](\tfrac{k-k_0}{\varepsilon})\Phi_{\mu,k}+\text{c.c.}\}\Big](x - c_g t), \tag{2.10}$$

where $\chi_{\pm k_0} : \mathbb{R}^d \to [0,1]$ is a smooth cut–off function which is equal 1 for $|k - k_0|$ or $|k + k_0| \le \delta/2$ and 0 for $|k - k_0|$ and $|k + k_0| \ge \delta$. Here B is again a solution of (2.8). We refer to (6.19) and (6.20) for the exact mathematical implementation of these Fourier transform methods.

2.2.2 The nonlinear part

Since the structure of the nonlinear terms in GLe (2.4) is clear from the outset, we can derive its coefficient without using modulations. It suffices to compare constant solutions of the GLe with periodic patterns in the original problem (2.1). The basic periodic pattern is $\mathbf{p}(t,x) = \mathbf{E}(t,x)\Phi_{0,k_0}$ where we use $\mathbf{E} = \mathbf{E}(t,x) = \mathrm{e}^{\mathrm{i}(\omega t + \langle k_0, x\rangle)}$. The nonlinear terms generate higher harmonics $\mathbf{E}^j$ of the basic (t,x)–pattern $\mathbf{E}$. If the amplitude of $\mathbf{E}$ is of order ε, then the nonlinearity produces amplitudes of $\mathbf{E}^j$ which are of order $\varepsilon^{|j|}$ for $j \ne 0$ and of order ε^2 for $\mathbf{E}^0 \equiv 1$. This leads to the ansatz

$$u(t,\cdot) = \varepsilon[A(\tau)\mathbf{E}\Phi_{0,k_0}+\text{c.c.}] + \varepsilon^2 \sum_{j=-1}^{1} \mathbf{E}^{2j}\Psi_{2j}(\tau) + \varepsilon^3 \sum_{j=-2}^{1} \mathbf{E}^{2j+1}\Psi_{2j+1}(\tau) + \mathcal{O}(\varepsilon^4), \tag{2.11}$$

where the only x–dependence occurs through $\mathbf{E} = \mathbf{E}(t,x)$. The unknowns A and Ψ_j just depend on the slow time τ and on the cross-sectional variable y in the latter case. Recall that we do not account for slow spatial modulations in this restricted ansatz.

To insert (2.11) we need to expand $L(\mu, \partial_x)$ and $N(\mu, \partial_x, u)$ as follows:

$$\begin{aligned} L(\varepsilon^2\rho, \partial_x)[\mathbf{E}^j\Psi] &= \mathbf{E}^j L(\varepsilon^2\rho, \mathrm{i}jk_0)\Psi, \\ N(0, \partial_x, \varepsilon \textstyle\sum_{-J}^{J} \mathbf{E}^j\Psi_j) &= \varepsilon^2 \textstyle\sum_{j,l=-J}^{J} \mathbf{E}^{j+l}\mathcal{B}_{(j,l)}[\Psi_j, \Psi_l] \\ &\quad + \varepsilon^3 \textstyle\sum_{j,l,n=-J}^{J} \mathbf{E}^{j+l+n}\mathcal{T}_{(j,l,n)}[\Psi_j, \Psi_l, \Psi_n] + \mathcal{O}(\varepsilon^4). \end{aligned}$$

Here the bilinear operator $\mathcal{B}$ and the trilinear operator $\mathcal{T}$ are symmetric with respect to interchanging the order of the arguments together with the associated indices. Substituting ansatz (2.11) into (2.1) and equating the coefficients of the terms $\varepsilon^j\mathbf{E}^l$ of lowest order to 0 we find

$$\begin{aligned} \varepsilon^1\mathbf{E}^1: &\quad \mathrm{i}\omega A\Phi_{0,k_0} &&= \mathrm{i}\omega A\Phi_{0,k_0}, \\ \varepsilon^2\mathbf{E}^2: &\quad 2\mathrm{i}\omega\Psi_2 &&= L(0, 2\mathrm{i}k_0)\Psi_2 + A^2\mathcal{B}_{(1,1)}[\Phi_{0,k_0}, \Phi_{0,k_0}], \\ \varepsilon^2\mathbf{E}^0: &\quad 0\Psi_0 &&= L(0,0)\Psi_0 + |A|^2 2\mathcal{B}_{(1,-1)}[\Phi_{0,k_0}, \overline{\Phi}_{0,k_0}], \\ \varepsilon^3\mathbf{E}^1: &\quad \partial_\tau A\Phi_{0,k_0} + \mathrm{i}\omega\Psi_1 &&= \rho\lambda_{0,1}A\Phi_{0,k_0} + L(0, \mathrm{i}k_0)\Psi_1 \\ & && \quad + A\, 2\mathcal{B}_{(1,0)}[\Phi_{0,k_0}, \Psi_0] + \overline{A}\, 2\mathcal{B}_{(2,-1)}[\Psi_2, \overline{\Phi}_{0,k_0}] \\ & && \quad + |A|^2 A\, 3\mathcal{T}_{(1,1,-1)}[\Phi_{0,k_0}, \Phi_{0,k_0}, \overline{\Phi}_{0,k_0}]. \end{aligned}$$

We have omitted the equations for $\varepsilon^j\mathbf{E}^{-l}$ as they are just the complex conjugates. From our spectral assumptions we know that Ψ_0 and Ψ_2 can be calculated from the second and third relation as $\Psi_0(\tau, y) = |A(\tau)|^2 W_0(y)$ and $\Psi_2(\tau, y) = A(\tau)^2 W_2(y)$ with

$$W_0 = 2L(0,0)^{-1}\mathcal{B}_{(1,-1)}[\Phi_{0,k_0}, \overline{\Phi}_{0,k_0}], \quad W_2 = (2\mathrm{i}\omega I - L(0, 2\mathrm{i}k_0))^{-1}\mathcal{B}_{(1,1)}[\Phi_{0,k_0}, \Phi_{0,k_0}].$$

Thus, the fourth equation takes the form

$$[L(0,\mathrm{i}k_0)-\mathrm{i}\omega I]\Psi_1 = (\partial_\tau A - \rho\lambda_{0,1}A)\Phi_{0,k_0} - |A|^2 A\Psi_* \tag{2.12}$$

with $\Psi_* = 2\mathcal{B}_{(1,0)}[\Phi_{0,k_0}, W_0] + 2\mathcal{B}_{(2,-1)}[W_2, \overline{\Phi}_{0,k_0}] + 3\mathcal{T}_{(1,1,-1)}[\Phi_{0,k_0}, \Phi_{0,k_0}, \overline{\Phi}_{0,k_0}]$. As $\mathrm{i}\omega$ is an algebraically simple eigenvalue of $L(0,\mathrm{i}k_0)$ this equation is solvable if and only if the right–hand side is orthogonal to the eigenvector Φ^*_{0,k_0} of the adjoint of $L(0,\mathrm{i}k_0)$. We choose Φ^*_{0,k_0} such that $\langle \Phi_{0,k_0}, \Phi^*_{0,k_0}\rangle = 1$ where $\langle\cdot,\cdot\rangle$ is the duality pairing. Because of our spectral assumption in Section 2.1 the solvability condition for (2.12) is

$$0 = \partial_\tau A - \rho\lambda_{0,1}A - \widehat{c}|A|^2 A \quad \text{with} \quad \widehat{c} = \langle \Psi_*, \Phi^*\rangle \in \mathbb{C}. \tag{2.13}$$

For generic problems the cubic coefficient $\widehat{c}$ is nonzero. Joining the linear and the nonlinear result we obtain the GLe (2.4). Of course, (2.13) is the lowest order expansion of the reduced ODE on a two–dimensional center manifold for (2.1) when the solutions are restricted to the spatial periodicity, see e.g. [IoA92]. This equation is often called Landau equation and takes the form

$$\frac{\mathrm{d}}{\mathrm{d}t}a = g(\mu,a) = \lambda(\mu,k_0)a + \widetilde{c}(\mu)|a|^2 a + \widehat{g}(\mu)|a|^4 a + \mathcal{O}(|a|^7), \quad a\in\mathbb{C}. \tag{2.14}$$

2.3 Transformations and scalings

The linear term in cGL can be considered as a parameter for most applications. On the one hand the real part of $a = \rho\lambda_{0,1}$ can be chosen via the bifurcation parameter $\mu = \varepsilon^2\rho$. On the other hand the imaginary part of a can be adjusted via replacing $A(\tau,\xi)$ by $\mathrm{e}^{-\mathrm{i}\widetilde{\omega}\tau}\widetilde{A}(\tau,\xi)$. Then, $\widetilde{A}$ solves cGL with the linear term $a = \lambda_{0,1}\rho - \mathrm{i}\widetilde{\omega}$. From our spectral assumption we have $\operatorname{Re}\lambda_{0,1} > 0$, and thus by choosing $\rho, \widetilde{\omega} \in \mathbb{R}$ suitably we can generate every $a \in \mathbb{C}$. Below we will restrict a to be real, namely equal to ρ.

The general form of GLe we derived so far is $\partial_\tau A = \operatorname{div}_\xi(\Lambda^0\nabla_\xi A) + \lambda_{0,1}\rho A + \widehat{c}|A|^2 A$ where the matrix $\Lambda^0 \in \mathbb{C}^{d\times d}$ is symmetric and has a real part which is positive definite. Setting

$$\widetilde{\tau} = r\tau, \quad \widetilde{\xi} = K\xi \quad \text{and} \quad \widetilde{A} = s\mathrm{e}^{\mathrm{i}\widetilde{\omega}\tau}A \tag{2.15}$$

with $r,s>0$, $\widetilde{\omega}\in\mathbb{R}$ and an invertible $K \in \mathbb{R}^{d\times d}$ we find the equivalent form

$$\partial_{\widetilde{\tau}}\widetilde{A} = \frac{1}{r}\operatorname{div}_{\widetilde{\xi}}(K^{\mathrm{t}}\Lambda^0 K\nabla_{\widetilde{\xi}}\widetilde{A}) + \frac{\lambda_{0,1}-\mathrm{i}\widetilde{\omega}}{r}\widetilde{A} + \frac{\widehat{c}}{rs^2}|\widetilde{A}|^2\widetilde{A}.$$

With $\Lambda^0 = S_1 + \mathrm{i}S_2$, $S_j \in \mathbb{R}^{d\times d}$ we choose $K = \sqrt{r}S_1^{-1/2}Q$ with $Q^{\mathrm{t}}Q = I$, then $1/r\, K^{\mathrm{t}}\Lambda^0 K = I + \mathrm{i}Q^{\mathrm{t}}S_1^{-1/2}S_2S_1^{-1/2}Q$ and a suitable choice of Q allows us to diagonalize the imaginary part. Next, we choose $\widetilde{\omega} = \rho\operatorname{Im}\lambda_{0,1}$ and $r = \operatorname{Re}\lambda_{0,1}$. For s we either take $s = \sqrt{|\widehat{c}|/r}$ or, if $\operatorname{Re}\widehat{c} < 0$, we can also take $s = \sqrt{-\operatorname{Re}\widehat{c}/r}$. Then, after dropping the tildes we arrive at

$$\partial_\tau A = \sum_{j=1}^{d}(1+\mathrm{i}\alpha_j)\partial^2_{\xi_j}A + \rho A + \widehat{d}(\beta)|A|^2 A \tag{2.16}$$

where either $\widehat{d}(\beta) = \mathrm{e}^{\mathrm{i}\beta}$ or $\widehat{d}(\beta) = -(1+\mathrm{i}\beta)$, respectively. We could also reduce to the case $\rho \in \{-1,0,1\}$ by choosing $r = |\rho|\lambda_{0,1}$, however, it is sometimes better to keep ρ as a parameter which may change sign.

The parameters $\alpha_1,\ldots,\alpha_d$ are called linear dispersion parameters, and β in $\widehat{d}(\beta)=-(1+\mathrm{i}\beta)$ is the nonlinear dispersion parameter. GLe is called isotropic if all α_j are the same. Taking the complex conjugate of the equation we can interchange the signs of all these dispersion parameters simultaneously.

If all α_j have a large modulus and have the same sign, e.g. $\alpha_j\gg 1$ and if $|\operatorname{Im}\widehat{c}\,|\gg|\operatorname{Re}\widehat{c}\,|$, then (2.16) can be considered as a perturbation of the nonlinear Schrödinger equation $\partial_\tau A=\mathrm{i}\Delta A\pm\mathrm{i}|A|^2A$. By rescaling accordingly we obtain

$$\partial_\tau A=\mathrm{i}\left[\sum_{j=1}^{d}\Big(1-\frac{\mathrm{i}}{\alpha_j}\Big)\partial_{\xi_j}^2A\right]+\rho A-\Big(\frac{1}{|\beta|}+\mathrm{i}\operatorname{sign}\beta\Big)|A|^2A. \tag{2.17}$$

For $1/\alpha_j,\rho,1/\beta\to 0$ we obtain the nonlinear Schrödinger equation; it is the focussing case if $\operatorname{sign}\beta=-1$ and the defocussing case for $\operatorname{sign}\beta=+1$. In the global existence theory in Section 5 this difference will be important.

Steady solutions or rotating waves of the form $A(\tau,\xi)=\mathrm{e}^{\mathrm{i}\widehat{\omega}\tau}\widetilde{A}(\xi)$ solve an elliptic problem. In the isotropic case $\alpha=\alpha_j$ the equation is $(1+\mathrm{i}\alpha)\Delta\widetilde{A}+(\rho-\mathrm{i}\widehat{\omega})\widetilde{A}+\widehat{d}(\beta)|A|^2=0$ which transforms via an appropriate scaling into

$$\Delta A-(1+\mathrm{i}\theta_1)^2A+(2+\mathrm{i}\theta_2)(1+\mathrm{i}\theta_2)|A|^2A=0 \tag{2.18}$$

as long as $\widehat{d}(\beta),\rho-\mathrm{i}\widehat{\omega}\notin(1+\mathrm{i}\alpha)[0,\infty)$. This form allows for the study of pulse solutions, i.e. solutions which decay exponentially, namely like $\mathrm{e}^{-|\xi|}$. For instance, for $d=1$ and $\theta_1=\theta_2$ there is the explicit pulse $A(\xi)=(\cosh\xi)^{-(1+\mathrm{i}\theta_1)}$, see Section 4.3.1 for more details. For $\theta_1=\theta_2=0$ we obtain the real GLe (rGL), and for $d=2$ there exist the steady Ginzburg–Landau vortices of degree $n\in\mathbb{Z}$. They have the form $A(x)=r_n(|x|)(x_1+\mathrm{i}x_2)^n/|x|^n$ with $r_n:[0,\infty)\to[0,\infty)$ satisfying $r_n(s)\sim s^n$ for $s\to 0$ and $r_n(s)\to 1/\sqrt{2}$ for $s\to\infty$, see Section 4.2.1.

2.4 Symmetries of the problem

Here we treat the question how symmetries of the full problem (2.1) reduce to symmetries for the associated GLe, where symmetries of GLe are related to restrictions on the coefficients Λ_0, $\lambda_{0,1}$ and $\widehat{c}$. Before doing this we recall the general symmetries of GLe, namely invariance under time and space translations (τ and ξ) as well as the phase invariance $A\mapsto\mathrm{e}^{\mathrm{i}\phi}A$, $\phi\in\mathbb{S}^1$. The first two arise from the original problem which is also invariant under time and space translations (t and x). However, the phase invariance has to be understood as a normal form symmetry; we refer to [IoMD89, Mel98] for a discussion. For a usage of the symmetries of cGL in the study of bifurcations and chaos on finite domains we refer to [AsL98] and the references therein.

We now return to the original question about further symmteries and first treat the case $k_0\neq 0$. Depending on the symmetries of the original problem the derived GLe also has symmetries. Assume that the original problem has a reflection symmetry $\mathcal{R}:(x,u)\mapsto(R_1x,R_2u)$ with $R_1^2=I_{\mathbb{R}^d}$ and $R_2^2=I_{\mathbb{R}^m}$. Reflecting the basic periodic pattern $\mathbf{p}(t,x)$ gives the reflected pattern $\widetilde{\mathbf{p}}(t,x)=R_2\mathbf{p}(t,R_1x)=\mathrm{e}^{\mathrm{i}(\omega t+\langle k_0,R_1x\rangle)}R_2\Phi_{0,k_0}$. We have either $\widetilde{\mathbf{p}}=\pm\mathbf{p}$ or $\widetilde{\mathbf{p}}=\pm\overline{\mathbf{p}}$. This follows since our assumption imply that, up to complex multiples $a\in\mathbb{C}$, there are only two basic patterns; and since $\mathcal{R}$ is a reflection, we have $a=\pm1$. Another way to see this is to use $\mathcal{R}L(\mu,\partial_x)=L(\mu,\partial_x)\mathcal{R}$ which gives

$$L(\mu,\mathrm{i}k)=R_2^{-1}L(\mu,\mathrm{i}R_1^{\mathrm{t}}k)R_2\quad\text{and thus}\quad\lambda(\mu,k)=\lambda(\mu,R_1^{\mathrm{t}}k). \tag{2.19}$$

From $\widetilde{\mathbf{p}} = \pm\overline{\mathbf{p}}$ we conclude $\omega = 0$, $R_1^{\mathrm{t}} k_0 = -k_0$ and $R_2\Phi_{0,k_0} = \pm\overline{\Phi}_{0,k_0}$. With the general fact $\lambda(\mu,-k) = \overline{\lambda(\mu,k)}$, (2.3) and (2.19) we obtain

$$\lambda_0(\mu) = \overline{\lambda_0(\mu)}, \quad \lambda_1(\mu) = -R_1\overline{\lambda_1(\mu)}, \quad \Lambda_0 = R_1\overline{\Lambda}_0 R_1^{\mathrm{t}}.$$

This implies $\lambda_{0,1} \in \mathbb{R}$ and $c_g = R_1 c_g$. Thus, GLe is symmetric with respect to the reflection $(\xi, A) \mapsto (R_1\xi, \pm\overline{A})$ and the coefficients Λ_0, $\lambda_{0,1}$ and $\widehat{c}$ satisfy

$$R_1\Lambda_0 R_1^{\mathrm{t}} = \overline{\Lambda}_0, \quad \lambda_{0,1} \in \mathbb{R}, \quad \widehat{c} \in \mathbb{R}.$$

From this we cannot conclude that Λ_0 is also a real matrix except if it is scalar. However, generically the present situation associated to the above Case 1 with $\lambda(\mu,k) \in \mathbb{R}$ for $|k - k_0| \leq \delta$, and then we know $\Lambda_0 \in \mathbb{R}^{d\times d}$.

The case $\widetilde{\mathbf{p}} = \pm\mathbf{p}$ implies $R_1^{\mathrm{t}} k_0 = k_0$ and $R_2\Phi_{0,k_0} = \pm\Phi_{0,k_0}$, and similar to the above we obtain $R_1 c_g = c_g$ and $R_1\Lambda_0 R_1^{\mathrm{t}} = \Lambda_0$. GLe is now invariant under reflection $(\xi, A) \mapsto (R_1\xi, \pm A)$ which has no consequences for the coefficients $\lambda_{0,1}, \widehat{c} \in \mathbb{C}$. This is the generic situation described in Case 2 above.

The Case 3 corresponds to $k_0 = 0$ where the basic periodic pattern $\mathbf{p}$ does not depend on x. Since $\mathbf{p}$ depends on t ($\omega \neq 0$), any reflection gives $\widetilde{\mathbf{p}} = \pm\mathbf{p}$. We obtain the same conclusions as for Case 2. However, in this situation the set of reflections is no longer restricted by $R_1^{\mathrm{t}} k_0 = k_0$ and it is possible to consider all reflections in the space $\mathbb{R}^d$. Then we have an isotropic situation and obtain $\Lambda_0 = bI_{\mathbb{R}^d}$ with $b \in \mathbb{C}$.

2.5 A simple example

To illustrate the above method we study a scalar problem on $\mathbb{R}^2$, namely

$$\partial_t u = -\Delta^2 u - 2\partial_1^2 u - u + \langle a, \nabla u\rangle + \langle b, \nabla\Delta u\rangle + \mu u + \gamma_1 u^3 + \langle\Gamma, \nabla u\rangle u^2 + \gamma_2 u|\nabla u|^2. \quad (2.20)$$

Here $u(t, x_1, x_2) \in \mathbb{R}$ and $\partial_1 = \partial/\partial x_1$, moreover $\gamma_1, \gamma_2 \in \mathbb{R}$ and $a, b, \Gamma \in \mathbb{R}^2$ are fixed coefficients while μ is the small parameter.

The operator $L(\mu, \mathrm{i}k)$ is simply a scalar and thus it coincides with the eigenvalue

$$\lambda(\mu,k) = -|k|^4 + 2k_1^2 - 1 + \mathrm{i}[\langle a,k\rangle - |k|^2\langle b,k\rangle] + \mu.$$

We find $k_0 = (1,0)^{\mathrm{t}}$ and the expansion

$$\begin{aligned}&\lambda(\varepsilon^2\rho, k_0 + \varepsilon\kappa) = \mathrm{i}\omega - \varepsilon\mathrm{i}\langle c_g, \kappa\rangle + \varepsilon^2\Big(\rho - \langle\Lambda_0\kappa,\kappa\rangle\Big) + \mathcal{O}(\varepsilon^3)\\ &\text{with } \omega = a_1 - b_1,\ c_g = b + (2b_1, 0)^{\mathrm{t}} - a,\ \Lambda_0 = \begin{pmatrix} 4-6\mathrm{i}b_1 & -2\mathrm{i}b_2\\ -2\mathrm{i}b_2 & 2-2b_1\end{pmatrix}.\end{aligned}$$

Since the problem is scalar and the nonlinearity is cubic the coefficient $\widehat{c} \in \mathbb{C}$ can be calculated simply by inserting the periodic pattern $\mathbf{p}(t,x) = \mathbf{E}(t,x) = \mathrm{e}^{\mathrm{i}(\omega t + x_1)}$ into the nonlinearity and identifying the term associated to $\mathbf{E}^j$ with $j = 1$. This gives the complex coefficient of the cubic term

$$\widehat{c} = 3\gamma_1 + \mathrm{i}\Gamma_1 + \gamma_2.$$

The last two formulae show that we can generate a rich class of GLes.

We are particularly interested in versions which have a reflection symmetry with respect to either $R^{(1)} : (x_1, x_2) \mapsto (-x_1, x_2)$ or $R^{(2)} : (x_1, x_2) \mapsto (x_1, -x_2)$. In the first case the coefficients have to satisfy $a_1 = b_1 = \Gamma_1 = 0$. This implies $\omega = 0$ and $\widehat{c} \in \mathbb{R}$. Moroever, the group velocity c_g is orthogonal to the wave vector of the periodic pattern, namely $c_g = (0, b_2 - a_2)^{\mathrm{t}}$. The diffusion matrix Λ_0 may have purely imaginary off–diagonal coefficients. Using the Ginzburg–Landau formalism we find approximate solutions of (2.20) in the form

$$u(t,x) \approx U_A(t,x) = \varepsilon A(\varepsilon^2 t, \varepsilon(x_1, x_2 - (b_2 - a_2)t))\mathrm{e}^{\mathrm{i}x_1} + \mathrm{c.c.},$$

where A solves the associated GLe.

In the case with reflection $R^{(2)}$ the coefficients must satisfy $a_2 = b_2 = \Gamma_2 = 0$. As a consequence $\omega = a_1 - b_1$ may be different from 0 and the group velocity is parallel to e_1, namely $c_g = (3b_1 - a_1, 0)^{\mathrm{t}}$. The diffusion matrix is diagonal but with nonreal entries. The approximative solutions take the form

$$u(t,x) \approx U_A(t,x) = \varepsilon A(\varepsilon^2 t, \varepsilon(x_1 - (3b_1 - a_1)t, x_2))\mathrm{e}^{\mathrm{i}((a_1 - b_1)t + x_1)} + \mathrm{c.c.},$$

where A solves the corresponding GLe.

2.6 A few other amplitude equations

The theory of modulation equations is much more general than presented in this work. In many different contexts it is possible to employ a multiple scaling ansatz in the form given above. Equating equal powers to 0 one finds an equation for the desired amplitude function. This applies also to cases where several spectral bands become unstable. For each band we obtain an amplitude equation, and they are coupled nonlinearly, see e.g. [MaV92, DoR97] and the references therein. Typically the coupling of these equations becomes nonlocal in ξ due to the different group velocities associated to the different spectral bands. The question of the mathematical justification of such systems is treated in [Schn97].

In hyperbolic systems, where the whole spectrum lies on the imaginary axis, the nonlinear Schrödinger equation

$$\partial_\tau A = \mathrm{i}[\Delta_\xi A + \rho A + \gamma |A|^2 A] \tag{2.21}$$

appears as a modulation equation ($\rho, \gamma \in \mathbb{R}$). The most important application in this case is the theory of water waves where modulation equations are used to study the nonlinear interaction of wave packages, see for instance [HaO72, ZaK86, CSS92]. Mathematical justifications in the sense of Section 6 for such kind of problems are given in [Kal89, KSM92, Schn98c, Schn98d].

We return to dissipative problems where the associated modulation equations are parabolic. Typical applications occur in hydrodynamics (with viscosity) or in chemical reactions. A first variant of the GLe occurs in the case when the linear theory is exactly as above, but the coefficient $\widehat{c}$ of the cubic term vanishes. Such a situation occurs typically in multi–parameter systems when the additional parameter is chosen appropriately. In the Landau equation (2.14) the cubic coefficient satisfies

$$\widetilde{c}(\mu) = \widehat{c} + \mu \widehat{d} + \mathcal{O}(\mu^2) \quad \text{for } \mu \to 0.$$

We are now interested in the case $\widehat{c} = 0$ and $\widehat{d} \neq 0$. With $\mu = \rho\varepsilon^2$ and the scaling $a(t) = \varepsilon^{1/2}A(\varepsilon^4 t)$ the Landau equation (2.14) produces the scaling limit $(\mathrm{d}/\mathrm{d}\tau)A = \rho\lambda_{0,1}A + \rho\widehat{d}|A|^2A + \widehat{g}(0)|A|^4\widetilde{A}$. If now modulationsa are also taken into account, then the linear part is the same as in the standard case by our sprectral assumptions. The modulation ansatz takes the form $u(t,x) = \varepsilon^{1/2}A(\varepsilon^2 t, \varepsilon(x-c_g t))\mathbf{E}(t,x)\Phi_{0,k_0}$ + c.c. and leads to the so–called **generalized GLe**

$$\begin{aligned}\partial_\tau A &= \mathrm{div}_\xi(\Lambda_0\nabla_\xi A) + \rho\lambda_{0,1}A + \rho\widehat{d}\,|A|^2A \\ &\quad + \widehat{g}(0)|A|^4A + \langle V_1, \nabla_\xi A\rangle|A|^2 + \langle V_2, \nabla_\xi\overline{A}\rangle A^2,\end{aligned} \tag{2.22}$$

where $V_1, V_2 \in \mathbb{C}^d$. This equation is sometimes also called the derivative GLe; in the case $V_1 = V_2 = 0$ it is also called the cubic–quintic GLe.

An analysis of this equation is given mainly for the spatially one–dimensional case, see e.g. [BeN85, EcI89, DoE91, vSH92, DTH92, Kap98, KaS98, She98]. The mathematical justification of the generalized cGL is established in [Schn96, She97].

In many physical models there is an additional conservation law (e.g. conservation of mass) which implies that the operator $L(\mu,0)$ has a an eigenvalue 0, which implies that for $L(\mu,k)$ there is an eigenvalue family $\lambda^{(0)}(\mu,k)$. If this family is real and has the form $\lambda^{(0)}(\mu,k) = \alpha(\mu)|k|^2 - \beta(\mu)|k|^4 + \mathcal{O}(|k|^6)$ where $\beta(0) > 0$, $\alpha(0) = 0$ and $\alpha'(0) > 0$, then the ansatz $u(t,x) = \varepsilon^r v(\varepsilon^4 t, \varepsilon x)\Phi_{0,0}$ and $\mu = \rho\varepsilon^2$ yields a modulation equation of Cahn–Hilliard or Kuramoto–Sivashinsky type:

$$\partial_\tau v = -\beta(0)\Delta_\xi^2 v - \rho\alpha'(0)\Delta_\xi v + f(v, \nabla_\xi v, \nabla_\xi^2 v),$$

where f is homogeneous with $f(\varepsilon^r v, \varepsilon^{r+1}V, \varepsilon^{r+2}W) = \varepsilon^{r+4}f(v,V,W)$ which determines r. In the Cahn–Hilliard equation we have $f = -\Delta(v^3)$ and $r = 1$; and in the Kuramoto–Sivashinsky equation we have $f = |\nabla v|^2$ which gives $r = 2$. We refer to [Man90, vHa95, BJ*98] and the references therein for derivations of these amplitude equations. The question of justification is treated in [Schn99c] for the Kolmogorov flow problem in an infinite strip.

For rotationally symmetric problems with $x \in \mathbb{R}^2$ the eigenvalue curve $\lambda(\mu,k)$ is necessarily rotationally symmetric, i.e. we have $\lambda(\mu,k) = \widetilde{\lambda}(\mu,|k|)$. If $k_0 \neq 0$ then $\mathrm{Re}\,\lambda(\mu,k) = 0$ along a whole circle. Assume $k_0 = (r,0)^{\mathrm{t}}$ and $\lambda(\mu,k) \in \mathbb{R}$, then it is natural to expand $k = k_0 + (\varepsilon\kappa_1, \varepsilon^{1/2}\kappa_2)$ and we obtain

$$\lambda(\rho\varepsilon^2, k_0 + (\varepsilon\kappa_1, \varepsilon^{1/2}\kappa_2)) = \varepsilon^2\Big(\lambda_{0,1}\rho - \Lambda^0(2r\kappa_1 + \kappa_2^2)^2\Big) + \mathcal{O}(\varepsilon^3),$$

with $\Lambda^0 > 0$. The parabola $\kappa_1 = -\kappa_2^2/(2r)$ approximates the critical circle $\{\, k : |k| = r \,\}$ in the point k_0. The associated two–dimensional modulation equation is called the Newell–Whitehead equation ([NeW69]) and reads

$$\partial_\tau A = \Lambda^0(2r\partial_{\xi_1} - \mathrm{i}\partial_{\xi_2}^2)^2 A + \lambda_{0,1}\rho A - \widehat{c}|A|^2A. \tag{2.23}$$

An analysis for the approximation properties of this formal equation is given in [Schn95b].

In hydrodynamical problems with two unbounded directions, like the Rayleigh–Bénard problem or the Poiseuille problem, there always appear additional modes due to the mean flow. To lowest order this additional equation couples an order parameter $\psi \in \mathbb{R}$ (associated to the mean pressure) with the equation for the amplitude parameter $A \in \mathbb{C}$. This additional equation is typically elliptic, since the pressure in the Navier–Stokes equation does not appear with a time derivative.

In [Ber94, DeP94] the following system is derived for the Rayleigh–Bénard problem between parallel plates:

$$\begin{aligned}\partial_\tau A &= (\partial_{\xi_1} - \mathrm{i}\partial_{\xi_2}^2)^2 A + \rho A - |A|^2 A + \mathrm{i} A\, \partial_{\xi_2}\psi, \\ 0 &= \partial_{\xi_2}^4 \psi + \gamma_1 \partial_{\xi_1}\partial_{\xi_2}|A|^2 + \gamma_2 \partial_{\xi_2} \operatorname{Im}(\overline{A}\, \partial_{\xi_2}^2 A).\end{aligned}$$

In Section 3.3 we repeat the arguments in [HoS72, DHS74] for the perturbations of the Poiseuille flow close to the threshold of instability. We obtain the coupled system

$$\begin{aligned}\partial_\tau A &= c_1 \partial_{\xi_1}^2 A + c_2 \partial_{\xi_2}^2 A + \rho c_3 A + c_4 |A|^2 A + c_5 A\, \partial_{\xi_1}\psi, \\ 0 &= \Delta_\xi \psi + \gamma \partial_{\xi_1}|A|^2,\end{aligned}$$

with $c_j \in \mathbb{C}$ and $\gamma \in \mathbb{R}$, where $\operatorname{Re} c_l > 0$ for $l = 1, \ldots, 4$. See (3.5) for the exact values of these constants. The mathematical justification of this system was given in [Bol96].

3 Applications in hydrodynamics

Historically the GLe was first derived in the context of hydrodynamics where the modulated periodic patterns can be seen in lots of experiments and in nature. Famous examples are cloud streets, water waves and the Taylor vortices of a flow between rotating concentric cylinders. First papers in this contexts were published around 1970, see for instance [NeW69, Seg69, DES71, NeW71, HoS72, DHS74]. For this reason it is suggested in [Man90] to use the name "Hocking–Stewartson equation" (after [HoS72]) rather than "Ginzburg–Landau equation". In fact, the latter name is not derived from the present context of mudulational theory but from its usage in the theory of superconductivity, see [GiL65, DGP92, CHO92]. The connection between superconductivity and the modulation theory arises from the fact that in both cases the rGL plays a central role.

In the following three subsections we present the basic cases in hydrodynamics where the Ginzburg–Landau theory is applicable. There are many other cases where the method of amplitude equations gives useful physical insight (e.g. in reaction–diffusion problems), however, due to the limited space we restrict ourselves. The fourth subsection introduces the questions of sideband instabilities which were one of the major questions which stimulated the development of the theory of modulation equations.

3.1 The Rayleigh–Bénard problem

We consider a two-dimensional fluid layer of an incompressible viscous fluid between two rigid plates which are kept at two fixed temperatures. After nondimensionalizing the velocity $u = (u_1, u_2)$, the pressure p and the temperature deviation θ from the linear profile between the two walls, we find the system

$$\left.\begin{aligned}\partial_t u + (u\cdot\nabla)u + \nabla p - R\theta\tbinom{0}{1} - \Delta u &= 0, \\ \operatorname{div} u &= 0, \\ P(\partial_t\theta + (u\cdot\nabla)\theta) - Ru_2 - \Delta\theta &= 0,\end{aligned}\right\} \quad \text{for } t \geq 0, (x,y) \in \mathbb{R}\times(0,\pi), \qquad (3.1)$$

together with the stress-free boundary conditions $\partial_y u_1 = u_2 = \theta = 0$ for $y \in \{0, \pi\}$, see [NeW69, DES71, Mie92, Ber94, Mi97b]. These boundary conditions are used only for mathematical simplicity, the more physical conditions $u_1 = u_2 = \theta = 0$ can be done similarly, however the calculations can no longer be done analytically, see [DeP94].

Here the parameter P is called the Prandtl number and R^2 is called the Rayleigh number. The latter parameter is considered as bifurcation parameter as it is proportional to the temperature difference at the two boundaries whereas P is a material parameter.

For the analysis we insert the ansatz

$$\begin{pmatrix} u_1(t,x,y) \\ u_2(t,x,y) \\ \theta(t,x,y) \\ p(t,x,y) \end{pmatrix} = \mathrm{e}^{\lambda_n t + \mathrm{i}kx} \begin{pmatrix} \alpha_1 \cos(ny) \\ \alpha_2 \sin(ny) \\ \alpha_3 \sin(ny) \\ \alpha_4 \cos(ny) \end{pmatrix}$$

with $\alpha_j \in \mathbb{C}$ into the linearization at $(u, \theta) = 0$. We obtain the relation

$$R^2 k^2 - (\lambda_n + n^2 + k^2)(P\lambda_n + n^2 + k^2)(n^2 + k^2) = 0$$

from which we derive that the critical parameter is $R = R_0 = 3\sqrt{3}/2$, that is for $R \leq R_0$ we have $\lambda_n(R, k) \leq 0$. The critical eigenvalue occurs for $n = 1$ with wavenumber $k_0 = 1/\sqrt{2}$, it is real and has the expansion

$$\lambda(\rho\varepsilon^2, k_0 + \varepsilon\kappa) = \varepsilon^2 \Big[\frac{4R_0\rho}{9(P+1)} - \frac{4}{P+1}\kappa^2\Big] + \mathcal{O}(\varepsilon^3),$$

where $R = R_0 + \rho\varepsilon^2$. The associated eigenvector Φ_{0,k_0} has the form

$$\Phi_{0,k_0}(y) = (\mathrm{i}\sqrt{2}\cos y, \sin y, \sqrt{3}\sin y, -3\cos y)^{\mathrm{t}}.$$

Yet, there is another critical mode occuring for $n = 0$ and $k = 0$, which derives from the fact that the Navier-Stokes equation is invariant under the addition of a constant to the pressure, or equivalently, from the freedom of superimposing a mean flow. In the case of one unbounded direction this degeneracy can be projected out by using the fact that the mass flux

$$Q(t, x) = \int_{y=0}^{\pi} u_1(t, x, y)\mathrm{d}x$$

is independent of x which follows easily from $\operatorname{div} u = 0$ and the boundary conditions.

Thus fixing $Q(t, x) \equiv 0$ in addition, the formal derivation of the Ginzburg–Landau formalism is valid and we obtain the following rGL as amplitude equation

$$\partial_\tau A = \frac{4}{P+1}\partial_\xi^2 A + \frac{2\rho}{\sqrt{3}(P+1)}A - \frac{P^2}{8(P+1)}|A|^2 A. \tag{3.2}$$

With Φ_{0,k_0} from above the approximations of the solutions to (3.1) read

$$\begin{pmatrix} u_1(t,x,y) \\ u_2(t,x,y) \\ \theta(t,x,y) \\ p(t,x,y) \end{pmatrix} = \varepsilon\, A(\varepsilon^2 t, \varepsilon x)\mathrm{e}^{\mathrm{i}x/\sqrt{2}} \begin{pmatrix} \mathrm{i}\sqrt{2}\cos y \\ \sin y \\ \sqrt{3}\sin y \\ -3\cos y \end{pmatrix} + \text{c.c.} + \mathcal{O}(\varepsilon^2)$$

where A solves (3.2) and ε is defined via $R = R_0 + \rho\varepsilon^2$.

The first mathematically rigorous justification theory for the Rayliegh–Bénard problem is provided in [Schn94b].

3.2 The Couette–Taylor problem

For this problem the Navier–Stokes equation is considered between two infinitely long concentric cylinders which are rotating with different speed.

Following [ChI94, Chap. II.1.2] we denote by r_{in}, r_{out}, ω_{in} and ω_{out} the radii and rotational velocities of the inner and outer cylinder and by ν the kinematical viscosity. The problem can be made nondimensional by introducing three similarity parameters:

$$\widetilde{\omega} = \omega_{\text{out}}/\omega_{\text{in}}, \quad \eta = r_{\text{in}}/r_{\text{out}} \in (0,1), \quad \mathcal{R} = r_{\text{in}}\omega_{\text{in}}(r_{\text{out}}-r_{\text{in}})/\nu.$$

The Navier-Stokes equations for the velocity $v \in \mathbb{R}^3$ take the form

$$\left.\begin{array}{rcl} \partial_t v + \mathcal{R}(v\cdot\nabla)v + \nabla p - \Delta v &=& 0, \\ \operatorname{div} v &=& 0, \end{array}\right\} \quad \text{in } \Omega_\eta = \mathbb{R}\times\Sigma_\eta, \tag{3.3}$$

where $\Sigma_\eta = \{\, y \in \mathbb{R}^2 \,:\, r_\eta < |y| < 1 \,\}$ with $r_\eta = \eta/(1-\eta)$ is the cross–section in form of an annulus. The boundary conditions are

$$v(x, r_\eta\cos\phi, r_\eta\sin\phi) = (0, -\sin\phi, \cos\phi)^{\text{t}} \text{ and } v(x, \cos\phi, \sin\phi) = \widetilde{\omega}/\eta(0, -\sin\phi, \cos\phi)^{\text{t}}$$

for all $\phi \in [0, 2\pi)$.

The basic Couette flow V_{Cou} is such that fluid particles stay on circles concentric and orthogonal to the axis $y = 0$. It has the explicit expression

$$V_{\text{Cou}}(x, r\cos\phi, r\sin\phi) = \begin{pmatrix} 0 \\ -\widehat{v}(r)\sin\phi \\ \widehat{v}(r)\cos\phi \end{pmatrix} \quad \text{with } \widehat{v}(r) = \widehat{a}r + \widehat{b}/r,$$

where $\widehat{a}, \widehat{b} \in \mathbb{R}$ are chosen such that $\widehat{v}(r_\eta) = 1$ and $\widehat{v}(1) = \widetilde{\omega}/\eta$.

Like in the Rayleigh–Bénard problem above we also impose that the mean flux through the cross–sections is 0, i.e. $\int_{\Sigma_\eta} v_1(t,x,y)\,\mathrm{d}y \equiv 0$. The Ginzburg–Landau formalism is now applied to the perturbations $u = v - V_{\text{Cou}}$ of the basic Couette flow. We will obtain a one–dimensional GLe, since the problem has only one unbounded direction. As the problem has the additional reflection symmetry $x \mapsto -x$ we know by Section 2.4 that if the Ginzburg–Landau formalism applies with $k_0 \neq 0$ that it leads to a real GLe.

The parameters η and the rotation ratio $\widetilde{\omega} \in \mathbb{R}$ are usually fixed and the Reynolds number $\mathcal{R}$ is used as the bifurcation parameter. Numerical computations (cf. [ChI94]) show that our assumptions on the leading unstable eigenvalue are always fulfilled if $\widetilde{\omega} > \widehat{\omega}(\eta)$ for some $\widehat{\omega}(\eta) < 0$. The critical Reynolds number is then denoted by $\mathcal{R}_0 = \mathcal{R}_0(\eta)$ and we have $k_0 > 0$. The associated eigenvalue is simple, and hence it is real. Moreover, the associated eigenvectors $\Phi_{\mu,k}$ must be rotationally invariant.

Thus, for $\mathcal{R} = \mathcal{R}_0 + \rho\varepsilon^2$ the GLe reads

$$\partial_\tau A = \gamma_1\partial_\xi^2 A + \gamma_2\rho A - \gamma_3|A|^2 A,$$

where the real coefficients γ_j are all positive, the first calculation of these coefficients aappears to be in [Kir66] using the Liapunov–Schmidt reduction as in Section 2.2.2. We refer to [IoMD89] and [ChI94, Chap. VII] for the first rigorous derivation of the steady part of the GLe in the context of the Couette–Taylor problem.

Since the eigenvector $\Phi_{\mu,k}$ is rotationally invariant it follows that all solutions which are constructed via the Ginzburg–Landau formalism

$$v(t,x,y) = V_{\mathrm{Cou}} + \varepsilon A(\varepsilon^2 t, \varepsilon x)\Phi_{0,k_0}(y) + \text{c.c.} + \mathcal{O}(\varepsilon^2)$$

are also rotationally invariant. It was established in [Schn99a, Schn99b] that every perturbation of V_{Cou} approaches the set of rotationally symmetric functions exponentially fast. Moreover there is an associated local attractor $\mathcal{A}$ in the set of all rotationally symmetric functions which attracts all solutions starting sufficiently close to V_{Cou}, see the above references or Section 6.7 for more details.

3.3 The Poiseuille flow

In contrast to the Rayleigh–Bénard problem and the Couette–Taylor problem the Poiseuille flow does not play such an important role in the theory of modulation equations for mainly two reasons. First, by now it is experimentally impossible to reach the theoretical threshold of instability which is at Reynolds number 5772.2. Typical experiments see a stable Poiseuille flow for Reynolds numbers below 2500, while for larger Reynolds numbers the domain of attraction of the Poiseuille solution is too small to be realizied by experimental conditions. The second reason involves the bifurcations which are predicted. It turns out that the real part of the cubic coefficient $\widehat{c}$ in the cGL is positive rather than negative. Hence the associated bifurcations are subcritical and lead to unstable phenomena. Nevertheless the Poiseuille problem was important historically (cf. [HoS72, DHS74]) and it still serves as a problem on which one can test numerical and analytical methods. We mainly include it here as it is one of the few hydrodynamical problems which leads to a cGL with two unbounded directions.

The Poiseuille flow occurs between two plates when a pressure gradient drives the system. Consider in nondimensional form the flow domain $\Omega = \mathbb{R}^2 \times (-1,1)$ with two unbounded directions x_1 and x_2 and one bounded direction $y \in (-1,1)$. The velocity $v(t,x,y)$ is now in $\mathbb{R}^3$ with components v_1, v_2 and v_3 in the direction x_1, x_2 and y. The basic equations are the Navier–Stokes equation with no–flux boundary conditions

$$\begin{array}{rll} \left.\begin{array}{r} \partial_t v + (v\cdot\nabla)v + \nabla p - \frac{1}{R}\Delta v = 0 \\ \operatorname{div} v = 0 \end{array}\right\} & \text{in} & \Omega, \\ v = 0 \qquad\qquad & \text{on} & \partial\Omega. \end{array} \tag{3.4}$$

Here $R = U_0 h/\nu$ is the Reynolds number, where U_0 is the typical velocity, h the height of the fluid layer and ν the kinematic viscosity. The basic Poiseuille solution is $V_{\mathrm{Pois}}(t,x,y) = (1-y^2,0,0)^{\mathrm{t}}$ which is associated to the pressure $p(x,y) = -2Rx_1$ which gives a constant pressure gradient in x_1–direction.

We are now interested in small solutions around the basic Poiseuille flow. Since V_{Pois} is independent of $x \in \mathbb{R}^2$ the classical ansatz $u(t,x,y) = \mathrm{e}^{\lambda t + \mathrm{i}\langle k,x\rangle}\Phi(y)$ can be inserted into the linearization at V_{Pois}. The classical Squire theorem (cf. [DHS74, AfM95]) shows that instability first occurs with a wave vector parallel to the x_1–direction. In fact, numerically one finds that the critical Reynolds number is $R_0 \approx 5772.2$ and the associated wave vector is $k_0 \approx (1.0206, 0)^{\mathrm{t}}$. The eigenvalue expansion can be found numerically and it reads

$$\lambda(R-R_0,k) = \mathrm{i}\omega + \lambda_{0,1}(R-R_0) - \mathrm{i}\langle c_g, k-k_0\rangle - \langle \Lambda_0(k-k_0), k-k_0\rangle + \text{h.o.t.}$$

with $\omega = -0.269$, $\lambda_{0,1} = (0.168{+}\mathrm{i}0.811)10^{-5}$, $c_g = (0.383, 0)^{\mathrm{t}}$, $\Lambda_0 = \mathrm{diag}(0.187 + \mathrm{i}0.0275, 0.00466 + \mathrm{i}0.0808)$, see [DHS74]. The associated eigenfunction can be expressed by the eigenfunction of the associated Orr–Sommerfeld equation $\varphi : (-1, 1) \to \mathbb{C}$ as $\Phi_{0,k_0}(y) = ((\mathrm{d}/\mathrm{d}y)\varphi(y), 0, -\mathrm{i}\langle k_0, e_1\rangle\varphi)^{\mathrm{t}}$, see [AfM95].

However, as in the other two cases above the mean flow has to be taken into account. Since we have two unbounded directions we cannot restrict the mean flow of the solutions. Thus, one either has to reduce the problem to a problem with only one unbounded direction, e.g. by making it periodic in one of the directions, or we have to generalize the Ginzburg–Landau formalism. We prefer the latter idea, as it allows us to find the other cases also by reinterpreting the obtained modulation equation accordingly.

To stay consistent with the normalization in [DHS74] we let $R = R_0 + \rho\varepsilon^2$ with $\rho = 1/\operatorname{Re}\lambda_{0,1}$, then the ansatz for the solutions $u = v - V_{\mathrm{Pois}}$ of (3.4) takes the form

$$\begin{array}{rcl} u(t, x, y) & = & \varepsilon A(\varepsilon^2 t, \varepsilon(x{-}c_g t))\mathbf{E}(t, x)\Phi_{0,k_0}(y) + \text{c.c.} + \text{h.o.t.}, \\ p(t, x, y) & = & p_0(t) - \frac{2x_1}{R} + \varepsilon\psi(\varepsilon^2 t, \varepsilon(x{-}c_g t))\mathbf{E}^0 + \text{h.o.t.}, \end{array}$$

and equating the coefficients for $\varepsilon^l\mathbf{E}^m$ to 0 one arrives at the coupled system

$$\begin{array}{rcl} \partial_\tau A - \Lambda_{11}^0\partial_{\xi_1}^2 A - \Lambda_{22}^0\partial_{\xi_2}^2 A - \frac{\lambda_{0,1}}{\operatorname{Re}\lambda_{0,1}}A - c_1|A|^2 A - c_2 A\,\partial_{\xi_1}\psi & = & 0, \\ \Delta_\xi\psi + \gamma\partial_{\xi_1}|A|^2 & = & 0, \end{array} \tag{3.5}$$

where $c_1 = 29.5 - \mathrm{i}144$, $c_2 = -28.0 + \mathrm{i}642$ and $\gamma = 0.0453$.

The appearance of a second equation for the pressure (which does not have a time derivative in it) is typical for all Navier–Stokes problems with two unbounded directions, see Subsection 2.6. We stayed within the frame of two unbounded directions as it serves as a model where the anisotropic cGL appears ($\arg\Lambda_{11}^0 \neq \arg\Lambda_{22}^0$). In fact, the formal approach here was made mathematically rigorous, in the sense of Secton 6, in [Bol96] for the full Poiseuille problem with two unbounded directions. However, to relate back to the pure GLe, we can reduce to the case of one unbounded direction by restriction to special solution classes and thus obtain a classical one–dimensional GLe.

First we consider solutions which are independent of x_2 and hence the solutions of (3.5) are independent of ξ_2. Then the second equation can be integrated to $\partial_{\xi_1}\psi = -\gamma|A|^2 + \delta$ and inserted into the first to obtain

$$\partial_\tau A = \Lambda_{11}^0\partial_{\xi_1}^2 A + [\tfrac{\lambda_{0,1}}{\operatorname{Re}\lambda_{0,1}} + \delta c_2]A + (c_1{-}\gamma c_2)|A|^2 A. \tag{3.6}$$

Here $\delta \in \mathbb{R}$ corresponds to an additional pressure gradient which can be set to 0 as it is equivalent to redefining the Reynolds number.

Second, we may follow [AfM95, AfM99b] and consider solutions of (3.4) which are periodic in downstream direction x_1 with a period close to that of $\mathbf{E}$. This amounts to looking for solutions of (3.5) of the form $A(\tau, \xi) = \mathrm{e}^{\mathrm{i}\beta\xi_1}\widetilde{A}(\tau, \xi_2)$ and $\psi(\tau, \xi) = \widetilde{\psi}(\tau, \xi_2)$ and we find

$$\partial_\tau\widetilde{A} = \Lambda_{22}^0\partial_{\xi_2}^2\widetilde{A} + [\tfrac{\lambda_{0,1}}{\operatorname{Re}\lambda_{0,1}}{-}\Lambda_{11}^0\beta^2]\widetilde{A} + c_1|\widetilde{A}|^2\widetilde{A}. \tag{3.7}$$

Below we will show that for (3.6) and (3.7) there exist pulse solutions of the form

$$A(\tau, \xi_j) = d_1\mathrm{e}^{\mathrm{i}\nu\tau}[\cosh(\eta\xi_j)]^{d_2}$$

with $d_1, d_2 \in \mathbb{C}$, $\nu \in \mathbb{R}$, $\eta > 0$ and $\operatorname{Re} d_2 < 0$. However, there is a major difference between the two cases with respect to the full Poiseuille problem (3.4).

To prove the existence of such pulse solutions for the full Poiseuille problem one needs to use the reflection symmetry $\xi_j \mapsto -\xi_j$ which is present in (3.6) and (3.7). However, the full problem (3.4) is only invariant under the reflection $x \mapsto (x_1, -x_2)$ but not with respect to $x \mapsto (-x_1, x_2)$. As a consequence it is only possible to prove existence of reflectionally symmetric pulse type solutions for the case associated to (3.7), see [AfM99a, AfM99b]. The case associated to (3.6) was treated in [Doe93] where existence of nonsymmetric pulses is shown by introducing a travel speed as an additional parameter.

3.4 Sideband instabilities

Historically the theory of sideband instabilities in hydrodynamics was one of the major motivations for the development of the modulation theory. Here we want to explain this phenomenon and relate it to the stability treatment of plane waves in Subsection 4.1.

We consider the full problem (2.1) for $\mu > 0$ and that the cubic coefficient $\widehat{c}$ in the GLe satisfies $\operatorname{Re}\widehat{c} > 0$. For the linearization $L(\mu, \partial_x)$ at $u \equiv 0$ there is already an open set of wave vectors unstable:

$$\mathcal{K}(\mu) = \{\, k \in \mathbb{R}^d \;:\; \operatorname{Re}\lambda(\mu, k) > 0, \quad \langle k, k_0\rangle \geq 0\,\},$$

where $\lambda(\mu, k)$ is the critical eigenvalue of $L(\mu, \mathrm{i}k)$ in (2.3). Under our spectral assumptions $\mathcal{K}(\mu)$ forms a smooth region of nearly elliptical shape with center in k_0 and diameter of order $\sqrt{\mu}$.

Using the Liapunov–Schmidt reduction (cf. e.g. [Mi97b]) it is easy to show that for all $k \in \mathcal{K}(\mu)$ there exists a periodic solution $u(t, x) = \mathcal{U}(\mu, k, t, x)$ of the nonlinear problem (2.1) in the form

$$U(\mu, k, t, x, y) = \mathcal{V}_{\mu,k}(\zeta, y) = \mathcal{V}_{\mu,k}(\zeta{+}2\pi, y) \quad \text{with} \quad \zeta = \widetilde{\omega}t{+}\langle k, x\rangle$$

where $\widetilde{\omega} \approx \omega$ is a new parameter. Such solutions are called plane travelling waves as they only depend on one coordinate in the (t, x)–space. The first approximation of $\mathcal{V}_{\mu,k}$ is given by $B\mathrm{e}^{\mathrm{i}\zeta}\Phi_{\mu,k}(y) + \text{c.c.}$ where $B \in \mathbb{C} \setminus \{0\}$ satisfies the algebraic equation

$$\mathrm{i}\widetilde{\omega}B = g(\mu, k, |B|^2)B \quad \text{with} \quad g(\mu, k, s) = \lambda(\mu, k) - \widehat{c}\, s + \text{h.o.t.},$$

cf. (2.14). This algebraic equation is of course compatible with the GLe since the lowest order term of $\mathcal{V}_{\mu,k}$ can be written as U_A with $A(\tau, \xi) = \mathrm{e}^{\mathrm{i}((\widetilde{\omega}-\omega)t+\langle k-k_0,x\rangle)}B$.

By the principle of exchange of stability it is known that the solutions $\mathcal{U}(\mu, k, t, x, y)$ are stable with respect to perturbations in the same function class, namely those which only depend on (ζ, y) with period 2π in ζ. However, in experiments not all of these periodic patterns are observable, since some of them are unstable with respect to perturbations having a slightly different wave length. Thus, one is lead to study the spectral properties of the linearization around the solution $\mathcal{U}(\mu, k, t, x, y)$ in general function spaces:

$$\partial_t w = L(\mu, \partial_x)w + \mathrm{D}_u N(\mu, \partial_x, \mathcal{U}(\mu, k, t, x, \cdot))[w].$$

This leads to a differential operator with spatially periodic coefficients. Using the Bloch–Floquet ansatz $w(t, x) = \mathrm{e}^{\widetilde{\lambda}t+\langle\sigma,x\rangle}W(\zeta, y)$ the problem reduces to the spectral problem for a family of Bloch operators

$$\mathbb{B}(\sigma, \mu, k, \partial_\zeta)W = \widetilde{\lambda}W$$

posed on the bounded domain $\mathbb{S}^1{\times}\Sigma$. The vector $\sigma \in \mathbb{R}^d$ is called the sideband vector of the perturbation, since $W(\zeta) = W(\widetilde{\omega}t{+}\langle k, x\rangle)$ already contains the underlying wave vector k.

By construction we know that $\mathbb{B}(0, \mu, k, \partial_\zeta)\partial_\zeta \mathcal{V}_{\mu,k} = 0$ such that $\lambda(\sigma, \mu, k) = 0$ for $\sigma = 0 \in \mathbb{R}^d$. Moreover, $\mathbb{B}$ has discrete eigenvalues which can be expanded in σ close to 0. Define the set of unstable sideband vectors

$$\mathbf{S}(\mu, k) = \{\, \sigma \in \mathbb{R}^d \ : \ \mathbb{B}(\sigma, \mu, k, \cdot) \text{ has an eigenvalue } \widetilde{\lambda} \text{ with } \operatorname{Re}\widetilde{\lambda} > 0 \,\}$$

associated to the solution $\mathcal{V}_{\mu,k}$. If $\mathbf{S}(\mu, k)$ is nonempty then the periodic pattern $\mathcal{V}_{\mu,k}$ is unstable. If $\mathbf{S}(\mu, k)$ intersects every neighborhood of $\sigma = 0$, then $\mathcal{U}$ is said to have an unstable sideband. (Note that $0 \neq \mathbf{S}(\mu, k)$ if $\mathcal{U}$ is stable with respect to periodic perturbations in ζ.)

A general analysis of the Floquet–Bloch ansatz in hydrodynamics is given in [Sca99]. Here we want to reveal the interplay with the Ginzburg–Landau formalism. In fact the solutions $\mathcal{U}(\mu, k, t, x, \cdot)$ constructed above have their counterparts in the plane waves

$$A_{\text{wave}}(\tau, \xi) = r\,\mathrm{e}^{\mathrm{i}(\delta\tau + \langle\kappa,\xi\rangle)} \quad \text{with } r^2 = \frac{1}{\operatorname{Re}\widehat{c}}[\rho \operatorname{Re}\lambda_{0,1} - \langle(\operatorname{Re}\Lambda_0)\kappa, \kappa\rangle].$$

The condition $r^2 > 0$ defines an ellipsoid which is the blowup of the set $\mathcal{K}(\mu) = \{\, k \ : \ \operatorname{Re}\lambda(\mu, k) > 0 \,\}$. The stability of these plane waves as solutions of the GLe can now be studied much easier due to the phase invariance $A \mapsto \mathrm{e}^{\mathrm{i}\alpha}A$ of GLe. Writing perturbations in the form $A(\tau, \xi) = A_{\text{wave}}(\tau, \xi)[1{+}B(\tau, \xi)]$ we obtain a parabolic equation for B which is homogeneous in space–time. Hence, the spectral problem can be solved algebraically by using Fourier transform, see Subsection 4.1.

In most physical papers the stability or instablity of a periodic pattern $\mathcal{U}(\mu, k, \cdot)$ is directly infered from the stability or instability of the associated plane wave for GLe. Yet, the given connection between the sideband instabilities of the periodic patterns $\mathcal{U}$ in the full problem and that of the plane waves in GLe is formal so far. However, in [Mie95, Mi97b, Mi97c] it is shown that this connection can be made rigorous in many cases by using the principle of reduced instability and thus avoiding the Ginzburg–Landau formalism.

4 Special solutions and dynamics of the Ginzburg–Landau equation

In the whole of this section we only consider the GLe without any reference to the full problem. This chapter is completely independent of the others. Hence we may use $t \geq 0$ and $x \in \mathbb{R}^d$ to denote the time and the space variable. We will also use the abbreviation $\partial_j = \partial/\partial_{x_j}$. Thus the GLe in its form (2.16) reads now

$$\partial_t A = \sum_{j=1}^{d}(1{+}\mathrm{i}\alpha_j)\partial_j^2 A + \rho A - (1{+}\mathrm{i}\beta)|A|^2 A. \tag{4.1}$$

For the mathematical theory we introduce a few function spaces. The main point here is that we need appropriate function spaces which enable us to study general nondecaying functions such as spatially periodic or quasiperiodic solutions as well as fronts, pulses or solutions with different periodic behavior at infinity.

For domains $\Omega \subset \mathbb{R}^n$ we define the weighted norm

$$\|A\|_{\mathrm{L}^p_w} = \left(\int_\Omega w(x)|A(x)|^p \,\mathrm{d}x\right)^{1/p} \tag{4.2}$$

and with $\mathrm{L}^p_w(\Omega)$ we denote the associated Banach space. The weight $w : \mathbb{R}^d \to (0,\infty)$ is assumed to be integrable over $\mathbb{R}^d$ and satisfies the estimate $|\nabla w(x)| \leq C w(x)$. Throughout it will be sufficient to consider the weight $w(x) = \mathrm{e}^{-|x|}$.

The basis of the theory for unbounded domains are the uniformly local Sobolev spaces $\mathrm{L}^p_{\mathrm{ul}}(\Omega)$ which are defined as follows. Let

$$\widetilde{\mathrm{L}}^p_{\mathrm{ul}}(\Omega) = \{\, u \in \mathrm{L}_{\mathrm{loc}}(\Omega) \ :\ \|u\|_{\mathrm{L}^p_{\mathrm{ul}}} < \infty \,\} \quad \text{with } \|u\|_{\mathrm{L}^p_{\mathrm{ul}}} = \sup_{y\in\mathbb{R}^d} \|w(\cdot - y)u\|_{\mathrm{L}^p}.$$

The final uniformly local space is defined as

$$\mathrm{L}^p_{\mathrm{ul}}(\Omega) = \text{closure of } \mathcal{C}^\infty_{\mathrm{bdd}}(\Omega) \text{ in } \widetilde{\mathrm{L}}^p_{\mathrm{ul}}(\Omega),$$

where $\mathcal{C}^\infty_{\mathrm{bdd}}(\Omega)$ is the space of all $\mathcal{C}^\infty$ functions whose derivatives are bounded in Ω. Note the embeddings $\mathcal{C}_{\mathrm{bdd,unif}}(\overline{\Omega}) \subset \mathrm{L}^q_{\mathrm{ul}}(\Omega) \subset \mathrm{L}^p_{\mathrm{ul}}(\Omega)$ for $1 \leq p < q < \infty$ with norm estimates $\|u\|_{\mathrm{L}^p_{\mathrm{ul}}} \leq C_w^{(q-p)/(qp)}\|u\|_{\mathrm{L}^q} \leq C_w^{1/p}\|u\|_{\mathrm{L}^\infty}$ where $C_w = \int_{\mathbb{R}^d} w(x)\mathrm{d}x$.

The uniformly local Sobolev spaces $\mathrm{W}^{k,p}_{\mathrm{ul}}(\Omega)$ are defined as usual:

$$\mathrm{W}^{k,p}_{\mathrm{ul}}(\Omega) = \{\, u \in \mathrm{L}^p_{\mathrm{ul}}(\Omega) \ :\ \mathrm{D}^\alpha u \in \mathrm{L}^p_{\mathrm{ul}}(\Omega) \text{ for } \alpha \in \mathbb{N}_0^n,\ |\alpha| \leq k \,\}.$$

For the Hilbert spaces we write $\mathrm{H}^k_{\mathrm{ul}}(\Omega) = \mathrm{W}^{k,2}_{\mathrm{ul}}(\Omega)$. In the case of a bounded domain the space $\mathrm{W}^{k,p}_{\mathrm{ul}}(\Omega)$ and its topology coincide with the usual Lebesgue space $\mathrm{W}^{k,p}(\Omega)$.

4.1 Plane waves and their stability

We have indicated in Subsection 3.4 that the plane waves and their stability properties are of central interest in the theory of pattern selection. In particular, they explain the phenomenon of sideband instability which is observed in hydrodynamical experiments on large spatial domains. Here we present the relevant algebraic calculations.

Because of the rotational symmetry $A \mapsto \mathrm{e}^{\mathrm{i}\gamma}A$ there is simple solution class given in the form $A(t,x) = r\,\mathrm{e}^{\mathrm{i}(\delta t + \langle\kappa,x\rangle)}$. Inserting this into (4.1) and recalling the notion $S = \mathrm{diag}(\alpha_1,\ldots,\alpha_d)$ we find the explicit solutions

$$A(t,x) = r\,\mathrm{e}^{\mathrm{i}(\delta t+\langle\kappa,x\rangle)} \quad \text{with } r = \sqrt{\rho - |\kappa|^2} \text{ and } \delta = \langle S\kappa,\kappa\rangle - \beta(\rho-|\kappa|^2). \tag{4.3}$$

These solutions play a central role in the theory of pattern formation. They correspond to space–time periodic solutions in the full problem whose frequency $\omega+\varepsilon\delta$ and wave vector $k = k_0+\varepsilon\kappa$ are small perturbations of those of the basic periodic pattern $\mathbf{p} = \mathbf{E}\Phi_{0,k_0}$. The question of pattern selection is thus intimately related to the stability of the solutions in (4.3).

This stability analysis is classical in the real case, where it is called the Eckhaus instability, since it was first derived in [Eck65]. The isotropic complex case was first treated in [New74]. We repeat the main steps here since we want to generalize the theory to the anisotropic case.

Stability is best tested by inserting the ansatz $A(t,x) = r\mathrm{e}^{\mathrm{i}(\delta t+\langle\kappa,x\rangle)}(1+b_1(t,x)+\mathrm{i}b_2(t,x))$ into (4.1) and to find the linearized equation for the perturbation $b=(b_1,b_2)\in\mathbb{R}^2$. This special ansatz and the rotational invariance lead to a linear problem with constant coefficients:

$$\partial_t\begin{pmatrix}b_1\\ b_2\end{pmatrix}=\sum_{j=1}^{d}\begin{pmatrix}1 & -\alpha_j\\ \alpha_j & 1\end{pmatrix}\begin{pmatrix}\partial_j^2 b_1-2\kappa_j\partial_j b_2\\ \partial_j^2 b_2+2\kappa_j\partial_j b_1\end{pmatrix}-2(\rho-|\kappa|^2)\begin{pmatrix}b_1\\ \beta b_1\end{pmatrix}.$$

We have to use a real representation as we now want to study the complex spectrum of this linear real operator. Inserting $b(t,x)=\mathrm{e}^{\lambda t+\mathrm{i}\langle\eta,x\rangle}(z_1,z_2)^{\mathrm{t}}$ with $\lambda,z_1,z_2\in\mathbb{C}$, we find solutions if and only if λ is an eigenvalue of

$$M(\rho,\kappa,\eta)=\begin{pmatrix}-|\eta|^2-2\mathrm{i}\langle S\kappa,\eta\rangle-2(\rho-|\kappa|^2) & -2\mathrm{i}\langle\kappa,\eta\rangle+\langle S\eta,\eta\rangle\\ 2\mathrm{i}\langle\kappa,\eta\rangle-\langle S\eta,\eta\rangle-2\beta(\rho-|\kappa|^2) & -|\eta|^2-2\mathrm{i}\langle S\kappa,\eta\rangle\end{pmatrix}.$$

The characterisic polynomial takes the form $\lambda^2+(a_1+\mathrm{i}a_2)\lambda+(a_3+\mathrm{i}a_4)=0$ with

$$\begin{aligned}&a_1=2|\eta|^2+2r^2,\quad a_2=4\langle S\kappa,\eta\rangle,\\ &a_3=|\eta|^2+\langle S\eta,\eta\rangle^2-4\langle S\kappa,\eta\rangle^2-4\langle\kappa,\eta\rangle^2+2|\eta|^2r^2,\\ &a_4=4[|\eta|^2+r^2]\langle S\kappa,\eta\rangle-4\langle\kappa,\eta\rangle\langle S\eta,\eta\rangle.\end{aligned}$$

The two roots λ_1 and λ_2 of this quadratic polynomial have nonpositive real part if and only if

$$a_1\geq 0,\quad a_1^2+a_2^2+4a_3\geq 0,\quad a_1^2a_3+a_1a_2a_4-a_4^2\geq 0.$$

The first condition is always satisfied and the other two lead after the substitution $\eta=r\xi$ to a necessary and sufficient condition for (linearized) stability:

$$\begin{aligned}&4\Big(\langle\kappa,\xi\rangle^2-\langle S\kappa,\xi\rangle^2\Big)\leq r^2\Big(5|\xi|^4+10|\xi|^2+1+\langle S\xi,\xi\rangle^2+2\beta\langle S\xi,\xi\rangle\Big)\\ &\text{and}\quad 4\langle\kappa,\xi\rangle^2\leq r^2\tfrac{(1+|\xi|^2)^2\left[|\xi|^4+2|\xi|^2+\langle S\xi,\xi\rangle^2+2\beta\langle S\xi,\xi\rangle\right]}{(1+|\xi|^2)^2+(\beta+\langle S\xi,\xi\rangle)^2}\quad\text{for all }\xi\in\mathbb{R}^d.\end{aligned}\tag{4.4}$$

We arranged the terms such that the left–hand side is quadratic in κ while the right–hand side is linear in $r^2=\rho-|\kappa|^2$. In the nonisotropic case the analysis of these conditions is rather complicated, however, we may derive a necessary condition for stability by considering those terms in the second condition which are quadratic in ξ, namely $4\langle\kappa,\xi\rangle^2\leq 2r^2[|\xi|^2+\beta\langle S\xi,\xi\rangle]/(1+\beta^2)$ for all $\xi\in\mathbb{R}^d$. This provides the *anisotropic necessary condition for stability*

$$1+\alpha_j\beta\geq 0\text{ for }j=1,\ldots,d\quad\text{and}\quad\sum_{j=1}^{d}\frac{\kappa_j^2}{1+\alpha_j\beta}\leq\frac{r^2}{2(1+\beta^2)}.\tag{4.5}$$

For the isotropic case the conditions simplify, see [DG*88]. In particular, it is possible to show that the first condition in (4.4) is a consequence of the second. Defining

$$\nu(\alpha,\beta)=\min_{y\in\mathbb{R}}\frac{(1+y^2)^2[y^2+2+\alpha^2y^2+2\alpha\beta]}{(1+y^2)^2+(\beta+\alpha y^2)^2}$$

we have stability if and only if $4|\kappa|^2 \leq r^2\nu(\alpha,\beta)$. Thus, stable rolls can only exist for $\nu(\alpha,\beta) \geq 0$ which holds if and only if the Newell criterion ([New74]) $1+\alpha\beta \geq 0$ is satisfied. Moroever, the rotating wave which loses stability last is the one with $\kappa = 0$. We arrive at the *isotropic stability criterion*

$$1+\alpha\beta \geq 0 \quad \text{and} \quad |\kappa|^2 \leq \rho\, \frac{\nu(\alpha,\beta)}{4+\nu(\alpha,\beta)}. \tag{4.6}$$

Since $\nu(0,0) = 2$ we find the Eckhaus criterion $|\kappa|^2 \leq \rho/3$ for rGL, [Eck65].

The ansatz $b(t,x) = \mathrm{e}^{\lambda t}\mathrm{e}^{\mathrm{i}\langle\eta,x\rangle}(z_1,z_2)^{\mathrm{t}}$ shows instability with respect to perturbations in the space $\mathrm{L}^2_{\mathrm{ul}}(\mathbb{R}^d)$. From the theory in [Mi97c, Sca99] it follows that this implies instability in the classical space $\mathrm{L}^2(\mathbb{R}^d)$. However, in the case of stability there is a slight difference: in $\mathrm{L}^2_{\mathrm{ul}}(\mathbb{R}^d)$ we have the perturbation $b(t,x) \equiv (0,z_2)^{\mathrm{t}}$ which does not decay whereas perturbations $b(t,\cdot)$ with $b(0,\cdot) \in \mathrm{L}^2(\mathbb{R}^d)$ decay to 0 for $t \to \infty$, namely $\|b(t,\cdot)\|_{\mathrm{L}^2} \to 0$ and $\|b(t,\cdot)\|_{\mathrm{L}^\infty} \leq Ct^{-d/4}$.

The stability of the rotating plane waves in the nonlinear problem is an even more delicate question because the spectrum of the linearization is continuous and contains a point on the imaginary axis. Thus, we can only expect marginal stability without exponential decay. In the present situation it can be shown that the rotating waves are diffusively stable in the sense that inital perturbations which are spatially localized (e.g. small in $\mathrm{H}^1(\mathbb{R}^d)$) will decay in $\mathrm{L}^\infty(\mathbb{R}^d)$ like $t^{-\alpha}$ for some $\alpha > 0$, see [BrK92, Gal95]. The nonlinear stability with respect to perturbations in $\mathrm{L}^\infty(\mathbb{R}^d)$ is an open problem.

4.2 The real Ginzburg–Landau equation

In our notation the real GLe is the GLe for the complex order parameter $A(t,x) \in \mathbb{C}$ when the equation has real coefficients. It can always be made isotropic and we normalize the coefficients to 1:

$$\partial_t A = \Delta A + A - |A|^2 A. \tag{4.7}$$

Sometimes the name 'real GLe' is also used for the real part of our equation, namely the scalar parabolic equation $\partial_t\psi = \Delta\psi + \psi - \psi^3$ which is often called Allen–Cahn equation.

If rGL is considered on a bounded domain $\Omega \subset \mathbb{R}^d$ with suitable boundary conditions, then it is a gradient system with respect to the Ginzburg–Landau energy functional

$$E(A) = \int_\Omega \Big(\frac{1}{2}|\nabla A|^2 - \frac{1}{2}|A|^2 + \frac{1}{4}|A|^4\Big)\,\mathrm{d}x + \int_{\partial\Omega} \frac{\eta}{2}|A|^2\,\mathrm{d}a. \tag{4.8}$$

We use $\eta \in \mathbb{R}$ to generate the Robin boundary conditions $\langle\nabla A,\nu\rangle + \eta A = 0$, which includes the Neumann case $\eta = 0$. For Dirichlet boundary conditions and for periodic boundary conditions we also let $\eta = 0$ and choose the appropriate function space. (In fact, also the isotropic complex GLe (4.1) with $\alpha_j = \beta$ and suitable boundary conditions has $E(A)$ as a strict Liapunov function.) Thus, we can conclude that the global attractor consists of steady states and heteroclinic connections between them, see [Hal88, BaV92, Tem88]. However, this theory does not give any information on the case of unbounded domains. There still exists a global attractor (cf. Section 5), however, most solutions have infinite energy and thus need not to settle down to an equilibrium, see e.g. our example of diffusive mixing in Section 4.2.3, coarsening dynamics discussed in [EcR99], or more general the discussion in Subsection 5.3.

Another important property involves the maximum principle which usually only works for scalar parabolic equations. The maximum $m(t)$ of $|A(t,x)|$ over Ω satisfies the scalar differential inequality $4\dot{m} \leq m - m^3$. This follows by the maximum principle if we write rGL with $A = r\mathrm{e}^{\mathrm{i}\phi}$:

$$\partial_t r = \Delta r - |\nabla\phi|^2 r + r - r^3, \quad \partial_t \phi = \Delta\phi + \frac{2}{r}\langle \nabla\phi, \nabla r\rangle.$$

As consequence we find that the global attractor $\mathcal{A}$, as constructed in Section 5, lies in the ball $\{\, A \in \mathrm{L}^\infty(\Omega) \,:\, \|A\|_\infty \leq 1 \,\}$.

Second we may consider nodal lines of $\mathrm{Re}\, A$ or more general of $u(t,x) = \mathrm{Re}\, [\mathrm{e}^{\mathrm{i}\gamma} A(t,x)]$ for any $\gamma \in \mathbb{R}$. If A is a solution of (4.7), then u solves the scalar linear parabolic equation

$$\partial_t u = \Delta u + (1 - |A(t,x)|^2)u. \tag{4.9}$$

Clearly, new nodal lines cannot be created. If for instance $u(t,x) > 0$ for $t \in [t_0 - \varepsilon, t_0)$ and $|x - x_0| \leq \varepsilon$, then $u(t_0, x_0) = 0$ is impossible becaause of the maximum principle. This concept will be especially useful in the one–dimensional setting since the nodal lines are just points, see Section 4.2.2.

4.2.1 Ginzburg–Landau vortices

Strictly speaking Ginzburg–Landau vortices are steady states of the real Ginzburg–Landau equation on very large two–dimensional domains, see [BBH94, JeS98, OvS98, LiX99]. For simplicity we state the problem only on $\mathbb{R}^2$ and refer to the references for the case of a domain which is large with respect to the natural length scale of the vortices. However, one still speaks of Ginzburg–Landau vortices when the solution looks locally like a steady state which moves slowly in space.

The existence theory of the vortices can be found in [Hag82] or the references in [BBH94, WeX96]. For each $n \in \mathbb{Z}$ there is a unique vortex (up to translations and phase invariance) of degree (or vorticity) n. It can be written in the form

$$\begin{array}{rcl} V_n(x) &=& \alpha_n(|x|)\mathrm{e}^{\mathrm{i}n\varphi(x)} \quad \text{where} \\ \varphi(x) &=& \arg(x_1 + \mathrm{i}x_2) \quad \text{such that } |x|^n \mathrm{e}^{\mathrm{i}n\varphi(x)} = (x_1 + \mathrm{i}x_2)^n, \\ \alpha_n(s) &>& 0 \text{ with } \alpha(s) \to 1 \text{ for } s \to \infty \text{ and} \\ 0 &=& \alpha'' + \frac{1}{s}\alpha' - \frac{n^2}{s^2} + \alpha - \alpha^3. \end{array}$$

For this we use the fact that $\Delta\varphi(x) = 0$ and $\nabla\varphi(x) = |x|^{-2}(-x_2, x_1)^t$ for $x \neq 0$ which implies $\langle \nabla\alpha_n(|x|), \nabla\varphi(x)\rangle \equiv 0$. Moreover, we have $\alpha_n(s) = c_n s^{|n|} + \mathcal{O}(s^{|n|+1})_{s\to 0}$ for some $c_n > 0$ and $\alpha_n(s) = 1 + \mathcal{O}(\mathrm{e}^{-s})_{s\to\infty}$. The case $n = 0$ is, of course, trivial with $V_0(x) \equiv 1$.

It was shown in [WeX96] that these steady states are unstable if $|n| \geq 2$ and that they are linearly stable for $|n| \leq 1$. In the unstable case the vortex generically breaks apart into $|n|$ vortices all having the degree $\mathrm{sign}(n)$.

Interaction of vortices can be studied if we place a finite number of stable vortices in the plane $\mathbb{R}^2$, say at the locations ζ_j, and ask for the behavior of the solution $A(t) = \mathcal{G}_t(A_0)$ of rGL with initial condition

$$A_0(x) = \prod_{j=1}^{N} V_{n_j}(x - \zeta_j(0)).$$

The locations $x_j(t)$ of the vortices for $t>0$ are then defined via $A(t,x_j(t))=0$.

The interaction can be studied well if the initial conditions are such that N vortices of degrees $n_j=\pm 1$ are spaced far apart from each other, namely $z(0)\gg 1$ where $z(t)=\min\{\,|\zeta_j(t)-\zeta_m(t)| : j\neq m\,\}$. It turns out that the solutions remain approximately of the above form, namely

$$A(t,x)=\prod_{j=1}^{N} V_{n_j}(x-\zeta_j(t))+\mathcal{O}(z(t)^{-1}) \quad \text{for } z(t)\to\infty. \tag{4.10}$$

The locations $x_j(t)$ of the vortices are approximated by $\zeta_j(t)$ which are obtained by solving an ODE for $\underline{\zeta}=(\zeta_1,\ldots,\zeta_N)\in\mathbb{R}^{2N}$. It is given as the gradient flow associated to the renormalized energy

$$H(\underline{\zeta})=-\sum_{j\neq m} n_j n_m \log|\zeta_j-\zeta_m|.$$

Thus the ODE $\dot{\underline{\zeta}}=-\nabla_{\underline{\zeta}}H(\underline{\zeta})$ takes the form

$$\dot{\zeta}_j=-\nabla_{\zeta_j}H(\underline{\zeta})=2n_j\sum_{m\neq j}\frac{n_m}{|\zeta_j-\zeta_m|^2}(\zeta_j-\zeta_m), \quad j=1,\ldots,N. \tag{4.11}$$

We refer to [JeS98] for a mathematical justification of this approximation result in the case of large bounded domains. It remains only valid in the limit $z(t)\gg 1$ which excludes collisions of the vortices. Typically a vortex of degree $+1$ and and a vortex of degree -1 attract each other such that $z(t)$ goes to 0 in finite time, the two vortices of opposite degree collide and annihilate each other, i.e. they disappear into a trivial vortex of degree 0.

A similar analysis can be made for the two–dimensional cGL with $\alpha_1=\alpha_2=\beta$, which we write in the form

$$\partial_t A=\Big(\mathrm{i}+\delta\Big)\Big(\Delta A+A-|A|^2A\Big),$$

where $\delta\geq 0$. In the case $\delta=0$ we obtain a form of the nonlinear Schrödinger equation. It is again possible to show that solutions of the form (4.10) exist, but now the locations $\zeta_j(t)$ satisfy the ODE

$$\dot{\zeta}_j=\begin{pmatrix}-\delta & n_j\\ -n_j & -\delta\end{pmatrix}\nabla_{\zeta_j}H(\underline{\zeta}). \tag{4.12}$$

In particular, this system is Hamiltonian for $\delta=0$, see [OvS98, LiX99] and the references therein.

4.2.2 One spatial direction

The one–dimensional rGL can either be considered on the real line with general initial data or on an interval $(0,\ell)$ with suitable boundary data. In the latter case we may have a gradient system with respect to the energy given in (4.8).

In the first case all stationary solutions, which are solutions of the four–dimensional ODE $A''+A-|A|^2A=0$, can be classified (up to phase and translation invariance) as follows, see [Gal95]:

a) real solutions $A(x)=r(x)\in\mathbb{R}$ where $r''+r-r^3=0$ (periodic or homoclinic $r(x)=$

$\tanh(x/\sqrt{2})$);
b) rotating plane waves $A(x) = \sqrt{1-\kappa^2}\,\mathrm{e}^{\mathrm{i}\kappa x}$ *with* $|\kappa| \leq 1$;
c) quasiperiodic solutions $A(x) = r(x)\mathrm{e}^{\mathrm{i}\phi(x)}$, *where* $r(\cdot)$ *and* $\phi'(\cdot)$ *are periodic with the same period, see [BrR94, DGJ95];*
d) homoclinic solutions $A(x) = \sqrt{2(\kappa^2+c^2\tanh^2(cx))}\,\mathrm{e}^{\mathrm{i}\phi(x)}$ *where* $c = \sqrt{(1-3\kappa^2)/2}$, $0 < \kappa^2 < 1/3$ *and* $\phi(x) = \kappa x + \arctan[\frac{c}{\kappa}\tanh(cx)]$.

Each of these solutions is unstable except for the rotating plane waves with $\kappa^2 \leq 1/3$, cf. [Gal95]. If some of these solutions are compatible with boundary conditions of the finite interval $(0,\ell)$, then they can be stable as the set of perturbations is reduced as well. It is an interesting question to ask which of these steady states are connected by heteroclinic solutions.

One of these results is obtained in [EGW95] by considering rGL with the quasiperiodic boundary conditions

$$A(t,\ell) = \mathrm{e}^{\mathrm{i}\gamma}A(t,0)$$

on the interval $\Omega = (0,\ell)$. The phase space is

$$Z_{\ell,\gamma} = \{\, A \in \mathrm{H}^1_{\mathrm{ul}}(\mathbb{R},\mathbb{C}) \;:\; \mathrm{e}^{-\mathrm{i}\gamma x/\ell}A(x) \text{ is periodic with period } \ell \,\}; \tag{4.13}$$

and rGL is a gradient system with respect to the energy E in (4.8). In addition to $A \equiv 0$ there are families of steady states $(\mathrm{e}^{\mathrm{i}\psi}U_{\kappa_j})_{\psi\in\mathbb{S}^1}$ where

$$U_\kappa(x) = \sqrt{1-\kappa^2}\,\mathrm{e}^{\mathrm{i}\kappa x} \quad \text{and} \quad \kappa_j\ell = \gamma \bmod 2\pi.$$

The energy of these steady rotating waves is $E(U_\kappa) = -\ell(1-\kappa^2)^2/4$.

In [EGW95] it is shown that there exists a solution $A :$ $R \to Z_{\ell,\gamma}$ starting at U_{κ_1} for $t \to -\infty$ and converging to $\mathrm{e}^{\mathrm{i}\psi}U_{\kappa_2}$ for $t \to \infty$, if κ_1 and κ_2 satisfy certain restrictions. First, U_{κ_1} has to be unstable with respect to perturbations in $Z_{\ell,\gamma}$. Second, there must be a solution U_{κ_2} which has lower energy, i.e. $E(U_{\kappa_1}) > E(U_{\kappa_2})$. This gives $\kappa_1^2 > (\ell^2+2\pi^2)/(3\ell^2)$ and $\kappa_1^2 > \kappa_2^2$. The third condition is the most difficult but it makes the proof straight forward. It is assumed that except for the family $\mathrm{e}^{\mathrm{i}\psi}B_2$ there are no other steady states B of rGL in $Z_{\ell,\gamma}$ with $E(B) < E(B_1)$. The following result is obtained by choosing $\kappa_2 = \kappa_1 - 2\pi/\ell$, see [EGW95].

Theorem 4.1
Assume that $\kappa_j \in (-1,1)$ *satisfies the relations* $1 < \kappa_1 - \kappa_2 < \sqrt{6\kappa_1 - 2}$. *Then there exists a solution* $A : \mathbb{R}\times\mathrm{H}^1_{\mathrm{ul}}(\mathbb{R},\mathbb{C})$ *of rGL and a* $\psi \in \mathbb{S}^1$ *such that*

$$\lim_{t\to-\infty} A(t,x) = U_{\kappa_1}(x) \quad \textit{and} \quad \lim_{t\to\infty} A(t,x) = \mathrm{e}^{\mathrm{i}\psi}U_{\kappa_2}(x)$$

uniformly in $x \in \mathbb{R}$.

The assumptions of this result imply $\kappa_1 \in (1/\sqrt{2}, 1)$ and $\kappa_2 < 0$. The solution A obtained in the theorem is not the one which is typically observed in computer experiments, when ℓ is taken much larger than $2\pi/\kappa_1$. In most cases a solution starting near U_{κ_1} with $\kappa_1 > 1/\sqrt{3}$ converges to U_{κ_j} for some $\kappa_j \in [0, 1/\sqrt{3})$. But there is no proof of such a statement.

There is a rather special property for rGL which generalizes the concept of "lap numbers" for scalar parabolic equations. We discuss this now for a bounded interval but it is clear that it can be transferred to the real line, if sufficient control over the behavior at infinity is obtained. Consider solutions A of rGL on $(0,\ell)$ and assume for the moment that $A(t,x) \neq 0$ on $(t_1,t_2)\times(0,\ell)$ then the angle $\phi(t,x)$ in $A = r\,\mathrm{e}^{\mathrm{i}\phi}$ is well defined up to multiples of 2π.

The key observation is that if ϕ has a local maximum at a point (t_0,x_0) then the maximal value will decrease in time. Indeed, consider $u(t,x) = \mathrm{Im}\,[\mathrm{e}^{-\mathrm{i}\phi(t_0,x_0)}A(t,x)]$ which solves (4.9) and has a double zero at (t_0,x_0). By the theory for scalar parabolic equations (cf. [Büg98]) this zero will disappear, which means $\phi(t,x) < \phi(t_0,x_0)$ for $t \in (t_0,t_0{+}\varepsilon)$ and $|x{-}x_0| < \varepsilon$ with sufficiently small ε. The analogous statement holds for minima of ϕ. However, new minima and maxima can appear when the solution undergoes a so–called phase slip, which means that $A(t_1,x_1) \in \mathbb{C}$ is 0. This occurs only at discrete points and typically $\partial_x^2 A(t_1,x_1) \neq 0$. In such a situation ϕ develops a jump of size $\sigma\pi$ with $\sigma \in \{1,-1\}$ before the phase slip and has a jump of size $-\sigma\pi$ directly after the phase slip.

In the case of Dirichlet or Neumann boundary conditions for the interval $(0,\ell)$ it is shown in [Büg98] that the *torsion function*

$$\psi(t) = \max_{x\in(0,\ell)} \phi(t,x) - \min_{x\in(0,\ell)} \phi(t,x)$$

is either constant along solutions or strictly decreasing, as long as no phase slips occur. As a consequence it can be shown that all solutions converge to a planar steady state (i.e., ϕ =const).

For quasiperiodic boundary conditions we have the boundary condition $\phi(t,\ell) = \phi(t,0) + \gamma + 2\pi N(t)$ with $N(t) \in \mathbb{Z}$. Moreover, the above arguments show that the total variation

$$\mathrm{Var}(\phi(t,\cdot)) = \sup\{ \sum_{j=1}^{n} |\phi(t,x_j){-}\phi(t,x_{j-1})| \;:\; 0 \le x_0 < x_1 < \ldots < x_n \le \ell \}$$

decreases along solutions of rGL even if phase slips occur. However, this does not imply that the function $|\phi(t,\ell){-}\phi(t,0)| = |\gamma{+}N(t)|$ decreases, in fact Theorem 4.1 gives examples where $\gamma{+}N(t)$ changes sign.

More informatio non phase slips can be found in [CoE92, EGW95, Büg98].

Another important solution class are traveling waves and oscillating fronts which take the form $A(t,x) = a(x,x{-}vt)$ where $v \in \mathbb{R}$ is the velocity of the front and $a(x,z)$ converges to steady states for $z \to \pm\infty$;

$$\lim_{z\to-\infty} a(x,z) = U_{\kappa_-}(x) \quad \text{and} \quad \lim_{z\to\infty} a(x,z) = \mathrm{e}^{\mathrm{i}\psi} U_{\kappa_+}(x).$$

In the case that $\kappa_+ = 0$ one can use the ansatz $a(x,z) = \mathrm{e}^{\mathrm{i}\kappa_- x} f(z)$ and finds the ODE

$$f'' + (v{+}2\mathrm{i}\kappa_-)f' + (1{-}\kappa_-^2)f - |f|^2 f = 0.$$

It is easily shown that for any $v > 0$ there is a unique solution (up to phase and z translation) satisfying $f(-\infty) = \sqrt{1{-}\kappa_-^2}$ and $f(\infty) = 0$.

The general case is much more difficult and only partial results are known, see [EcG93].

4.2.3 Diffusive mixing

Here we are interested in solutions which behave like one stable pattern at $x \to +\infty$ and like another one at $x \to -\infty$:

$$A(t,x) - \mathrm{e}^{\mathrm{i}\beta_\pm} U_{\kappa_\pm}(x) \to 0 \quad \text{for } x \to \pm\infty, \tag{4.14}$$

where $\kappa_\pm^2 < 1/3$. Clearly, if the initial condition satisfies these asymptotic boundary conditions, then the solution does for all times. Our concern is the dynamics of the system in the intermediate regime.

The structure of (4.7) becomes clearer when using polar coordinates $A = r\mathrm{e}^{\mathrm{i}\phi}$ giving

$$r_t = r_{xx} + r(1-r^2-\phi_x^2), \quad \phi_t = \phi_{xx} + 2r_x\phi_x/r.$$

The underlying idea of diffusive repair and diffusive mixing is that for $t \to \infty$ the derivatives r_t and r_{xx} tend to 0 in suitable norms, see [CoE92, GaM98]. In particular, this implies that asymptotically the amplitude r is slaved to the phase derivative ϕ_x such that $r^2 = 1-\phi_x^2 + \mathcal{O}(t^{-1/4})$. Hence, asymptotically the so–called phase diffusion equation

$$\phi_t = a(\phi_x)\phi_{xx} \quad \text{with } a(s) = (1-3s^2)/(1-s^2) \tag{4.15}$$

is relevant.

The boundary conditions (4.14) imply $\phi_x(t,x) \to \kappa_\pm$ for $x \to \pm\infty$. Hence, for $\eta(t,x) = \phi_x(t,x)$ we obtain from (4.15) the problem

$$\eta_t = (a(\eta)\eta_x)_x, \quad \eta(t,x) \to \kappa_\pm \text{ for } x \to \pm\infty. \tag{4.16}$$

It is shown in [GaM98, Thm. 3.1] that (4.16) has, for all $\kappa_+, \kappa_- \in (-1/\sqrt{3}, 1/\sqrt{3})$, a unique similarity solution $\eta(t,x) = \widetilde{\eta}(x/\sqrt{t})$ where $\widetilde{\eta}$ is monotone and attains its limits κ_+ and κ_- faster than exponential. It satisfies the ODE

$$[a(\widetilde{\eta}(\xi))\widetilde{\eta}'(\xi)]' + \frac{\xi}{2}\widetilde{\eta}'(\xi) \quad \text{for } \xi \in \mathbb{R}. \tag{4.17}$$

With $\widetilde{N}(\xi) = \kappa_-\xi + \int_{-\infty}^{\xi} [\widetilde{\eta}(s)-\kappa_-]\mathrm{d}s$ we define the time–dependent profile

$$\widetilde{U}(t,x) = \sqrt{1-\widetilde{\eta}(x/\sqrt{t})^2}\, \mathrm{e}^{\mathrm{i}\sqrt{t}\,\widetilde{N}(x/\sqrt{t})}. \tag{4.18}$$

The mathematical theory of diffusive mixing consists in showing that this profile is a good asymptotic approximation for the limiting behavior of suitable solutions A of rGL satisfying the boundary conditions (4.14). A first result was obtained in [BrK92] for the case $\kappa_\pm \approx 0$, the general case with $\kappa_\pm^2 < 1/3$ is treated in [GaM98] and provides the following result.

Theorem 4.2
Let $\kappa_+, \kappa_- \in (-1/\sqrt{3}, 1/\sqrt{3})$ with $\kappa_+ \neq \kappa_-$. Then there exist $t_0 > 0$ and $\varepsilon > 0$ such that for all $A_0 \in \mathrm{W}^{2,2}_{\mathrm{ul}}(\mathbb{R})$ satisfying $\|A_0 - \widetilde{U}(t_0,\cdot)\|_{\mathrm{W}^{2,2}(\mathbb{R})} \leq \varepsilon$, the unique solution $A(t) \in \mathrm{W}^{2,2}_{\mathrm{ul}}(\mathbb{R})$ of rGL with $A(0) = A_0$ satisfies, for $t \to \infty$,

$$\|A(t) - \widetilde{U}(t_0+t,\cdot)\|_{\mathrm{W}^{1,2}_{\mathrm{ul}}(\mathbb{R})} = \mathcal{O}(t^{-1/5}) \quad \textit{and} \quad \|\,|A(t)| - |\widetilde{U}(t_0+t,\cdot)|\,\|_{\mathrm{W}^{1,2}_{\mathrm{ul}}(\mathbb{R})} = \mathcal{O}(t^{-2/3}).$$

Note that the assumptions use the nonweighted classical $\mathrm{W}^{2,2}$ norm while the results are stated in the uniformly local norm. This is due to the diffusive character which transports initially localized perturbations out to infinity. The second result in the theorem manifests the fact that the slaving of the amplitude $r(t,x) = |u(t,x)|$ occurs faster than the diffusive mixing which is mainly driven by the phase diffusion which has the decay rate $1/\sqrt{t}$.

The above result says that the limit behavior of the solutions for $t \to \infty$ is completely determined by the time–dependent profile $\widetilde{U}$. From its explicit form we find the relation

$$\|\widetilde{U}(t,\cdot) - \mathrm{e}^{\mathrm{i}\phi_*\sqrt{t}}\, U_{\kappa_*}(\cdot)\|_{\infty,w} = \mathcal{O}(t^{-1/2}) \quad \text{for } t \to \infty.$$

Here we again have to use the weighted norm, since a similar estimate in $\mathrm{L}^2_{\mathrm{ul}}(\mathbb{R})$ cannot hold due to the different limit behavior for $x \to \infty$ of $\widetilde{U}(t,x))$ and $\mathrm{e}^{\mathrm{i}\beta}U_{\kappa_*}$. The values κ_* and ϕ_* are given via $\kappa_* = \widetilde{\eta}(0)$ and $\phi_* = \widetilde{N}(0) \neq 0$ and hence are well–defined functions of the limit values κ_+ and κ_-. As a consequence we have uniform convergence on each bounded interval $(-x_1, x_1)$.

The results of Theorem 4.2 mean that the solutions $A(t)$ converge to the circle

$$\mathbf{S} = \{\, \mathrm{e}^{\mathrm{i}\beta}\, U_{\kappa_*} \in \mathrm{L}^2_{\mathrm{ul}}(\mathbb{R}) \,:\, \beta \in \mathbb{R} \,\}$$

of equilibria in the sense that $\mathrm{dist}_w(A(t,\cdot), \mathbf{S}) \to 0$ for $t \to \infty$. The solutions $u(t)$ do not converge to a particular point in $\mathbf{S}$ but they slide along $\mathbf{S}$ with speed $\frac{1}{2}\phi_* t^{-1/2}$ and thus never settle down completely.

It is interesting to consider the flow of the local energy density

$$e(t,x) = \tfrac{1}{2}|\nabla A|^2 - \tfrac{1}{2}|A|^2 + \tfrac{1}{4}|A|^4 = \tfrac{1}{2}[|\nabla r|^2 + r^2|\nabla\phi|^2]^2 - \tfrac{1}{2}r^2 + \tfrac{1}{4}r^4$$

associated with the Ginzburg–Landau functional (4.8). Here we have given the multi–dimensional case for later reference. As the total energy $E(A(t,\cdot))$ of the solution is not defined, the energy density can be used to study the energy dissipation. Moreover, we may compare the relative energy

$$E_{\mathrm{rel}}(A(t)) = \int_{\mathbb{R}} [e(t,x) - e(0,x)]\,\mathrm{d}x,$$

which is finite as the boundary conditions are time–independent for all times. As a result we will find that this relative energy tends to $-\infty$ like $-\sqrt{t}$ for $t \to \infty$.

The local energy density satisfies the following conservation law having a dissipative term:

$$\partial_t e + \mathrm{div}_x f = -g, \tag{4.19}$$

where the energy flux $f(t,x) \in \mathbb{R}^d$ and the dissipated energy $-g(t,x)$ are given by

$$\begin{aligned}
f(t,x) &= -\mathrm{Re}\,[(\Delta A + A - |A|^2 A)\nabla\overline{A}] \\
&= -(\Delta r - r|\nabla\phi|^2 + r - r^3)\nabla r - (r^2\Delta\phi + 2r\nabla r\cdot\nabla\phi)\nabla\phi, \\
g(t,x) &= |\partial_t A|^2 = |\Delta A + A - |A|^2 A|^2 = (\Delta r - r|\nabla\phi|^2 + r - r^3)^2 + (r\Delta\phi + 2\nabla r\cdot\nabla\phi)^2.
\end{aligned}$$

Returning to the one–dimensional case of diffusive mixing we employ the asymptotics derived in Theorem 4.2 and (4.18) and obtain the expansions

$$e(t,x) = h(\widetilde{\eta}) + \mathcal{O}(1/t), \quad g(t,x) = \frac{(1-3\widetilde{\eta}^2)^2}{t(1-\widetilde{\eta}^2)}\widetilde{\eta}'^2 + \mathcal{O}(1/t^2)$$

where $h(\eta) = -(1-\eta^2)^2/4$ and $\widetilde{\eta}$ and its derivative are evaluated at $\xi = x/\sqrt{t}$. Using (4.17) we find

$$\frac{\mathrm{d}}{\mathrm{d}t}E_{\mathrm{rel}}(A(t)) = \int_{\mathbb{R}} \partial_t e(t,x)\,\mathrm{d}x = -\int_{x\in\mathbb{R}} g(t,x)\,\mathrm{d}x = -\frac{1}{2}\int_{\xi\in\mathbb{R}} \xi \frac{\mathrm{d}}{\mathrm{d}\xi} h(\widetilde{\eta}(\xi))\,\mathrm{d}\xi + \mathcal{O}(1/t^{3/2}).$$

Defining $\eta_\infty(\xi) = \kappa_\pm$ for $\pm\xi > 0$ we obtain via integration by parts

$$\frac{\mathrm{d}}{\mathrm{d}t}E_{\mathrm{rel}}(A(t)) = -\frac{B(\kappa_-,\kappa_+)}{2\sqrt{t}} + \mathcal{O}(1/t^{3/2}), \text{ with } B(\kappa_-,\kappa_+) = \int_{\mathbb{R}} [h(\eta_\infty(\xi)) - h(\widetilde{\eta}(\xi))]\,\mathrm{d}\xi.$$

Clearly $B(\kappa_-,\kappa_+)$ is strictly positive for $\kappa_- \neq \kappa_+$.

The energy density in the middle region tends to $h(\kappa_*)$ which always lies below the average of $h(\kappa_-)$ and $h(\kappa_+)$, since the function h is convex on the η–interval $(-1/\sqrt{3}, 1/\sqrt{3})$. In a situation with $h(\kappa_-) < h(\kappa_*) < h(\kappa_+)$ the energy flux has to transport energy from the region $x \gg 1$ via the region $x \approx 0$ to the region with $x \ll -1$.

The same result could be obtained in a simpler way (but with less insight) by considering the relative energy $\int_{\mathbb{R}} [e(t,x) - h(\eta_\infty(x))]\,\mathrm{d}x$. The leading term of this integral is scaling invariant and gives $-\sqrt{t}B(\eta_-,\eta_+)$.

4.3 Special solutions for the complex Ginzburg–Landau equation

There is a very rich literature on different aspects of cGL, however, we focus here on solutions defined on $\mathbb{R}^d$, in particular traveling and rotating waves. In the one–dimensional case one can look for solutions having a fixed profile which rotates in phase with frequency $\widetilde{\omega}$ and travels along the x–axis with speed v, namely $A(t,x) = \mathrm{e}^{i\widetilde{\omega}t}a(x-vt)$. This leads to the ODE

$$(1+\mathrm{i}\alpha)a'' + va' + (\rho-\mathrm{i}\widetilde{\omega})a + \widehat{c}|a|^2a = 0. \tag{4.20}$$

Because of the explicit algebraic structure of this equation there is a variety of explicit solutions in addition to the family of rotating plane waves discussed in Section 4.1. One of these families are the so–called Bekki–Nozakki solutions ([BeN85]), which are surveyed and generalized in [vSH92]. These solutions are heteroclinic connections between two rotating plane waves at infinity. However, the solution is not structurally stable, see [Doe96]. This is in contrast to the pulse solutions considered in the next subsection.

Since (4.20) is a four–dimensional ODE with phase invariance, it can be reduced to a three–dimensional ODE and studied in detail. The real variables $\rho, \widetilde{\omega}$ and v serve as parameters. In particular the nearly integrable case (i.e., $\arg(\rho-\mathrm{i}\widetilde{\omega}) \approx \arg\widehat{c} \approx \arg(1+\mathrm{i}\alpha)$ and $v \approx 0$) is studied in great detail, see e.g. [Hol86, DoE91, Doe93, KaM96, Doe96, Kap96, AfM99a]

4.3.1 One–dimensional pulse solutions

Pulses are considered mainly in the one–dimensional case. The simplest pulses are rotating waves in time and decay exponentially for $|x| \to \infty$. To make the pulse a stationary solution we write cGL in the form

$$\partial_t A = (1+\mathrm{i}\alpha)\Big[\partial_x^2 A - (1+\mathrm{i}\theta_1)^2 A + (1+\mathrm{i}\theta_2)(2+\mathrm{i}\theta_2)|A|^2 A\Big]. \tag{4.21}$$

If θ_1 and θ_2 are near zero, then this equation can be seen as a perturbation of both, the real GLe if $\alpha \approx 0$ and the nonlinear Schrödinger equation if $|\alpha| \gg 1$. It was first observed in [StS71] that this equation has an explicit solution for $\theta = \theta_2 = \theta_1$, namely

$$H_\theta(x) = \mathrm{e}^{\mathrm{i}\gamma}[\cosh x]^{-(1+\mathrm{i}\theta)}. \tag{4.22}$$

There are more general steady pulse solutions of (4.21). In particular, in applications of cGL to laser optics (cf. [MHK98, Kap98]) one is interested in multi–pulse solutions, i.e. solutions with $|A(\cdot)|^2$ having several well–distinguished maxima. Such solutions were found in cGL in [KaM96] for the case $0 < |\theta_2| \ll 1$ and $|\theta_1 - \theta_2| \ll |\theta_2|$. This work was generalized in [AfM99a] and then applied to the Poiseuille problem in [AfM99b] where θ_2 between 4 and 30 is relevant. The application to the Poiseuille problem involves the justicfication of the Ginzburg–Landau formalism by the help of the Kirchgässner reduction, see Section 6.1.

For the construction of multi–pulse solutions one analyzes the stationary problem. This is a four–dimensional ODE which is reversible because of the reflection symmetry $x \mapsto -x$ of cGL and is SO(2) invariant. After factorization, the problem can be treated as a three–dimensional problem which is singular at the origin. We fix $\theta = \theta_1$ and treat $\varepsilon = \theta_2 - \omega$ as a small bifurcation parameter:

$$A'' - (1+\mathrm{i}\theta^2)A + (1+\mathrm{i}(\theta+\varepsilon))(2+\mathrm{i}(\theta+\varepsilon))|A|^2 A. \tag{4.23}$$

The explicit pulse (4.22) exists for $\varepsilon = 0$ and it breaks apart for $\varepsilon \neq 0$ if the pulse is nondegenerate with respect to the parameter ε. The analysis shows that nondegeneracy holds for all $\theta \in \mathbb{R}$ except for an at most countable set $\{0, \pm\widehat{\theta}_1, \pm\widehat{\theta}_2, \ldots\}$ with $\widehat{\theta}_j \to \infty$, and numerical calculations give $\widehat{\theta}_1 \approx 8.032$ and $\widehat{\theta}_2 \approx 9.51$.

Using appropriate local and global Poincaré maps it is possible to study the flow in a tubular neighborhood of the primary pulse solution. One obtains existence of shift dynamics as well as reversible and nonreversible periodic orbits for all small ε. It is clear that the existence of a homoclinic orbit means that the stable and the unstable manifolds of $(A, A') = 0 \in \mathbb{C}^2 \sim \mathbb{R}^4$ (both are two–dimensional) coincide, since homoclinic orbits occur as one–parameter families. Such a homoclinic solution is called n–pulse solution if it has the form

$$A(x) \approx \sum_{m=1}^{n} \mathrm{e}^{\mathrm{i}\gamma_m} H_\theta(x - y_m)$$

with the phases $\gamma_m \in \mathbb{S}^1$ and shifts $y_m \in \mathbb{R}$. Here we have $y_{m+1} - y_m \gg 1$ which then justifies the name 'multi–pulse'. We do not call these solutions n–homoclinic as they do follow n different homoclinic orbits rather than just one.

Denote by $E \subset (-\varepsilon_0, \varepsilon_0)$ the set of all ε such that (4.23) has a homoclinic orbit. Then we have $E = \cup_{n=1}^{\infty} E^n$ where E^n is the set of ε for which (4.23) has an n–pulse solution. The results obtained in [KaM96, AfM99a] can be summarized as follows:

Theorem 4.3
Assume that $\theta \notin \{0, \pm\widehat{\theta}, \ldots\}$, then the following holds.
(a) Both sets, E^2 and E^3, consist of two sequences converging monotonically to $\varepsilon = 0$ with rate $\mathrm{e}^{-\pi/|\theta|}$, one sequence from above and one from below. Moreover the two sequences separate each other. The associated homoclinic orbits are transversal again.
(b) For all n the sets E^n are infinite and contain $\varepsilon = 0$ in their closure.
(c) If $n = pq$ with $p, q \geq 2$ then $E^q \subset$ closure(E^n).

Considering H_θ as a stationary solution of cGL in the form (4.21) we may ask about the stability of these solutions. Clearly, it can only be stable if the asymptotic state $A \equiv 0$ is stable as well. Since the continuous spectrum of the linear operator $A \mapsto (1+\mathrm{i}\alpha)[A'' - (1+\mathrm{i}\theta)A]$ is given by

$$\{ -1+2\alpha\theta+\theta^2 - s^2 \pm \mathrm{i}[2\theta+\alpha(s^2+1-\theta^2)] \,:\, s \in \mathbb{R} \},$$

we obtain the necessary condition for stability $-1+2\alpha\theta+\theta^2 \leq 0$.

However, there are also discrete eigenvalues, in particular we obtain always two eigenvalues 0 due to translation and rotation invariance. In the real case $\alpha = \theta = 0$ with $H_0(x) = 1/(\cosh x)$ it is easy to see that there is also one unstable eigenvalue since the eigenvalue problem for $A = a_1+\mathrm{i}a_2$ takes the decoupled form

$$a_1'' - (1-6H_0^2(x))a_1 = \lambda a_1, \quad a_2'' - (1-2H_0^2(x))a_2 = \lambda a_2.$$

These are two selfadjoint operators with real spectrum whose continuous part is $(-\infty, -1]$. The second equation has the lowest eigenvalue 0 with eigenfunction H_0, however, the first equation has the second eigenvalue 0 with eigenfunction H_0' (which has one zero). For small α and θ the pulse solution remains unstable. In fact, the numerical calculations in [BeL98, BKT99] show that this positive real eigenvalue persists in a large region of the (α, θ)–plane.

Moreover, it can be shown analytically that for $-1+2\alpha\theta+\theta^2 = 0$ the eigenvalue 0 is in fact threefold. It is easy to check, that (4.21) with $\alpha = (1-\theta^2)/(2\theta)$ has a family of rotating pulse solutions:

$$V_\gamma(t,x) = \tfrac{1}{\gamma}\mathrm{e}^{\mathrm{i}\psi t}[\cosh(\gamma x)]^{-(1+\mathrm{i}\theta)} \quad \text{with } \psi = (\gamma^2-1)\tfrac{(1+\theta^2)^2}{2\theta}.$$

For $\gamma = 1$ we find the standing pulse and the derivative with respect to γ at $\gamma = 1$ gives that $t\frac{(1+\theta^2)^2}{\theta}V_1(x) - W(x)$ with $W(x) = [1+(1+\mathrm{i}\theta)x\tanh x]V_1(x)$ solves the linear problem arising from (4.21) by linearizing around V_1. As V_1 is an eigenfunction to the eigenvalue 0 we have found the generalized eigenfunction W.

The numerics in [BeL98, BKT99] suggests that this additional real eigenvalue passes through 0 from left to right upon leaving the region $-1+2\alpha\theta+\theta^2 \leq 0$. Further off this region the new real unstable eigenvalue meets the old one to form a complex pair which then may move back into left complex halfplane. However, the pulses remain unstable due to the continuous spectrum. By now it is an open problem to show that all these pulses are unstable. Stable pulses are known to exist only in the generalized GLe (2.22), see [Kap96, Kap98, KaS98].

4.3.2 Spiral waves

Spiral waves are solutions on $\mathbb{R}^2$ whose spatial pattern (e.g. $\mathrm{Re}\, A(t,\cdot)$) looks like a spiral, while the whole pattern rotates with a fixed angular velocity γ around a fixed center x_*. We insert the ansatz

$$A(t,x) = B(R(\gamma t)[x-x_*]) \quad \text{with} \quad R(s) = \begin{pmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{pmatrix}$$

into the isotropic two–dimensional cGL $\partial_t A = (1+\mathrm{i}\alpha)\Delta A + \lambda_{0,1}\rho A - \widehat{c}|A|^2 A$ and find the elliptic PDE

$$(1+\mathrm{i}\alpha)\Delta_y B + \gamma\langle\nabla_y B, Jy\rangle + \lambda_{0,1}\rho B - \widehat{c}|B|^2 B = 0, \tag{4.24}$$

where $y = R(\gamma t)[x - x_*]$ and $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The phase invariance of cGL allows us to construct spiral wave solutions using ODE methods by searching B in the form

$$B(y) = b(|y|)\mathrm{e}^{\mathrm{i}n\varphi(y)} \quad \text{where} \quad \varphi(y) = \arg(y_1 + \mathrm{i}y_2).$$

Such solutions are called n–armed spirals since the real part of A has n positive and n negative regions when we circle around the center of the spiral. Problem (4.24) reduces to the nonautonomous ODE

$$(1+\mathrm{i}\alpha)\Big[b'' + \tfrac{1}{s}b' - \tfrac{n^2}{s^2}b\Big] + (\lambda_{0,1}\rho + \mathrm{i}\gamma n)b - \widehat{c}|b|^2 b = 0, \quad s = |y| \in [0,\infty).$$

This construction is a straightforward generalization of the Ginzburg–Landau vortices considered in Section 4.2.1, however, now the coefficients and $b(s)$ are complex, and we have the rotation velocity γ as an additional parameter. After division by $1+\mathrm{i}\alpha$ we arrive at the system

$$b'' + \frac{1}{s}b' - \frac{n^2}{s^2}b + \widetilde{\lambda}\,b - \widetilde{c}\,|b|^2 b = 0 \tag{4.25}$$

where $\widetilde{\lambda} = \frac{\lambda_{0,1}\rho + \mathrm{i}\gamma n}{1+\mathrm{i}\alpha}$ and $\widetilde{c} = \frac{\widehat{c}}{1+\mathrm{i}\alpha}$. Note that it is always possible to make $\widetilde{\lambda}$ real by choosing the appropriate rotation speed γ. However, $\widetilde{c}$ is fixed due to α and $\widehat{c}$ which can not be varied through the bifurcation parameters.

The case of real $\widetilde{\lambda}$ and $\widetilde{c}$ in (4.25) is well understood. In fact, for $\widetilde{\lambda}, \widetilde{c} > 0$ we obtain exactly rescaled versions of the Ginzburg–Landau vortices, see Subsection 4.2.1. For the case $\widetilde{\lambda}, \widetilde{c} < 0$ there are infinitely many solutions for each n which decay exponentially for $s \to \infty$ and have $k \in \mathbb{N}_0$ simple zeros, see [Sche98]. By perturbation arguments it is then possible to prove that all these solutions persist for small imaginary part of $\widetilde{c}$, in the sense that there is an appropriate rotation speed γ such that (4.25) still has a solution, see [Sche98, Sche99].

Existence results for spiral waves in general reaction–diffusion systems (which do not have the phase invariance of cGL) are given in [Sche99]. Stability questions and further bifurcations of spiral waves are treated in [SSW99].

4.4 Global existence for the complex Ginzburg–Landau equation

The local existence of solutions is classical on either bounded domains or unbounded domains when suitable boundary conditions are added. Here we discuss the main ideas typical for GLe and refer to the literature for the standard technical details of parabolic systems of PDEs, see [DG*88, BC*90, DoT93, DGL94, GiV96, LeO96, Mi97a, Mie98].

4.4.1 Bounded periodic domain

We consider the domain $\Omega = \Omega_\ell = (-\ell_1, \ell_1)\times \ldots \times(-\ell_d, \ell_d)$, which has volume $2^d\ell_1 \ldots \ell_d$, and complement cGL

$$\partial_t A = \sum_{j=1}^{d}(1+\mathrm{i}\alpha_j)\partial_j^2 A + \rho A - (1+\mathrm{i}\beta)|A|^2 \quad \text{on } \Omega_\ell \tag{4.26}$$

by periodic boundary conditions. This will allow us to do integrations by parts without worrying about boundary terms.

It is well–known that smooth solutions exist locally in time and moreover, if a solution stays bounded in $\mathrm{L}^p(\Omega)$ for some $p>d$, then it exists globally and is analytic in space and time, see e.g. [DoT93, LeO96]. In certain cases even distribution–valued initial conditions can be allowed ([LeO97]). Thus, global existence is established if we find a priori bounds of an appropriate L^p–norm. The standard energy estimate reads

$$\begin{aligned}\tfrac{1}{p}\tfrac{\mathrm{d}}{\mathrm{d}t}\textstyle\int_\Omega |A|^p\,\mathrm{d}x &= \operatorname{Re}\textstyle\int_\Omega |A|^{p-2}\overline{A}\,\partial_t A\,\mathrm{d}x\\ &= -\textstyle\sum_{j=1}^d Q^p_{j,1+\mathrm{i}\alpha}(A)+\int_\Omega[\rho|A|^p-|A|^{p+2}]\,\mathrm{d}x\\ \text{where } Q^p_{j,b}(A) &= \operatorname{Re} b\textstyle\int_\Omega \partial_j(|A|^{p-2}\overline{A})\partial_j A\,\mathrm{d}x.\end{aligned}$$

Simple algebraic manipulations (cf. [Mi97a, Mie98]) show that

$$Q^p_{j,b}(A)\geq 0,\quad \text{if } b\in\mathbb{C} \text{ satisfies}\quad \operatorname{Re} b\geq \frac{p-2}{2\sqrt{p-1}}|\operatorname{Im} b|. \tag{4.27}$$

(In fact, under this condition the integrand $\operatorname{Re} b\partial_j(|A|^{p-2}\overline{A})\partial_j A\geq 0$ of $Q^p_{j,b}$ is pointwise nonnegative.) From this we derive a simple L^p estimate if the linear dispersion parameters are not too large. Using

$$az-bz^{1+\gamma}\leq \gamma a([a/b]^{1/\gamma}-z)\quad \text{for all } a,b,z,\gamma>0, \tag{4.28}$$

we obtain the following result which implies that the L^p–norm cannot blow up.

Lemma 4.4
Assume $2\leq p\leq \frac{2\sqrt{\alpha_j^2+1}}{\sqrt{\alpha_j^2+1}-1}$ *for* $j=1,\ldots,d$, *then we have the estimate*

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\|A(t)\|^p_{\mathrm{L}^p}\leq \tfrac{2\rho}{p}\Big(\rho^{p/2}\mathrm{vol}(\Omega)-\|A(t)\|^p_{\mathrm{L}^p}\Big). \tag{4.29}$$

This settles the global existence questions in dimensions $d=1$ and 2 since it is always possible to choose $p>2\geq d$ in the above lemma. For $d\geq 3$ one obtains the restrictions $|\alpha_j|<2\sqrt{d-1}/(d-2)$.

There is another method involving cancellation properties in the nonlinear energy $\int_\Omega[\frac{\delta}{2}|\nabla A|^2+\frac14|A|^4]\,\mathrm{d}x$. This works especially well in the defocussing case ($\operatorname{sign}(\alpha_j\beta)>0$) but also in some limited regime of the focussing case. These estimates are also very useful in dimensions 1 and 2, when one wants to derive estimates on the L^∞–norm of solutions, see Section 4.4.3. Here we give the main idea and refer to [LeO96, Mi97a, Mie98] for a more detailed analysis leading to a wider range of allowed dispersion parameters. The first derivatives are estimated via

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_j A\|^2_{\mathrm{L}^2}=-\sum_{l=1}^d\|\partial_j\partial_l A\|^2_{\mathrm{L}^2}+\rho\|\partial_j A\|^2_{\mathrm{L}^2}-Q^4_{j,1+\mathrm{i}\beta}(A),$$

where we have used $\operatorname{Re}(1+\mathrm{i}\alpha_l)\int_\Omega \partial_j^2\overline{A}\,\partial_l^2 A\,\mathrm{d}x=\|\partial_j\partial_l A\|^2_{\mathrm{L}^2}$ which follows by integration by parts. It is an immediate consequence of this and the above L^2–estimate that the H^1–norm of $A(t,\cdot)$ cannot blow up if the nonlinear dispersion parameter β satisfies $|\beta|\leq\sqrt{3}$, since $Q^4_{j,1+\mathrm{i}\beta}(A)\geq 0$ by (4.27).

In the defocussing case we may assume, without loss of generality, that $\alpha_j, \beta \geq 0$. By the above we may further assume that $\beta \geq \sqrt{3}$. With $\delta_j = (1+\alpha_j)/\beta$ we let $F(A) = \sum_{l=1}^{d} \frac{\delta_j}{2}\|\partial_j A\|_{\mathrm{L}^2}^2 + \frac{1}{4}\|A\|_{\mathrm{L}^4}^4$, and by adding L^4 and the derivative estimates we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}F(A) \leq -\sum_{j,l=1}^{d} \delta_j \|\partial_j\partial_l A\|_{\mathrm{L}^2}^2 + \rho\Big(\sum_{j=1}^{d} \delta_j\|\partial_j A\|_{\mathrm{L}^2}^2 + \|A\|_{\mathrm{L}^4}^4\Big) - \|A\|_{\mathrm{L}^6}^6 - \sum_{j=1}^{d} Q_{j,z_j}^4(A)$$

where $z_j = 1-\mathrm{i}\alpha_j + \delta_j(1+\mathrm{i}\beta)$. The choice of δ_j guarantees $Q_{j,z_j}^4(A) \geq 0$. With $0 < \delta_* \leq \delta_j \leq \delta^*$ and $\|\partial_j A\|_{\mathrm{L}^2}^2 \leq \|\partial_j^2 A\|_{\mathrm{L}^2}\,\|A\|_{\mathrm{L}^2}$ we obtain

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}F(A) &\leq -\delta_* \textstyle\sum_1^d \|\partial_j^2 A\|_{\mathrm{L}^2}^2 + 2\rho\delta^* \sum_1^d \|\partial_j^2 A\|_{\mathrm{L}^2}\|A\|_{\mathrm{L}^2} + 2\rho\|A\|_{\mathrm{L}^4}^4 - \rho F(A) - \|A\|_{\mathrm{L}^6}^6 \\ &\leq -\rho F(A) + \tfrac{d\rho^2\delta^{*2}}{\delta_*}\|A\|_{\mathrm{L}^2}^2 + 2\rho\|A\|_{\mathrm{L}^4}^4 - \|A\|_{\mathrm{L}^6}^6 \;\leq\; -\rho F(A) + \rho^3 C\mathrm{vol}(\Omega), \end{aligned} \tag{4.30}$$

where $C = 4+4d^{3/2}(\delta^*/\delta_*)^3$. Again we conclude that $F(A(t,\cdot))$ cannot blow up and since $F(A)$ controls the norm in L^4 and H^1 we obtain global existence also for $d = 3$.

We summarize this result in the following statement and refer to [LeO96, GiV96, Mie98] where more dispersion parameters are allowed and the generalized nonlinearity $-(1+\mathrm{i}\beta)|A|^{2\sigma}A$ is considered.

Theorem 4.5
We consider dimensions $d \in \{1, 2, 3\}$ where in the case $d = 3$ additionally one of the following three conditions holds:

$$\text{(i) } |\beta| \leq \sqrt{3}, \quad \text{(ii) } \max_{j=1,\dots,d} |\alpha_j| < \sqrt{8}, \quad \text{(iii) } \min_{j=1,\dots,d} \alpha_j\beta \geq 0.$$

Then the solutions $A(t,\cdot)$ of (4.26) exist globally in time.

The above energy estimates also provide the existence of an absorbing ball, namely

$$\limsup_{t\to\infty} \|A(t)\|_{\mathrm{L}^p} \leq \rho^{1/2}[\mathrm{vol}(\Omega)]^{1/p}, \quad \limsup_{t\to\infty} \|\nabla A(t)\|_{\mathrm{L}^2}^2 \leq \rho C[\mathrm{vol}(\Omega)]^{1/2}, \tag{4.31}$$

which follows from (4.29) and (4.30), respectively.

4.4.2 Unbounded domain

To generalize the existence theory to unbounded domains we follow an approach initiated in [Col94] and developed further in [MiS95, Mi97a, Mie98] using weighted norms. Alternative approaches are given in [BaV90, GiV96]. Here we restrict ourselves to the case $\Omega = \mathbb{R}^d$ and refer to the given references for more general unbounded domains.

The main idea is to obtain a priori bounds in the uniformly local norms $\mathrm{L}_{\mathrm{ul}}^p(\mathbb{R}^d)$ via weighted energy estimates. We show this in the simplest case of the $\mathrm{L}_{\mathrm{ul}}^2$–bound from which it will become clear that introducing the weight will introduce only a few lower order terms in the energy estimates. Thus, the sign properties of the leading terms are not changed. Moreover, the volume of the domain will no longer be relevant but only the integral of the weight over the domain.

Recall the weight $w(x) = \mathrm{e}^{-|x|}$ which satisfies the two important properties $|\nabla w(x)| \leq C_1 w(x)$ and $\int_{\mathbb{R}^d} w(x)\,\mathrm{d}x \leq C_w$. With this and $\widehat{\alpha} = \max\{\,|1+\mathrm{i}\alpha|\;:\; j=1,\ldots,d\,\}$ we have

$$\begin{aligned}
\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} w|A|^2\,\mathrm{d}x \;&=\; \mathrm{Re}\int_{\mathbb{R}^d}\overline{A}\,\partial_t A\,\mathrm{d}x\\
&=\; -\int_{\mathbb{R}^d}\{w|\nabla A|^2 + \mathrm{Re}\,\overline{A}\textstyle\sum_1^d(1+\mathrm{i}\alpha_j)\partial_j w\,\partial_j A\}\,\mathrm{d}x + \int_{\mathbb{R}^d} w[\rho|A|^2 - |A|^4]\,\mathrm{d}\xi\\
&\leq\; \int_{\mathbb{R}^d}\{-w|\nabla A|^2 + \widehat{\alpha}|A||\nabla w||\nabla A|\}\,\mathrm{d}x + \int_{\mathbb{R}^d} w[\rho|A|^2 - |A|^4]\,\mathrm{d}\xi\\
&\leq\; \int_{\mathbb{R}^d} w[C_3|A|^2 - |A|^4]\,\mathrm{d}x \;\leq\; \int_{\mathbb{R}^d} w\,C_3(C_3 - |A|^2)\,\mathrm{d}x \;\leq\; C_3[C_3C_w - \int_{\mathbb{R}^d} w|A|^2\,\mathrm{d}x],
\end{aligned}$$

where $C_3 = (\widehat{\alpha}C_1/2)^2 + \rho$. This implies

$$\|A(t)\|^2_{\mathrm{L}^2_w} \leq \mathrm{e}^{-2C_3 t}\|A(0)\|^2_{\mathrm{L}^2_w} + C_3 C_w.$$

Now we can use the fact that the same estimate holds, if the weight is replaced by the translated weight $w(\cdot - y)$. Taking the supremum over $y \in \mathbb{R}^d$ we find the estimate

$$\|A(t)\|_{\mathrm{L}^2_{\mathrm{ul}}} \leq \mathrm{e}^{-C_3 t}\|A(0)\|_{\mathrm{L}^2_{\mathrm{ul}}} + (C_3 C_w)^{1/2}.$$

Similarly the estimates for $\|A\|_{\mathrm{L}^p_{\mathrm{ul}}}$ and $\|\partial_j A\|_{\mathrm{L}^2_{\mathrm{ul}}}$ can be obtained by generalizing the estimates with the constant weight given in the previous subsection. The following result is a special case of [Mi97a, Thm. 4.3].

Theorem 4.6
Let $\Omega \subset \mathbb{R}^d$ be a general domain and impose either Dirichlet or Neumann boundary conditions for the solutions of GLe. Under the same assumptions as in Theorem 4.27 the GLe (4.26) defines a global semiflow on $\mathrm{L}^p_{\mathrm{ul}}(\Omega)$ for all $p > d$. Moreover, this semiflow has an absorbing ball.

The use of weighted spaces has several positive effects. First, we are able to capture phenomena which are typical in the theory of unbounded domains, namely spatially quasi-periodic solutions, fronts, and pulses. Second, the energy estimates done above with the constant weight 1 can be generalized easily to the present case, as we explained above. Third, we are able to find absorbing balls for GLe in $\mathrm{L}^p_{\mathrm{ul}}(\mathbb{R}^d)$ whereas in $\mathrm{L}^p(\mathbb{R}^d)$ we only have a global semiflow where solutions, however, may grow to infinity. This is already seen in rGL (4.7) with $d = 1$, since there are solutions $A(t,x) \approx b(|x| - 2t)$ for $\tau \gg 1$ where b is the traveling front profile solving $b'' + 2b' + b - b^3 = 0$ with $b(\infty) = 0$ and $b(-\infty) = 1$.

The fourth advantage of weighted spaces is that they allow even for improvements on the estimates for bounded domains. We come back to this point at the end of this section. However, before that we state a result on the existence of the global attractor. For bounded domains, e.g. in the periodic case this theory is classical (cf. [Hal88, BaV92, Tem88]). In Section 5 we present a theory which is directly applicable to GLe even for unbounded domains. It uses the weighted norm $\mathrm{L}^p_w(\Omega)$. Together with the smoothing properties of parabolic systems the weighted norm generates compactness; viz., a bounded set in $\mathrm{W}^{1,p}_{\mathrm{ul}}(\Omega)$ is compact in $\mathrm{L}^p_w(\Omega)$.

The following result is proved in [MiS95, Mi97a] and is a special case of Theorem 5.1 below.

Theorem 4.7
Under the assumptions of Theorem 4.6 the semiflow of the GLe has a global attractor $\mathcal{A} = \mathcal{A}^\Omega \subset \mathrm{L}^p_{\mathrm{ul}}(\Omega)$ *with the following properties:*

(i) $\mathcal{A}$ *is nonempty, bounded and closed in* $\mathrm{L}^p_{\mathrm{ul}}(\Omega)$*; and* $\mathcal{A}$ *is compact in* $\mathrm{L}^p_w(\Omega)$*.*
(ii) $\mathcal{A}$ *is invariant under the semiflow* $(\mathcal{G}_t(\cdot))_{t\geq 0}$ *of GLe, i.e.* $\mathcal{G}_t(\mathcal{A}) = \mathcal{A}$ *for all* $t \geq 0$*.*
(iii) $\mathcal{A}$ *attracts all bounded sets in* $\mathrm{L}^p_{\mathrm{ul}}(\Omega)$ *in the semidistance induced by the norm* $\|\cdot\|_{\mathrm{L}^p_w}$ *(see (5.2)).*
(iv) $\mathcal{A}$ *is invariant under all symmetries of GLe and the domain, in particular under the phase invariance* $A(\cdot) \mapsto \mathrm{e}^{\mathrm{i}\gamma}A(\cdot)$*. If* $\Omega = \mathbb{R}^d$ *then* $\mathcal{A} = \mathcal{A}^{\mathbb{R}^d}$ *is translationally invariant.*

The conditions (i) to (iii) are the definition of an $(\mathrm{L}^p_{\mathrm{ul}}(\Omega), \mathrm{L}^p_w(\Omega))$–attractor.

4.4.3 Bounds, length scales and turbulence

The cGL serves as the simplest model to study turbulence, see [BC*90, Man90, LuD93, BJ*98]. The notions 'soft' and 'hard turbulence' are used to distinguish two different regimes of temporal chaos. In the case of soft turbulence the spatial behavior of the solutions remains rather simple whereas in the case of hard turbulence the solutions display rapid spatial changes like spikes and vortices which have much smaller length scales than the other typical structures. Often soft turbulence is also called 'weak' turbulence and hard turbulence is called 'fully developed' turbulence.

A connection between the smallest length scale in turbulence and bounds in the L^∞–norm for solutions is derived in [BGO96] for the cGL and the Navier–Stokes equations. Hence, it is desireable to derive upper bounds for the solutions of cGL. Turbulence is related to the flow on the attractor, so the diameter of the attractor $\mathcal{A}$ has to be estimated in dependence of the dispersion parameters:

$$\mathcal{D}(\rho, \alpha_1, \ldots, \alpha_d, \beta, \Omega) = \sup\{\, \|A\|_{\mathrm{L}^\infty} \;:\; A \in \mathcal{A}^\Omega(\rho, \alpha_1, \ldots, \alpha_d, \beta)\,\}.$$

A measure of turbulence is now obtained by the size of $\mathcal{D}$. As follows from (4.29) with $p = 2$ we have the simple bound $\|A\|_{\mathrm{L}^2} \leq [\rho \mathrm{vol}(\Omega)]^{1/2}$ for all $A \in \mathcal{A}^\Omega$. Hence, if $\mathcal{D}$ is significantly larger than $\rho^{1/2}$ then the solutions must display spatially localized behavior (spikes), and we might expect hard turbulence.

In the case of a finite domain with periodic boundary conditions the estimates (4.31) provide exponents $\gamma, \delta > 0$ such that

$$\mathcal{D}(\rho, \alpha_1, \ldots, \alpha_d, \beta, \Omega) \leq \widetilde{C}(\alpha_1, \ldots, \alpha_d, \beta)\rho^\gamma[\mathrm{vol}(\Omega)]^\delta.$$

To this end the Gagliardo–Nirenberg estimate $\|u\|_\infty \leq C\|u\|^\theta_{\mathrm{W}^{k,q}_{\mathrm{ul}}}\|u\|^{1-\theta}_{\mathrm{L}^p_{\mathrm{ul}}}$ with suitable θ is employed. Additionally, the scaling (see (2.15)) $(\widetilde{\xi}, \widetilde{\tau}, \widetilde{\rho}, \widetilde{A}) = (r\xi, r^2\tau, r^{-2}\rho, r^{-1}A)$ with $\mathrm{vol}(\widetilde{\Omega}) = r^d\mathrm{vol}(\Omega)$ gives $1 - 2\gamma + d\delta = 0$. Since $\delta \geq 0$ we find $\gamma \geq 1/2$. Indeed, when the weight $w \equiv 1$ is employed, one often obtains $\gamma > 1/2$, see [BC*90, BGO96].

The periodic solutions can also be treated as a special subclass in $\mathrm{L}^p_{\mathrm{ul}}(\mathbb{R}^d)$. Our a priori bounds in this space and the scaling invariance of cGL on all of $\mathbb{R}^d$ immediately

$$\mathcal{D}(\rho, \alpha_1, \ldots, \alpha_d, \beta, \mathbb{R}^d) = \rho^{1/2}\overline{C}(\alpha_1, \ldots, \alpha_d, \beta). \tag{4.32}$$

In [Mie98] the constants $\overline{C}(\alpha,\ldots,\alpha,\beta)$ (isotropic case) were estimated leading to the upper bounds

$$d=1: \quad \overline{C}(\alpha,\beta) \le \begin{cases} C_0[(1+\alpha^2)^{1/4}(1+\beta^2)^{-1/12} + (1+\beta^2)^{1/6}] & \text{for } 1+\alpha\beta + \frac{5}{3}|\alpha-\beta| \ge 0, \\ C_0(1+\alpha^2)^{1/2}(1+\beta^2)^{1/4} & \text{else,} \end{cases}$$

$$d=2: \quad \overline{C}(\alpha,\alpha,\beta) \le \begin{cases} C_0(1+\alpha^2)^{1/6}(1+\alpha^2+\beta^2) & \text{for } 1+\alpha\beta + \frac{5}{3}|\alpha-\beta| \ge 0, \\ C_0[(1+\alpha^2)^2(1+\beta^2)]^{|\alpha|+1/8} & \text{else.} \end{cases}$$

For $d=1$ the upper bound is polynomial in $|\alpha|$ and $|\beta|$ since we are still in the subcritical regime. For $d=2$ the upper bound in the focussing regime is exponential in $|\alpha|$.

It is interesting to note that, using perturbation arguments for the nonlinear Schrödinger equation (nlS), there is certain evidence for an exponential lower bound. For the focussing nlS with $d=2$ (that is (2.17) with $\alpha_1=\alpha_2, -\beta\to\infty$) one knows that solutions blow up in finite time. In [FiL98] the nlS was perturbed by adding small dissipation and small linear forcing in the form $\partial_t A = \mathrm{i}(\Delta A + |A|^2 A) + \varepsilon_1 A + \varepsilon_2 \Delta A - \varepsilon_3 |A|^2 A$. Using formal perturbation arguments along the blow–up solutions they establish that for small $\varepsilon_j > 0$ the solution behaves similarly to the blow–up solutions until the amplitude reaches a value approximately given by

$$\|A(0,\cdot)\|_\infty \approx c_0(\varepsilon_2+2\varepsilon_3)^{-1/6}\mathrm{e}^{c_1/(\varepsilon_2+2\varepsilon_3)},$$

for some positive constants c_0 and c_1. After that value is reached the dissipation wins over the focussing effect and the solution defocusses. See also [LuD93, Wil98] for numerical simulations showing formation and decay of such spikes. The numerical evidence suggests that these spike–forming solutions lie on the unstable manifold of the spatially homogeneous solution $A(t,x) = (\varepsilon_1/\varepsilon_3)^{1/2}\mathrm{e}^{\mathrm{i}\varepsilon_1 t/\varepsilon_3}$. Hence, these spike forming solutions lie in the global attractor $\mathcal{A}^\Omega$. Inserting our definitions $\varepsilon_1=\rho$, $\varepsilon_2 = 1/\alpha$ and $\varepsilon_3 = 1/\beta$ this suggests a lower bound for $\mathcal{D}$ of the form

$$C_0(\rho|\beta|)^{1/2}\Big(\frac{\alpha|\beta|}{\alpha+|\beta|}\Big)^{1/6}\mathrm{e}^{C_1\alpha|\beta|/(\alpha+|\beta|)}.$$

Recent numerical results in [Wil99] support the exponential lower bound.

Another way to measure the chaotic or turbulent behavior of cGL is obtained through the dimension of its attractor $\mathcal{A}$ rather than its diameter. Of course, most estimates for the dimension also depend on good estimates on the size of the solutions. For bounded domains the finite–dimensionality of the attractor was proved in [GhH87]. There it was shown that the dimension of the attractor is proportional to the volume of the domain when all other parameters are kept fixed, namely

$$\dim_{\mathrm{H}}(\mathcal{A}^\Omega) \le C(\rho,\alpha,\beta)\mathrm{vol}(\Omega).$$

Another approach to the dimension of attractors on large and unbounded domains is given in [CoE99a]. There cGL was studied directly on $\mathbb{R}^d$. For $\varepsilon,\ell>0$ one defines $N(\varepsilon,\ell)$ to be the minimal number of balls in $\mathrm{L}^\infty((0,\ell)^d)$ which are needed to cover the subset $\{\,A|_{(0,\ell)^d} : A\in\mathcal{A}^{\mathbb{R}^d}\}$. Then one shows that the limit $H(\varepsilon) = \lim_{\ell\to\infty}\ell^{-d}\log N(\varepsilon,\ell)$ exists. Finally one can show that the following *dimension per unit volume* is finite:

$$\mathrm{dim.u.v}(\mathcal{A}^{\mathbb{R}^d}) = \limsup_{\varepsilon\to 0+}\frac{\log H(\varepsilon)}{\log(1/\varepsilon)}.$$

It is interesting to note that this new approach does not use any volume forms in lower dimensions and thus does not need any Hilbert space structure. It is based on estimates which characterized the influence of the initial conditions in one place on the solution at later times at distant places. One such estimate is given in (5.3). In the more recent work [CoE99b] also the topological entropy per unit volume is defined.

5 Attractors for large and unbounded domains

5.1 Existence of attractors

For this section we generalize our point of view and consider general parabolic PDEs

$$u_t = \operatorname{div}(D\nabla u) + f(u) \quad \text{with } x \in \Omega \subset \mathbb{R}^d,\ u(t,x) \in \mathbb{R}^m, \tag{5.1}$$

on a large domain $\Omega \subset \mathbb{R}^d$ where 'large' means $\{\, x \in \mathbb{R}^d \,:\, |x| < \ell \,\} \subset \Omega$ for some $\ell \gg 1$. This also includes general unbounded domains.

Throughout we assume that the diffusion matrix D, the nonlinearity f, the boundary $\partial\Omega$ and the boundary conditions 'bc' are such that we have global existence of smooth solutions defining a global semiflow $\mathcal{S}^{\Omega}_{\text{bc}}(t,\cdot)$ on $\mathrm{L}^p_{\text{ul}}(\Omega)$ for a suitable $p \in (d,\infty)$. Moreover, we assume that there exists a radius R which is independent of Ω such that $B_{\text{abs}} = \{\, u \in \mathrm{L}^p_{\text{ul}}(\Omega) \,:\, \|u\|_{p,\text{ul}} \leq R \,\}$ is an absorbing set for the semigroup (see [Col94, Mi97a, GiV96, Mie98] for a construction of such bounds).

Under these assumptions we have the following existence result for the global attractor $\mathcal{A}^{\Omega}_{\text{bc}}$. The case of bounded Ω is classical ([BaV92, Hal88, Tem88]) and that of unbounded Ω is treated in [BaV90, MiS95, Fei96, Mi97a]. For the formulation we recall the definition (4.2) of the weighted norm $\|\cdot\|_{\mathrm{L}^p_w}$ and the associated weighted L^p–space $\mathrm{L}^p_w(\Omega)$.

Theorem 5.1
Under the above assumptions the semiflow $\mathcal{S}^{\Omega}_{\text{bc}}(t,\cdot)$ has a global $(\mathrm{L}^p_{\text{ul}}(\Omega), \mathrm{L}^p_w(\Omega))$–attractor $\mathcal{A}^{\Omega}_{\text{bc}}$ with the following properties:

(a) it is flow invariant, i.e., $\mathcal{S}^{\Omega}_{\text{bc}}(t,\mathcal{A}^{\Omega}_{\text{bc}}) = \mathcal{A}^{\Omega}_{\text{bc}}$ for all $t \geq 0$,
(b) it is nonempty, bounded and closed in $\mathrm{L}^p_{\text{ul}}(\Omega)$, and compact in $\mathrm{L}^p_w(\Omega)$,
(c) it attracts bounded sets B in $\mathrm{L}^p_{\text{ul}}(\Omega)$ in the L^p_w–distance, i.e.,

$$\operatorname{dist}_{\mathrm{L}^p_w}(\mathcal{S}^{\Omega}_{\text{bc}}(t,B), \mathcal{A}^{\Omega}_{\text{bc}}) \to 0 \text{ for } t \to \infty \tag{5.2}$$

where $\operatorname{dist}_{\mathrm{L}^p_w}(F,G) = \sup\{\, \inf\{\, \|u-v\|_{\mathrm{L}^p_w} \,:\, v \in G \,\} \,:\, u \in F \,\}$.

The important feature here is the usage of two different norms. The weighted norm localizes the distance measure and thus restores compactness whenever we are on an unbounded domain, viz., bounded sets in $\mathrm{W}^{1,p}_{\text{ul}}(\Omega)$ are precompact in $\mathrm{L}^p_w(\Omega)$. For $\Omega = \mathbb{R}^d$ it is shown in [BaV90, CoE99a] that the attractor $\mathcal{A}^{\mathbb{R}^d}$ is typically infinite dimensional.

There are other cases of attractors for reaction–diffusion equations on unbounded domains or for the Navier–Stokes equation on unbounded cylinders which are different from the theory above. There the nonlinearity is such that the solutions converge exponentially to $u \equiv 0$ if no external forcing is added. Now the system is forced by a function $g(x)$ which decays exponentially for $|x| \to \infty$. Then it is shown that the semiflow has a compact attractor of finite Hausdorff dimension. The functions in the attractor decay all exponentially for $|x| \to \infty$, see e.g. [BaV90, EfS97, EfM99] and the references therein.

5.2 Comparison of attractors

There are two different cases of comparing attractors of PDEs. Either we study the same PDE on two different domains, which is done in this section, or we may study two different PDEs on the same domain. The latter point of view will be addressed in Section 6.5 where the attractor of GLe is compared with that of the full problem.

If the same PDE is posed on two different domains, we want to compare the functions in the attractors on the intersection of these domains. To this end we simply embed $\mathrm{L}^p_{\mathrm{ul}}(\Omega)$ into $\mathrm{L}^p_{\mathrm{ul}}(\mathbb{R}^d)$ by extending all functions with 0 outside of Ω. Clearly this embedding preserves the norm.

The basic comparison result between the flows on different domains or with different boundary conditions can be roughly formulated as follows, see [Mi97a, CoE99a] for more precise statements. Take two domains $\Omega_j \subset \mathbb{R}^d$ and associated boundary conditions (bc_j) such that the semiflows $\mathcal{S}^j_t = \mathcal{S}^{\Omega_j}_{\mathrm{bc}_j}(t,\cdot)$ are well–defined. Furthermore assume $\{\, x \in \mathbb{R}^d \; : \; |x| < \ell \,\} \subset \Omega_j$ for $j = 1, 2$. Then, for each $R > 0$, there exists a constant $C > 1$ which is independent of Ω_1 and Ω_2 such that

$$\|\mathcal{S}^1_t(u_1) - \mathcal{S}^2_t(u_2)\|_{\mathrm{L}^p_w} \le C\mathrm{e}^{Ct}(\|u_1 - u_2\|_{\mathrm{L}^p_w} + [\ell^{d-1}\mathrm{e}^{-\ell}]^{1/p}) \tag{5.3}$$

for all $u_j \in \mathrm{L}^p_{\mathrm{ul}}(\Omega_j)$ with $\|u_j\|_\infty \le R$. Here the second term on the right–hand side contains all the influence of the different domains and boundary conditions whereas the first term is a standard Lipschitz condition. In particular, if $u_1(x) = u_2(x)$ for $|x| \le r < \ell$ then the first term is less than $C[r^{d-1}\mathrm{e}^{-r}]^{1/p}$ which provides a good control over the difference of the solutions over even smaller balls.

Using (5.3) and certain variants of it, it is not difficult to show the following upper semicontinuity results, see [Mi97a].

Theorem 5.2
Under the above assumptions for each $\varepsilon > 0$ there exists an $\ell > 0$ such that for all domains $\Omega \subset \mathbb{R}^d$ with $\{\, x : |x| < \ell \,\} \subset \Omega$ and all $\mathrm{bc} \in \{\mathrm{Dir}, \mathrm{Neu}, \mathrm{Per}\}$ we have $\mathrm{dist}_{\mathrm{L}^p_w}(\mathcal{A}^\Omega_{\mathrm{bc}}, \mathcal{A}^{\mathbb{R}^d}) \le \varepsilon$.

The remaining open problem is under what conditions we also have lower semicontinuity of the attractor, i.e., $\mathrm{dist}_{\mathrm{L}^p_w}(\mathcal{A}^{\mathbb{R}^d}, \mathcal{A}^\Omega_{\mathrm{bc}}) \le \varepsilon$ for suitably large domains Ω. In the next section we give a simple example showing that lower semicontinuity is in general false. In our example the limit of $\mathcal{A}^{(-\ell,\ell)}_{\mathrm{bc}}$ for $\ell \to \infty$ exists for each $\mathrm{bc} \in \{\mathrm{Dir}, \mathrm{Neu}, \mathrm{Per}\}$, but it depends on bc and is strictly contained in $\mathcal{A}^{\mathbb{R}}$.

The problem of convergence of $\mathcal{A}^{\Omega_n}_{\mathrm{bc}_n}$ to $\mathcal{A}^{\mathbb{R}^d}$ is a problem of interchanging the limit $t \to \infty$ in the definition of the attractors with the limit $\Omega_n \to \mathbb{R}^d$. The interchange of such limits would only be possible, if the attraction rates for the attractors $\mathcal{A}^{\Omega_n}_{\mathrm{bc}_n}$ are uniform in n; however, this is in general not the case. In [HaR93] a generalized limit attractor was introduced which allows for double limits where $t_n \to \infty$ simultaneously with the parameter $\varepsilon_n \to 0$. In our situation this leads to the following definition. Consider a sequence $(\mathrm{bc}_n, \Omega_n)_{n\in\mathbb{N}}$ of boundary conditions and domains which approach $\mathbb{R}^d$ in the sense that $\{\, x \; : \; |x| < \ell_n \,\} \subset \Omega_n$ with $\ell_n \to \infty$. Now set

$$\mathcal{A}^* = \Big\{\, u \in \mathrm{L}^p_{\mathrm{ul}}(\mathbb{R}^d) \;\; : \;\; \exists\, t_n \text{ with } t_n \to \infty \;\exists\, u_n \in \mathrm{L}^p_{\mathrm{ul}}(\Omega_n) \text{ with } \sup_{n\in\mathbb{N}} \|u_n\|_{p,\mathrm{ul}} < \infty \\ \text{such that } \|\mathcal{S}^{\Omega_n}_{\mathrm{bc}_n}(t_n, u_n) - u\|_{\mathrm{L}^p_w(\mathbb{R}^d)} \to 0 \text{ for } n \to \infty \,\Big\}.$$

The motivation to consider $\mathcal{A}^*$ rather than w–lim $\mathcal{A}^{\Omega_n}_{\mathrm{bc}_n}$ is that for practical purposes (e.g., in experiments or numerics) we always have to consider finite time and bounded domains. Thus, one should neither prescribe the limit $t \to \infty$ before $n \to \infty$ nor the opposite, see [HaR93] for a further discussion.

Theorem 5.3
For all sequences $(\mathrm{bc}_n, \Omega_n)_{n\in\mathbb{N}}$ *as above we have* $\mathcal{A}^* = \mathcal{A}^{\mathbb{R}^d}$.

Proof: We first show $\mathcal{A}^{\mathbb{R}^d} \subset \mathcal{A}^*$. To this end consider $u \in \mathcal{A}^{\mathbb{R}^d}$. As the sequence $\ell_n \to \infty$ for $n \to \infty$ is given as above, we can choose a sequence t_n with $t_n \to \infty$ such that the right–hand side in (5.3) tends to 0 if we use $t = t_n$, $\ell = \ell_n$ and $u_1 = u_2$. By the invariance of $\mathcal{A}^{\mathbb{R}^d}$ there exist $v_n \in \mathcal{A}^{\mathbb{R}^d} \subset \mathrm{L}^p_{\mathrm{ul}}(\mathbb{R}^d)$ such that $\mathcal{S}^{\mathbb{R}^d}(t_n, v_n) = u$. Defining $u_n \in \mathrm{L}^p_{\mathrm{ul}}(\Omega_n)$ via $u_n = v_n|_{\Omega_n}$ and using (5.3) we obtain the desired result $u = w\text{–}\lim_{n\to\infty} \mathcal{S}^{\Omega_n}_{\mathrm{bc}_n}(t_n, u_n)$, since $\|v_n - u_n\|_{p,w}$ can also be estimated by $C[\ell_n^{d-1}\mathrm{e}^{-\ell_n}]^{1/d}$.

For the opposite inclusion we note that $\mathcal{A}^*$ is contained in B_{abs} and hence it is a bounded set in $\mathrm{L}^p_{\mathrm{ul}}(\mathbb{R}^d)$. If we show that $\mathcal{A}^*$ is also negatively invariant (i.e., $\mathcal{A}^* \subset \mathcal{S}^{\mathbb{R}^d}(t, \mathcal{A}^*)$ for all $t > 0$), then we have the desired inclusion $\mathcal{A}^* \subset \mathcal{A}^{\mathbb{R}^d}$. Let $u \in \mathcal{A}^*$ and $t > 0$, then we have to find $\widetilde{u} \in \mathcal{A}^*$ such that $u = \mathcal{S}^{\mathbb{R}^d}(t, \widetilde{u})$. Take the sequence (t_n, u_n) with $u = w\text{–}\lim_{n\to\infty} v_n$ where $v_n = \mathcal{S}^{\Omega_n}_{\mathrm{bc}_n}(t_n, u_n)$ according to the definition of $\mathcal{A}^*$. The sequence $\widetilde{v}_n = \mathcal{S}^{\Omega_n}_{\mathrm{bc}_n}(t_n - t, u_n)$ is precompact in $\mathrm{L}^p_w(\mathbb{R}^d)$ and, after taking a subsequence if necessary, we have $\widetilde{u} = w\text{–}\lim_{n\to\infty} \widetilde{v}_n$. With $v_n = \mathcal{S}^{\Omega_n}_{\mathrm{bc}_n}(t, \widetilde{v}_n)$ and (5.3) we conclude

$$\begin{aligned} \|u - \mathcal{S}^{\mathbb{R}^d}(t, \widetilde{u})\|_{\mathrm{L}^p_w} &\leq \|u - \mathcal{S}^{\Omega_n}_{\mathrm{bc}_n}(t, \widetilde{v}_n)\|_{\mathrm{L}^p_w} + \|\mathcal{S}^{\Omega_n}_{\mathrm{bc}_n}(t, \widetilde{v}_n) - \mathcal{S}^{\mathbb{R}^d}(t, \widetilde{u})\|_{\mathrm{L}^p_w} \\ &\leq \|u - v_n\|_{\mathrm{L}^p_w} + C\mathrm{e}^{Ct}\Big[\|\widetilde{v}_n - \widetilde{u}\|_{\mathrm{L}^p_w} + [\ell_n^{d-1}\mathrm{e}^{-\ell_n}]^{1/p}\Big] \to 0 \end{aligned}$$

for $n \to \infty$. This gives the desired result $u = \mathcal{S}^{\mathbb{R}^d}(t, \widetilde{u})$. □

5.3 An example with different limit attractors

We consider the scalar parabolic equation

$$u_t = u_{xx} + f(u) \quad \text{with } f(u) = -u(1-u)^2 \tag{5.4}$$

for $x \in (-\ell, \ell)$ where $\ell > 0$. The essential feature of the nonlinearity f is that the ODE $\dot{u} = f(u)$ has exactly two equilibria, namely $u = 0$ and $u = 1$. While $u = 0$ is a stable equilibrium, the second equilibrium $u = 1$ is degenerate and semistable.

If $\ell < \infty$ we take one of the following boundary conditions:
(Dir): $u(-\ell) = u(\ell) = 0$,
(Neu): $u_x(-\ell) = u_x(\ell)$,
(Per): $u(-\ell) = u(\ell)$ and $u_x(-\ell) = u_x(\ell)$.
The initial value problem is well–posed and we have a global semiflow on $\mathrm{L}^2_{\mathrm{ul}}((-\ell, \ell))$ which we denote by $(\mathcal{S}^\ell_{\mathrm{bc}}(t, \cdot))_{t\geq 0}$. The associated attractors are denoted by $\mathcal{A}^\ell_{\mathrm{bc}}$. More general boundary conditions could be allowed, but this would complicate the notations and the analysis considerably.

For $\ell < \infty$ we can study the attractor explicitly as the flow is a gradient flow with respect to the energy

$$E(u) = \int_{-\ell}^{\ell} \Big[u_x^2/2 - F(u)\Big]\mathrm{d}x \quad \text{with } F(u) = \int_0^u f(v)\mathrm{d}v = -(6u^2 - 8u^3 + 3u^4)/12.$$

The standard theory for attractors for flows with a Liapunov function (see [Hal88, Tem88, BaV92]) tells us that the attractor consists exactly of the equilibria and their heteroclinic connections. Denote by $\mathcal{E}_{\mathrm{bc}}^\ell$ the set of equilibria of the semiflow $\mathcal{S}_{\mathrm{bc}}^\ell$, then

$$\mathcal{E}_{\mathrm{Dir}}^\ell = \{0\} \quad \text{and} \quad \mathcal{E}_{\mathrm{Neu}}^\ell = \mathcal{E}_{\mathrm{Per}}^\ell = \{0, 1\}.$$

We obtain the following result for the global attractors.

Theorem 5.4
For all $0 < \ell < \infty$ we have $\mathcal{A}_{\mathrm{Dir}}^\ell = \{0\}$ and $\mathcal{A}_{\mathrm{Neu}}^\ell = \mathcal{A}_{\mathrm{Per}}^\ell = \mathcal{A}_*$ where $\mathcal{A}_* = \{\, u \in \mathrm{L}^p((-\ell, \ell)) \,:\, \exists\, \theta \in [0,1] : u \equiv \theta \,\}$.

For (Dir) the result is clear since the only equilibrium $u \equiv 0$ is exponentially stable. For (Neu) and (Per) we use the fact that the linearization at $u \equiv 1$ has the eigenvalues $-(k\pi/\ell)^2$, $k \in \mathbb{Z}$. There is a one–dimensional center manifold associated to the eigenvalue 0, which, in fact, is given by the x–independent functions. Thus, the only solution connecting $u \equiv 1$ with $u \equiv 0$ is the spatially constant solution $u(t,x) = \theta(t)$, where $\dot\theta = f(\theta)$.

From Theorem 5.4 we immediately obtain the limits

$$\underset{\ell\to\infty}{w\text{–}\lim}\, \mathcal{A}_{\mathrm{Dir}}^\ell = \{0\}, \quad \underset{\ell\to\infty}{w\text{–}\lim}\, \mathcal{A}_{\mathrm{Neu}}^\ell = \underset{\ell\to\infty}{w\text{–}\lim}\, \mathcal{A}_{\mathrm{Per}}^\ell = \mathcal{A}_* \,. \tag{5.5}$$

Here $w\text{–}\lim_{\ell\to\infty} A(\ell) = B$ means $\mathrm{dist}_{\mathrm{L}^2_w}(A(\ell), B) + \mathrm{dist}_{\mathrm{L}^2_w}(B, A(\ell)) \to 0$ for $\ell \to \infty$.

Finally we show that the attractor $\mathcal{A}^\infty$ as defined above is larger than $\mathcal{A}_*$. This establishes the example where the limits of the attractors exist, depend on the boundary conditions and are strictly smaller than the limit attractor. Clearly we have $\mathcal{A}_* \subset \mathcal{A}^\infty$.

However, there are more solutions, for instance traveling waves of the form $u(t,x) = z(x - ct)$ with z satisfying the ODE $z'' + cz' + f(z) = 0$. If $c = 1/\sqrt{2}$ then we have the explicit solution $z(x) = \mathrm{e}^{x/\sqrt{2}}/(1 + \mathrm{e}^{x/\sqrt{2}})$. Moreover, for each $c \geq 1/\sqrt{2}$ there is a unique solution z_c which is monotone and satisfies $z_c(0) = 1/2$, $z_c(-\infty) = 0$ and $z_c(\infty) = 1$. For $|c| < 1/\sqrt{2}$ there are no nonconstant bounded solutions and for $c \leq -1/\sqrt{2}$ we have the solution $z_c(x) = z_{-c}(-x)$.

This shows that the limit attractor $\mathcal{A}^\infty$ contains at least the two–dimensional set

$$\mathcal{B} = \mathcal{A}_* \cup \{\, z_c(\cdot - y) \,:\, |c| \geq c_*,\ y \in \mathbb{R} \,\}.$$

In the space $\mathrm{L}^2_w(\mathbb{R})$ this set forms a closed subset which is homeomorphic to the closed unit disk since $|z_c'(x)| \leq K/|c|$ implies that $z_c(\cdot - cy)$ converges for $c \to \infty$ to a constant function which depends smoothly and monotonically on y. Together we have shown

$$\underset{\ell\to\infty}{w\text{–}\lim}\, \mathcal{A}_{\mathrm{Dir}}^\ell \subsetneq \underset{\ell\to\infty}{w\text{–}\lim}\, \mathcal{A}_{\mathrm{Neu}}^\ell = \underset{\ell\to\infty}{w\text{–}\lim}\, \mathcal{A}_{\mathrm{Per}}^\ell \subsetneq \mathcal{B} \subset \mathcal{A}^\infty.$$

We conjecture that in this special situation we have $\mathcal{B} = \mathcal{A}^\infty$. A similar conjecture was formulated in [Hof99] for the attractor in the KPP equation $u_t = u_{xx} + u - u^2$ when only nonnegative solutions are considered. There, the infinite–dimensional unstable manifold of $u \equiv 0$ is intersected with the cone of positive functions; and it seems that only a finite–dimensional global attractor remains.

6 Justification of the Ginzburg–Landau formalism

Finally we treat the question how good the formally obtained Ginzburg–Landau approximation really is. We recall the full problem

$$\partial_t u = L(\mu, \partial_x)u + N(\mu, \partial_x, u) \tag{6.1}$$

from (2.1) where $u(t, \cdot) : \Omega \to \mathbb{R}^m$ is the unknown function on the unbounded cylindrical domain $\Omega = \mathbb{R}^d \times \Sigma$ with Σ being a bounded cross–section. We will denote the solutions by $u(t) = \mathcal{S}_t^\varepsilon(u_0)$ where $(\mathcal{S}_t^\varepsilon)_{t\geq 0}$ is a (local) semiflow.

With $\mu = \rho\varepsilon^2$ the modulation ansatz takes roughly the form

$$u(t,x) = U_A(t,x) + \mathcal{O}(\varepsilon^2) \quad \text{where } U_A(t,x) = \varepsilon A(\varepsilon^2 t, \varepsilon(x - c_g t))\mathrm{e}^{\mathrm{i}(\omega + \langle k_0, x\rangle)}\Phi_{0,k_0} + \text{c.c.}. \tag{6.2}$$

Here $\mathrm{i}\omega\Phi_{0,k_0} = L(0, \mathrm{i}k_0)\Phi_{0,k_0}$ and $A(\tau, \xi)$ solves the associated GLe

$$\partial_\tau A = \operatorname{div}_\xi \Big(\Lambda_0 \nabla_\xi A\Big) - \rho\lambda_{0,1} A + \widehat{c}|A|^2 A. \tag{6.3}$$

The solutions of the GLe are denoted by $A(\tau) = \mathcal{G}_\tau(A_0)$ where $(\mathcal{G}_\tau)_{\tau\geq 0}$ is the (local) semiflow.

The questions we want to ask are the following.

1. What informations do we obtain for the full problem when studying the GLe?
2. Do solutions A of GLe generate via the modulation ansatz good approximations of solutions of the full problem?
3. Are the solutions obtained from the Ginzburg–Landau formalism the typical ones which are seen in experiments?
4. Do special solution classes in GLe correspond to similar solution classes in the full problem? If yes, can we deduce stability information for the full problem?
5. Are inertial manifolds or the attractor of GLe related to those of the full problem?

Here we are, of course, interested in rigorous mathematical estimates. Our aim is to show existence of solutions which can be described by the GLe. Moreover, we want to emphasize that the Ginzburg–Landau formalism allows us to prove global existence results for the full problem, e.g. for the three–dimensional Navier–Stokes equations, see below.

We insist that it is not enough for the justification procedure to take nice solutions of GLe, insert them in the modulation ansatz and then show that the error in the full problem is small. Without further assumptions this does not say that the full problem has a solution which really behaves like this. A warning example is given in [Schn95b] where it is shown that the Newell–Whitehead equation (2.23) is a formal modulation equation which does not correctly predict the dynamics of the full problem.

6.1 Kirchgässner reduction, spatial center-manifold theory

The first mathematically satisfactory justifications of the GLe where given for the spatially one–dimensional case and were restricted to either time–independent solutions of time–periodic solutions for the case $\lambda(\mu, k) \in \mathbb{R}$ and the case $\lambda(0, k_0) = \mathrm{i}\omega \neq 0$, respectively.

The main idea is to consider the full problem (6.1) as a spatial dynamical system where $x \in \mathbb{R}$ plays the role of time. Thus, we solve for the derivatives in x–direction and write the system as

$$\frac{\mathrm{d}}{\mathrm{d}x} v = K(\mu, \partial_y, \partial_t)v + F(\mu, v) \tag{6.4}$$

where F is the nonlinearity and K the leading linear operator. The new variable is constructed using u and some of its x–derivatives, e.g. $v = (u, \partial_x u)^{\mathrm{t}}$ if (6.1) was second order.

Because of the relation between the two linear problems $\partial_t u = L(\mu, \partial_x, \partial_y)u$ and $(\mathrm{d}/\mathrm{d}x)v = K(\mu, \partial_y, \partial_t)v$ we know that $\lambda = \mathrm{i}\widetilde{\omega}$ is an eigenvalue of $L(\mu, \mathrm{i}k)$ if and only if $\mathrm{i}k$ is an eigenvalue of $K(\mu, \partial_y, \mathrm{i}\widetilde{\omega})$. We temporarily reintroduced the cross–sectional derivatives ∂_y to display the generality of the approach. But further on we will drop ∂_y again.

The spectral assumption for $L(\mu, \mathrm{i}k)$ in Section 2 immediately provides spectral assumptions on $K(\mu, \mathrm{i}\omega)$, namely $\operatorname{spec} K(0, \mathrm{i}\omega) \cap \mathrm{i}\mathbb{R} = \{\mathrm{i}k_0, -\mathrm{i}k_0\}$. These eigenvalues are geometrically simple, however, their algebraic multiplicity may be higher. It was proved in [AfM95, Thm. 2.1] that the multiplicity is m if and only if λ has the expansion

$$\lambda(0, k) = \mathrm{i}\omega + c(k - k_0)^m + \mathcal{O}(|k - k_0|^{m+1}),$$

for some $c \neq 0$. Thus, in the real case (Case 1 in Section 2) and in the complex case with $k_0 = 0$ (Case 3) we immediately obtain $m = 2$. In the latter case we use $\lambda(0, -k) = \lambda(0, k)$. In Case 2 we typically have $c_g = (\mathrm{d}/\mathrm{d}k)\lambda(0, k_0) \neq 0$ and the eigenvalues are simple.

The spatial center manifold reduction was first developed in [Kir82], which is why it is also called Kirchgässner reduction. This method reduces the original elliptic partial differential equation on a cylindrical domain to an ODE with respect to the axial variable. The method was further developed in [Mie88] for quasilinear or non–autonomous problems. Time–periodic problems are treated in [Kir84, IoM91, IoM92]. To apply it we choose an appropriate phase space X, e.g. a closed subspace of $[\mathrm{L}^2(\Sigma)]^{2m}$ in the time–independent setting and of $[\mathrm{L}^2(\Sigma \times \mathbb{S}^1)]^{2m}$ for the time–periodic case. Here $\Sigma \subset \mathbb{R}^n$ is the bounded cross–section of the cylinder Ω and $\mathbf{S}^1$ denotes the time axis modula the period. Thus, in the time–periodic case the real time variable t takes the role of a cross–sectional variable while $x \in \mathbb{R}$ is considered as unbounded time–like variable instead.

To apply the center manifold theory we split the phase space X into $X_1 \oplus X_2$, where X_1 is the spectral part of $K(0, \partial_t)$ associated to the eigenvalues on the imaginary axis whereas X_2 associates with all the remaining spectrum. Thus we have $\dim X_1 = 4$ in Case 1 and Case 3 and $\dim X_1 = 2$ for Case 2. If $P_j : X \to X_j$ are the associated spectral projectors, then (6.4) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}x} v_1 = K_1 v_1 + F_1(\mu, \omega, v_1 + v_2), \quad \frac{\mathrm{d}}{\mathrm{d}x} v_2 = K_2 v_2 + F_2(\mu, \omega, v_1 + v_2),$$

where $K_j = K(0, \partial_t)|_{X_j}$, $F_j(\mu, \omega, v) = P_j(K(\mu, \partial_t) - K(0, \partial_t) + F(\mu, v))$.

Since the eigenvalues of K_2 are bounded away from the imaginary axis, one expects that all small bounded solutions can be described by the critical variable v_1 alone. In fact, the spatial center–manifold reduction states that all small bounded solutions lie on a locally invariant manifold which is a smooth graph over a neighborhood of $0 \in X_1$:

$$\mathcal{M}_{\text{center}} = \{\, v_1 + h(\mu, \omega, v_1) \;:\; \|v_1\| \leq \delta \,\}.$$

This now allows us to reduce the study of all small solutions which are bounded in x–direction to the ODE

$$\frac{\mathrm{d}}{\mathrm{d}x}v_1 = K_1 v_1 + F_1(\mu,\omega,v_1{+}h(\varepsilon,\omega,v_1)), \tag{6.5}$$

where v_2 was eliminated by the help of the reduction function $h(\varepsilon,\omega,\cdot): X_1 \to X_2$.

We now consider the three spectral cases separately.

Case 1: $\lambda(\mu,k)\in\mathbb{R}$ **and** $k_0\neq 0$.

For this case $\dim X_1 = 4$ and K_1 has the eigenvalues $\pm\mathrm{i}k_0$ with a nontrivial Jordan block. Thus after choosing appropriate coordinates $(A_1,A_2)\in\mathbb{C}^2$ in X_1 the ODE reads

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix}A_1\\A_2\end{pmatrix} = \begin{pmatrix}\mathrm{i}k_0 & 1\\ 0 & \mathrm{i}k_0\end{pmatrix}\begin{pmatrix}A_1\\A_2\end{pmatrix} + \begin{pmatrix}N_1(\mu,A_1,A_2,\overline{A}_1,\overline{A}_2)\\ N_2(\mu,A_1,A_2,\overline{A}_1,\overline{A}_2)\end{pmatrix}, \tag{6.6}$$

where we have omitted the complex conjugate equations. Doing a normal form transformation up to cubic order we may assume that the nonlinearities N_j have the form

$$\begin{aligned} N_1(\mu,\ldots) &= c_1A_1 + \mathcal{O}(|A_1|^3{+}|A_2^3|),\\ N_2(\mu,\ldots) &= c_2A_1 + c_3A_2 + c_4|A_1|^2A_1 + \mathcal{O}(|A_1|^5{+}|A_1^2A_2|{+}|A_2|^3), \end{aligned}$$

where $c_j = c_j(\mu)$ are some complex constants with $c_j(0)=0$ for $j=1,2,3$. As in the Ginzburg–Landau modulation ansatz we may now let $A_j(x) = \mathrm{e}^{\mathrm{i}k_0x}B_j(x)$ and insert this into (6.6). The obtained ODE becomes x–dependent, however, only in the higher order terms:

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix}B_1\\B_2\end{pmatrix} = \begin{pmatrix}c_1 & 1\\ c_2 & c_3\end{pmatrix}\begin{pmatrix}B_1\\B_2\end{pmatrix} + \begin{pmatrix}\text{h.o.t.}\\ c_4|B_1|^2B_1 + \text{h.o.t.}\end{pmatrix}.$$

Now we can introduce the same scaling as in the modulation ansatz $B_1(\varepsilon\xi) = \varepsilon A(\xi)$, $B_2(\varepsilon\xi) = \varepsilon^2 C(\xi)$ and $\mu = \rho\varepsilon^2$ giving an ODE for $(A,C)\in\mathbb{C}^2$. Solving the first equation for $C = (\mathrm{d}/\mathrm{d}\xi A) + \mathcal{O}(\varepsilon)$ we arrive at a second order equation for $A(\xi)$:

$$\tfrac{\mathrm{d}^2}{\mathrm{d}\xi^2}A - c_2'(0)\rho A - c_4(0)|A|^2A + g(\varepsilon,\rho,\xi/\varepsilon,A,\overline{A},\tfrac{\mathrm{d}}{\mathrm{d}\xi}A,\tfrac{\mathrm{d}}{\mathrm{d}\xi}\overline{A}) = 0 \tag{6.7}$$

where $g(\varepsilon,\ldots) = \mathcal{O}(\varepsilon)$ for $\varepsilon\to 0$. Thus, after neglecting the term of order ε we have derived the steady one–dimensional GLe.

Of course, the final steps in this procedure are also formal like the derivation of the GLe itself. However, here we are only dealing with an ODE. By this way, neglecting certain terms of higher order can be justified more easily. However, the main purpose in using the Kirchgässner reduction for the justification of the GLe is **not** to derive (6.7). We should rather say that (6.6) is the correct reduced ODE and this ODE can be studied rigorously by taking into account all higher order terms. For instance it is possible to show the existence of a family of periodic orbits which corresponds exactly to the family of rotating waves. Using the principle of reduced stability as discussed in [Mi97b, Mi97c] it is also possible to proof the stability results given in Section 4.1. We refer to [IoMD89] for more details and for a treatment of the Couette–Taylor problem as described in Section 3.2.

A further discussion of the GLe as a normal form equation is given in [Mel98]. There it is shown that the nonautonomous terms in (6.7) involving ξ/ε can be pushed to arbitrarily high order.

Case 2: $\lambda(0,k) = \mathrm{i}\omega + \mathrm{i}c_g(k{-}k_0) + \text{h.o.t.}$ **with** $c_g, k_0 \neq 0$

In this situation the center manifold is two–dimensional and the ODE can be written as

$$\frac{\mathrm{d}}{\mathrm{d}x}B = \mathrm{i}k_0 B + N(\mu, \widetilde{\omega}, B, \overline{B})$$

on the center manifold and it cannot be directly related to the GLe, see [IoM91, Sec. 4]. This system is SO(2)–invariant under the transformation $B \mapsto \mathrm{e}^{\mathrm{i}\phi}B$ as a consequence to the time–invariance of (6.4). We obtain trivially solutions of the from $B(x) = r\mathrm{e}^{\mathrm{i}k}$ with $k \approx k_0$ which correspond to the rotating waves of GLe.

Case 3: $k_0 = 0$ **and** $\lambda(0,0) = \mathrm{i}\omega$

We again have $\dim X_1 = 4$. The operator K_1 has an algebraically four–fold eigenvalue which is geometrically two–fold. After some normal form transformation the ODE can be written as

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} c_1(\mu,\widetilde{\omega}) & 1 \\ c_2(\mu,\widetilde{\omega}) & c_3(\mu,\widetilde{\omega}) \end{pmatrix} + \begin{pmatrix} \text{h.o.t.} \\ c_4(\mu,\widetilde{\omega})|A_1|^2 A_1 + \text{h.o.t.} \end{pmatrix}. \tag{6.8}$$

Again the system is SO(2)–invariant under the transformation $B \mapsto \mathrm{e}^{\mathrm{i}\phi}B$ as a consequence of the time–invariance of (6.4).

We employ the scaling $\xi = \varepsilon x$, $A_1 = \varepsilon A$, $A_2 = \varepsilon^2 C$, $\mu = \rho\varepsilon^2$ and $\widetilde{\omega} = \omega + \varepsilon\sigma$, then we may solve the first equation for C as a function of A and $\mathrm{d}A/\mathrm{d}\xi$ and obtain the following second order ODE for $A(\xi)$:

$$\tfrac{\mathrm{d}^2}{\mathrm{d}\xi^2}A + b_1\sigma\tfrac{\mathrm{d}}{\mathrm{d}\xi}A + (b_2\rho{+}b_3\sigma^2)A + b_4|A|^2A + \widetilde{g}(\varepsilon,\rho,\sigma,A,\overline{A},\tfrac{\mathrm{d}}{\mathrm{d}\xi}A,\tfrac{\mathrm{d}}{\mathrm{d}\xi}\overline{A}) = 0. \tag{6.9}$$

This case was treated in [IoM92] an applied to a reaction–diffusion system. In [AfM95, AfM99b] the spatial center–manifold reduction was used to prove existence of periodic and multi–pulse solutions (cf. Section 4.3.1) for the Poiseuille problem as described in Section 3.3.

6.2 The approximation property

We now return to the full time–dependent problem and want to study the approximation properties of the modulation ansatz when evaluated along solutions of GLe.

The general idea is to use the modulation ansatz as a first guess of the solution, insert the function $u(t,x) = V_A(t,x) + r(t,x)$ in the full problem and then try to show that the remainder remains small for all $x \in \mathbb{R}^d$ and for sufficiently long times. Here it is important to recall that the solutions $A(\tau,\xi)$ have to be rescaled via $t = \varepsilon^2\tau$ when they are inserted in the full problem. Hence, to see the action of GLe on an initial condition $A(0,\cdot)$ in the GLe we have to observe the solution in an interval $\tau \in [0,\tau_0]$ with $\tau_0 > 0$ independent of ε. Hence in the full problem the relevant time interval is $t \in [0,\tau_0/\varepsilon^2]$ which is a very large time scale.

Inserting $u = V_A + r$ in (6.1) we obtain the equation

$$\begin{aligned} \partial_t r =& L_\varepsilon r + N(\mu, V_A{+}r) - N(\mu, V_A) - \mathrm{Res}(V_A)(t,x), \\ &\text{with } \mathrm{Res}(V_A) = \partial_t V_A - L(\mu,\partial_x)V_A - N(\mu,V_A), \end{aligned} \tag{6.10}$$

where $L_\varepsilon = L(\rho\varepsilon^2, \partial_x)$. The term $\mathrm{Res}(V_A)$ is the residual which we obtain upon insertion of V_A. Assuming that A is a solution of the GLe (6.3) and by adding corrections if necessary we can arrange things such that R is sufficiently small, let us say $\mathcal{O}(\varepsilon^m)$. Moreover, the nonlinearity $N(\mu,\cdot)$ satisfies a Lipschitz continuity

$$\|N(\mu,u) - N(\mu,\widetilde{u})\| \leq C_N[\|u\|+\|\widetilde{u}\|]^n \|u-\widetilde{u}\|$$

for sufficiently small $u, \widetilde{u}$. Here $n = 1$ if N starts with quadratic terms and $n = 2$ if it starts with cubic ones. The linear operator L_ε generates a semigroup $(\mathrm{e}^{tL_\varepsilon})_{t\geq 0}$ which satisfies the bound

$$\|\mathrm{e}^{tL_\varepsilon}\| \leq C_1 \mathrm{e}^{c_2\varepsilon^2 t} \quad \text{for } t \geq 0.$$

With this and $\|V_A\| = \mathcal{O}(\varepsilon)$ we try to estimate r by the help of variation of constants formula for semigroups

$$r(t) = \mathrm{e}^{tL_\varepsilon} r(0) + \int_{s=0}^t \mathrm{e}^{(t-s)L(\varepsilon)}[N(\mu, V_A(s)+r(s)) - N(\mu, V_A(s)) - \mathrm{Res}(V_A)(s)]\,\mathrm{d}s.$$

Letting $m_r(t) = \sup_{s\in[0,t]} \|r(s)\|$ and similarly m_{Res} and m_{V_A} we obtain the estimate

$$m_r(t) \leq C_1 \mathrm{e}^{c_1\varepsilon^2 t}\|r(0)\| + \textstyle\int_{s=0}^t C_1 \mathrm{e}^{c_1\varepsilon^2(t-s)}(m_{\mathrm{Res}}(s)+C_N[m_r(s)+m_{V_A}(s)]^n\|m_r(s)\|)\,\mathrm{d}s.$$

For times $t \in [0, \tau_1/\varepsilon^2]$ with $\tau_1 > 0$ fixed and $m_{\mathrm{Res}}(t) \leq C_R\varepsilon^j$, $m_{V_A}(t) \leq C_A\varepsilon$ on this interval we arrive at

$$m_r(t) \leq D_1\|r(0)\| + \frac{D_2}{\varepsilon^2}\Big(C_R\varepsilon^j + C_N[m_r(t)+C_A\varepsilon]^n m_r(t)\Big).$$

It is immediately clear that this simple approach only works if $n \geq 2$. Indeed, to show that $m_r(t)$ remains small for $t \in [0, \tau_1/\varepsilon^2]$ we need $n \geq 2$ since the prefactor $D_2\varepsilon^{-2}C_N(C_A\varepsilon)^n$ must be bounded independently of ε.

These simple arguments can be made rigorous for the problems with cubic nonlinearities, see [CoE90, KSM92] and [MiS95, Sect. 3.1]. However, for general problems, e.g. those in hydrodynamics, another approach is needed. This was introduced in [vHa91] and further developed in [Schn94a, Schn95c, BSvH95]. Applications to several hydrodynamical problems are now available, namely for the Rayleigh–Bénard problem in [Schn94b], the Couette–Taylor problem in [Schn99a, Schn99b] and for the Poiseuille problem in [Bol96].

The general observation which is behind the analysis is that the modulated solutions of the basic pattern have Fourier transforms which are strongly concentrated around the two points $\pm k_0$. Through nonlinear interactions these Fourier clusters generate new modes around the points mk_0 with $m \in \mathbb{Z}$. However, modes with $|m| \neq 1$ are not critical and hence they are exponentially damped. In particular, quadratic interactions of the basic pattern generate only modes with $m \in \{-2, 0, 2\}$. Thus, the basic observation is:

$$\textit{Quadratic interactions of critical modes are noncritical.} \tag{6.11}$$

So far the argument was only true under the assumption $k_0 \neq 0$. However, the case $k_0 = 0$ with $\lambda(0,0) = \mathrm{i}\omega \neq 0$ can be treated as well since a similar mode coupling occurs in space–time, see [Schn98a].

We now give the mathematics in more detail. We consider (6.1) in suitable function spaces $\widehat{X} \subset X$, which contain modulated solutions. Typically one takes $\widehat{X}$ and X in the form $[\mathrm{W}^{k,p}_{\mathrm{ul}}(\Omega)]^m$ with $k_{\widehat{X}} \geq k_X$. Now assume that $A : [0, \tau_1]\times\mathbb{R}^d \to \mathbb{C}$ is a given solution of the associated GLe. Moreover, we use a modulation ansatz $V_A(t, x) = U_A + \mathcal{O}(\varepsilon^2)$ such that the associated residual $\mathrm{Res}(V_A)$, see (6.10), satisfies an estimate

$$\|\mathrm{Res}(V_A)(t, \cdot)\|_X \leq C_R\varepsilon^j \quad \text{for } t \in [0, \tau_1/\varepsilon^2].$$

This can always be achieved by using the fact that solutions of GLe are arbitrarily smooth and then doing the formal expansion of Section 2.2.2 to the desired high order, see e.g. [IoMD89, MiS96].

The main functional analytic tool involves the splitting of the critical modes (also called Ginzburg–Landau modes) and the remaining stable modes. This can't be done in a simple manner as we have to cut through the continuous spectrum of the operator L_ε. Hence we do not define an exact splitting using projections but use filters E_{c} and E_{s} instead of projections P_{c} and P_{s}. (Here 'c' and 's' are abbreviations for 'central' and 'stable'.) The filters are defined such that $E_{\mathrm{c}} + E_{\mathrm{s}} = I_X$ and, if we define the spaces $X_\alpha = \mathrm{closure}(E_\alpha X)$ for $\alpha \in \{\mathrm{c}, \mathrm{s}\}$, then $E_\alpha X_\alpha = X_\alpha$. Note that $E_{\mathrm{c}}E_{\mathrm{s}} \neq 0$, and hence $X_{\mathrm{c}} \cap X_{\mathrm{s}}$ is still an infinite dimensional subspace containing some stable modes.

The construction of E_{c} is done by the help of Fourier transform. We choose a $\delta > 0$ independent of ε and a smooth cut–off function $\chi_{\pm k_0} : \mathbb{R}^d \to [0, 1]$ which is equal to 1 for $|k\pm k_0| \leq \delta/2$ and is 0 if $|k-k_0|$, $|k+k_0| \geq \delta$. Using the eigenfunction $\Phi_{\rho\varepsilon^2,k}$ of $L(\rho\varepsilon^2, \mathrm{i}k)$ associated to the eigenvalue $\lambda(\rho\varepsilon^2, k)$ together with the eigenvector $\Phi^*_{\rho\varepsilon^2,k}$ of the adjoint problem with $\langle\Phi_{\rho\varepsilon^2,k}, \Phi^*_{\rho\varepsilon^2,k}\rangle = 1$ we define E_{c} through its Fourier transform

$$\mathcal{F}_x[E_{\mathrm{c}}u](k) = \chi_{\pm k_0}(k)\langle\mathcal{F}_x[u](k), \Phi^*_{\rho\varepsilon^2,k}\rangle\Phi_{\rho\varepsilon^2,k}. \tag{6.12}$$

Here the functions Φ and Φ^* of k need only be defined in the two balls of radius δ around k_0 and $-k_0$.

This formula is certainly well–defined for functions $u \in \mathrm{L}^2(\Omega)$ and it is established in [Schn94b, Lem. 5] that the operator E_{c} is a bounded linear operator from $\mathrm{L}^2_{\mathrm{ul}}(\Omega)$ into $\mathrm{W}^{k,2}_{\mathrm{ul}}(\Omega)$. This arises from the fact that all $\Phi_{\varepsilon,k}$ are smooth functions of $y \in \Sigma$ and that the Fourier transform in x–direction of $E_{\mathrm{c}}u$ is localized. Additionally the construction is such that E_{c} commutes with L_ε, i.e. $E_{\mathrm{c}}L_\varepsilon = L_\varepsilon E_{\mathrm{c}}$. By this definition E_{c} depends on ε, but for notational convenience we suppress the dependence.

We can now write solutions $u(t)$ of (6.1) in the form $u(t) = u_{\mathrm{c}}(t)+u_{\mathrm{s}}(t)$ with $u_\alpha(t) \in X_\alpha$. However, we warn the reader that generally $u_\alpha \neq E_\alpha u$ since we define the two parts via

$$\partial_t u_{\mathrm{c}} = L_\varepsilon u_{\mathrm{c}} + E_{\mathrm{c}}N(\varepsilon, u_{\mathrm{c}} + u_{\mathrm{s}}), \quad \partial_t u_{\mathrm{s}} = L_\varepsilon u_{\mathrm{s}} + E_{\mathrm{s}}N(\varepsilon, u_{\mathrm{c}} + u_{\mathrm{s}}). \tag{6.13}$$

The quantitative version of the quadratic interaction principle (6.11) takes the quantitative form

$$\|E_{\mathrm{c}}N(\varepsilon, u_{\mathrm{c}})\| = \mathcal{O}(\|u_{\mathrm{c}}\|^3).$$

To be more precise we make the following assumptions on the nonlinearity:

$$\begin{aligned} &\|E_\alpha[N(\varepsilon, u_{\mathrm{c}}+u_{\mathrm{s}}) - N(\varepsilon, \widetilde{u}_{\mathrm{c}}+\widetilde{u}_{\mathrm{s}})]\|_X \\ \leq\;& C_N\Big([\|u_{\mathrm{c}}\|_Y+\|\widetilde{u}_{\mathrm{c}}\|_Y]^{n_\alpha}+\|u_{\mathrm{s}}\|_Y+\|\widetilde{u}_{\mathrm{s}}\|_Y\Big)\|u_{\mathrm{c}}-\widetilde{u}_{\mathrm{c}}\|_Y \\ &+ C_N\Big(\|u_{\mathrm{c}}\|_Y+\|\widetilde{u}_{\mathrm{c}}\|_Y+\|u_{\mathrm{s}}\|_Y+\|\widetilde{u}_{\mathrm{s}}\|_Y\Big)\|u_{\mathrm{s}}-\widetilde{u}_{\mathrm{s}}\|_Y, \end{aligned} \tag{6.14}$$

where $n_c = 2$ and $n_s = 1$. Of course, this is assumed to hold only for $\|u_\alpha\|_Y, \|\widetilde{u}\|_Y \leq \delta_1$.

The linear operator L_ε generates a semigroup which, by construction, satisfies the estimates

$$\|e^{tL_\varepsilon}\|_{X_c \to Y_c} \leq C e^{c\varepsilon^2 t}, \ \|e^{tL_\varepsilon}\|_{X_s \to Y_s} \leq C t^{-\beta} e^{-\gamma t}, \ \|e^{tL_\varepsilon}\|_{Y_s \to Y_s} \leq C e^{-\gamma t}, \tag{6.15}$$

for $t > 0$, where $C, \gamma > 0$ and $\beta \in [0, 1)$.

The approximation result can now be formulated as follows.

Theorem 6.1
Let the above assumptions be satisfied. Choose any $\tau_1 > 0$ and any solution $A = A(\tau, \xi) : [0, \tau_1] \to \mathbb{R}^d \to \mathbb{C}$ of the GLe (6.3). Assume that there are $\kappa > 1$, $\varepsilon_0 > 0$ and $C_R, C_A > 0$ such that V_A is an approximate solution of (6.1) with

$$\|V_A(t, \cdot)\|_Y \leq C_A \varepsilon, \quad \|E_c \mathrm{Res}(V_A)(t, \cdot)\|_X \leq C_R \varepsilon^{\kappa+2}, \quad \|E_s \mathrm{Res}(V_A)(t, \cdot)\|_X \leq C_R \varepsilon^{\kappa+1},$$

for all $\varepsilon \in (0, \varepsilon_0)$ and all $t \in [0, \tau_1/\varepsilon^2]$.

Then for all $D > 0$ there exist $\varepsilon_1 \in (0, \varepsilon_0)$ and $C > 0$ such that for all $\varepsilon \in (0, \varepsilon_1)$ the following holds. If the initial datum $u_0 \in Y$ satisfies

$$\|E_c[u_0 - V_A(0, \cdot)]\|_Y \leq D\varepsilon^\kappa \quad \text{and } \|E_s[u_0 - V_A(0, \cdot)]\|_Y \leq D\varepsilon^{\max\{1, \kappa-1\}},$$

then the unique solution $u(t) = \mathcal{S}_t^\varepsilon(u_0)$ of (6.1) exists for $t \in [0, \tau_1/\varepsilon^2]$ and satisfies the estimate

$$\|E_c[u(t, \cdot) - V_A(t, \cdot)]\|_Y \leq C\varepsilon^\kappa, \text{and } \|E_s[u(t, \cdot) - V_A(t, \cdot)]\|_Y \leq C[e^{-\gamma t}\varepsilon^{\max\{1, \kappa-1\}} + \varepsilon^{\kappa+1}]$$

for all $t \in [0, \tau_1/\varepsilon^2]$.

If $\kappa > 1$ then the errors are relatively small compared to the solution u as well as compared to the approximation V_A.

Proof: We insert the ansatz $u = (E_c V_A + r_c) + (E_s V_A + r_s)$ in (6.1) and obtain by the variation of constant formula

$$r_\alpha(t) = e^{tL_\varepsilon} r_\alpha(0) + \int_0^t e^{(t-s)L_\varepsilon} E_\alpha\Big(\mathrm{Res}(V_A) + N(\varepsilon, V_A(s) + r(s)) - N(\varepsilon, V_A(s))\Big)\,\mathrm{d}s$$

for $\alpha \in \{c, s\}$. Using the abbreviation $m_c(t) = \sup_{s\in[0,t]} \|r_c\|_Y$ and $m_s(t) = \|r_s(t)\|_Y$, (6.14) and (6.15) together with the temporary assumptions

$$m_c(t) \leq C_c \varepsilon^\kappa \leq C_A \varepsilon \leq \delta_1, \quad m_s(t) \leq C_s \varepsilon \leq \delta_1, \tag{6.16}$$

we find the estimates

$$\begin{aligned} m_c(t) &\leq C e^{c\varepsilon^2 t} m_c(0) + \textstyle\int_0^t C e^{c\varepsilon^2(t-s)} (C_R \varepsilon^{\kappa+2} + C_N[4C_A^2 \varepsilon^2 m_c + 2\varepsilon(C_A + C_s) m_s])\,\mathrm{d}s, \\ m_s(t) &\leq C e^{-\gamma t} m_s(0) + \textstyle\int_0^t C(t-s)^{-\beta} e^{-\gamma(t-s)} (C_R \varepsilon^{\kappa+1} + C_N[2C_A \varepsilon m_c + \delta_1 m_s])\,\mathrm{d}s. \end{aligned}$$

The second estimate has the advantage of an exponentially decaying term in the integrand. With this, one can show that there exists a constant $\widehat{c}$ which depends only on $\delta_1, \beta, \gamma, C$ and C_N (but not on C_c and C_s) such that

$$m_s(t) \leq \widehat{c}(e^{-\gamma t} m_s(0) + C_R \varepsilon^{\kappa+1} + C_N C_A \varepsilon^{\kappa+1}). \tag{6.17}$$

Clearly, the estimate for m_s in (6.16) is satisfied when letting $C_\mathrm{s} = \widehat{c}(D + C_R + C_N C_A)$ and assuming $\varepsilon_1 \leq \min\{1, \delta_1/C_\mathrm{s}\}$.

Estimate (6.17) can now be inserted in the first inequality. For $M_\mathrm{c}(t) = \mathrm{e}^{-c\varepsilon^2 t} m_\mathrm{c}(t)$ we obtain the estimate

$$\begin{aligned} M_\mathrm{c}(t) \ \leq \ & a_1 + a_2 t + b \int_0^t M_\mathrm{c}(s)\,\mathrm{d}s \quad \text{with } b = 4\varepsilon^2 C_N C_A^2,\ a_1 = C m_\mathrm{c}(0) + \varepsilon C_2 m_\mathrm{s}(0), \\ & \text{and } a_2 = C_R \varepsilon^{\kappa+2} + 2CC_N(C_A + C_\mathrm{s})\widehat{c}\,\varepsilon(C_R \varepsilon^{\kappa+1} + C_N C_A \varepsilon^{\kappa+1}). \end{aligned}$$

Using Gronwall's inequality we obtain the estimate $m_\mathrm{c}(t) \leq \mathrm{e}^{c\varepsilon^2 t} M_\mathrm{c}(t) \leq \mathrm{e}^{c\varepsilon^2 t}[a_1 \mathrm{e}^{bt} + a_2(\mathrm{e}^{bt} - 1)/b)]$ and for $t \in [0, \tau_1/\varepsilon^2]$ we obtain

$$m_\mathrm{c}(t) \leq C_3[m_\mathrm{c}(0) + \varepsilon\, m_\mathrm{s}(0) + \varepsilon^\kappa], \tag{6.18}$$

where C_3 is still independent of $\varepsilon \leq \varepsilon_1$ and C_c. We now choose $C_\mathrm{c} = C_3(2D + 1)$ and by making ε_1 smaller if necessary we are able to fulfill the assumptions for m_c in (6.15).

The estimates (6.17) and (6.18) establish the result. $\square$

6.3 The attractivity of the set of modulated patterns

Here we treat the question why initial conditions in the modulated form $u_0 = U_A$ are of such an importance. We say that such a u_0 has a modulated structure. The GLe is formally derived to describe the evolution of the linearly unstable modes. In weakly nonlinear theory one expects that these modes dominate the dynamics of the full problem under arbitrary initial conditions. In the previous section the error estimates were shown under the assumption that the initial conditions already possess the modulated structure. Here we explain why, after the time $t_1 = T_1/\varepsilon^2$, all small solutions of the full problem develop this mode structure and, henceforth, can be described by the solutions of the associated GLe.

The first result in this direction was obtained in [Eck93]. For our choice of function spaces the result is given in [Schn94c]. Since the Ginzburg–Landau formalism is only a local theory, this attractivity can only be expected for initial conditions in a small neighborhood of the trivial solution.

Our Theorem 6.1 given above already gives a first result in that direction. If we start with u_0 such that $\|E_\mathrm{s} u_0\|_Y \leq (C_A + D)\varepsilon$ and a suitable part $E_\mathrm{c} u_0$, then after time $t = \tau_1/\varepsilon^2$ we have $\|E_\mathrm{s} u(t, \cdot)\|_Y \leq \|E_\mathrm{s} V_A\| + C\varepsilon^{\kappa+1}$ and it is easy to see that $E_\mathrm{s} V_A$ is of order ε^2. Since $\kappa > 1$ we conclude that the stable part is automatically reduced from size ε down to size ε^2. However, the attractivity result we need to be able to apply the Ginzburg–Landau formalism is much stronger since we also have to show that the modes of the central part, considered in Fourier space, concentrate around the critical modes $\pm k_0$.

To formulate the results more precisely we have to define an operator which extracts the modulation function A from a modulated pattern. The key step for isolating the Ginzburg–Landau mode $A \in Z$ from a general function $u \in Y$ is to find an approximate inverse of the mapping $A \mapsto U_A$. We introduce the linear operator

$$K_\varepsilon : \begin{cases} Z & \to \quad Y, \\ A & \mapsto \quad \varepsilon^{1-d}\Big[\mathcal{F}_k^{-1}\{\chi_0(k - k_0)[\mathcal{F}_\xi A](\tfrac{k-k_0}{\varepsilon})\Phi_{\rho\varepsilon^2, k}\} + \text{c.c.}\Big], \end{cases} \tag{6.19}$$

see also (2.10). Here χ_0 is a cut–off function with $\chi_0(k) = 1$ for $|k| \leq \delta/2$ and $= 0$ for $|k| \geq \delta$. The approximate inverse is defined as

$$M_\varepsilon : \begin{cases} Y & \to & Z, \\ u & \mapsto & \varepsilon^{d-1}\mathcal{F}_\kappa^{-1}\{\chi_0(\varepsilon\kappa)\langle[\mathcal{F}_x u](k_0{+}\varepsilon\kappa), \Phi^*_{\rho\varepsilon^2,k_0+\varepsilon\kappa}\rangle\}. \end{cases} \tag{6.20}$$

For sufficiently smooth $A \in Z$ we have $A = M_\varepsilon[K_\varepsilon A] + \mathcal{O}(\varepsilon)$ for $\varepsilon \to 0$ and $E_\mathrm{c} u = K_\varepsilon[M_\varepsilon u] + \mathcal{O}(\varepsilon)$, compare to (6.12) and see [Schn94c] for proofs.

In order to formulate the following results as simple as possible we restrict our attention to one particular equation for u, namely the one–dimensional version of (2.20):

$$\partial_1 u = -(1{+}\partial_x^2)^2 u + b(3{+}\partial_x^2)\partial_x u + \varepsilon^2 u - u^3 + g u^2 \partial_x u, \tag{6.21}$$

which has the associated one–dimensional cGL

$$\partial_\tau A = (4{+}\mathrm{i}b)\partial_\xi^2 A + A - (3{-}\mathrm{i}g)|A|^2 A. \tag{6.22}$$

We formulate the following result with the notation of the general setting, since most of them carry over up to additional technicalities (e.g., due to Sobelev embeddings or different scaling factors depending on the number of unbounded directions). This might change the exponent σ and the underlying spaces but not the general structure of the theory. See Section 6.7 for an application of these principles to the Navier–Stokes equations.

For the above equations the function spaces can be chosen as $X = \mathrm{L}^2_{\mathrm{ul}}(\mathbb{R},\mathbb{R})$, $Y = \mathrm{H}^1_{\mathrm{ul}}(\mathbb{R},\mathbb{R})$ and $Z = \mathrm{H}^1_{\mathrm{ul}}(\mathbb{R},\mathbb{C})$. In particular, Theorem 6.1 is applicable with $\beta = 1/4$ for these choices. The operators K_ε and M_ε (which, in the present case, simplify a lot due to $\Phi_{\varepsilon^2,k} = \Phi^*_{\varepsilon^2,k} = 1$) satisfy

$$\|K_\varepsilon A\|_Y \leq C\varepsilon\|A\|_Z, \quad \|M_\varepsilon u\|_Z \leq \frac{C}{\varepsilon^2}\|u\|_Y, \quad \|M_\varepsilon[\mathrm{e}^{(\tau/\varepsilon^2)L_\varepsilon}u]\|_Z \leq \frac{C(\tau)}{\varepsilon}\|u\|_Y \tag{6.23}$$

where $\tau > 0$ and all constants C and $C(\tau)$ are independent pf $\varepsilon > 0$. While the first estimate is trivial, the second estimate shows that rescaling yields an additional bad factor ε^{-1}. However, this factor disappears if we let act the linear semiflow for a suitable long time, see Section 2.2.1. Thus, for the linearized flow we have shown that any solution $u(t) = \mathrm{e}^{tL_\varepsilon}u_0$ is attracted to a modulated pattern, namely

$$u(\tau/\varepsilon^2) = K_\varepsilon A_1 + \mathcal{O}(\varepsilon^2) \quad \text{with } A_1 = M_\varepsilon u(\tau/\varepsilon^2).$$

The following result (see [MiS95, MiS96]) is the nonlinear version of this basically linear effect.

Theorem 6.2
Consider equations (6.21) and (6.22) with the given function spaces X, Y and Z. For all $C_1 > 0$ there exist $C_0, \varepsilon_0, \tau_0 > 0$, such that the following is true. For all $\varepsilon \in (0, \varepsilon_0)$ and $u_0 \in Y$ with $\|u_0\|_Y \leq C_1\varepsilon$ the solution $u(t) = \mathcal{S}^\varepsilon_t(u_0)$ exists for $t \in [0, \tau_0/\varepsilon^2]$ and $A_1 = M_\varepsilon u(\tau_0/\varepsilon^2)$ satisfies

$$\|A_1\|_Z \leq C_0 \quad \textit{and} \quad \|u(\tau_0/\varepsilon^2) - K_\varepsilon A_1\|_Y \leq C_0\varepsilon^{5/4}. \tag{6.24}$$

We define the set of modulated patterns,

$$\mathbf{MP} = \{\, K_\varepsilon A \,:\, \|A\|_Z \le C_0 \,\} \subset Y,$$

which is a small, very flat ellipsoid in the phase space Y. Then $\mathbf{MP}$ attracts all solutions starting in the ball of radius $C_1\varepsilon$ around $0 \in Y$, at least up to an error of order $\varepsilon^{5/4}$. This attractivity property of the modulated pattern gave rise to the title "*The Ginzburg–Landau manifold is an attractor*" for [Eck93].

The development of the mode structure is essentially a linear effect, e.g. from the linear theory in Section 2.2.1 we see that the time scale $\mathcal{O}(1/\varepsilon^2)$ is necessary for the solutions $u(t)$ to develop the scaled mode structure. For the nonlinear result it is then necessary to show that the solutions $u(t) = \mathcal{S}^\varepsilon_t(u_0)$ exist over the desired time interval $t \in [0, \tau_1/\varepsilon^2]$. For general problems with quadratic nonlinearity, solutions of order ε exist only on the time scale $1/\varepsilon$, but again the fact can be used that quadratic interactions of critical modes generate damped modes. The time τ_1 has to be chosen sufficiently small to avoid blow–up of solutions, which may occur in the case when $\mathrm{Re}\,\widehat{c} \le 0$ in (6.3). Under suitable assumptions on Λ_0 and $\widehat{c}$ in (6.3) (e.g. $\mathrm{Re}\,\widehat{c} > 0$ and $d \le 2$) the solution $A(\tau) = \mathcal{G}_\tau(A_0)$ exists for all time (see Section 4.4) and τ_1 can be chosen arbitrary.

Having Theorem 6.2 it is possible to apply the approximation theory of the previous section. In particular, the assumptions of Theorem 6.1 are fulfilled if we choose $u(0) = \mathcal{S}^\varepsilon_{\tau_0/\varepsilon^2}(u_0)$, and $A(0) = A_1$ and take any time $\tau_1 > 0$ such that $A(\tau) = \mathcal{G}_\tau(A_1)$ exists for $\tau \in [0, \tau_1]$.

6.4 Shadowing by pseudo–orbits and global existence

Finally we consider the case where the GLe has a global semiflow with a global attractor $\mathcal{A}_{\mathrm{GL}}$, see Sections 4.4 and 6.5. Hence, we may use the approximation property as well as the attractivity to control all small solutions $u(t) = \mathcal{S}^\varepsilon_t(u_0)$ of the original problem (6.1) for arbitrarily long time intervals. The idea is to use the attractivity property in a first step to find the modulated structure $u(\tau_0/\varepsilon^2) = [K_\varepsilon \widehat{A}_1] + \mathcal{O}(\varepsilon^2)$ where $\widehat{A}_1 = M_\varepsilon u(\tau_0/\varepsilon^2)$. In a second step the solution can be described over a time interval $[\tau_0/\varepsilon^2, (\tau_0{+}\tau_1)/\varepsilon^2]$ by the Ginzburg–Landau approximation $U_{A_1}(t)$ with $A_1(\tau) = \mathcal{G}_{\tau-\tau_0}(\widehat{A}_1)$. At the end of this interval the error between the true solution and the approximation might be somewhat larger; however, from the attractivity we know that $u((\tau_0{+}\tau_1)/\varepsilon^2)$ is close to a different modulated pattern $K_\varepsilon \widehat{A}_2$ from which the approximation theory may start on the next time interval. The exponential attractivity of the absorbing ball for the GLe controls the size of the solution A_j, $j \in \mathbb{N}$, and guarantees their uniform boundedness for all $\tau \ge \tau_0$.

In this way we can shadow the true orbit $u(t) = \mathcal{S}^\varepsilon_t(u_0)$ for all $t > 0$ by a sequence of solutions of the GLe.

Definition 6.3
Let $\tau_* > 0$ and $\delta > 0$. A function $A \in \mathrm{L}^\infty((0,\infty), Z)$ is called a (τ_*, δ)–pseudo–orbit in Z for (6.1) if for all $n \in \mathbb{N}$ the relations

$$\begin{aligned} &A((n{-}1)\tau_* + \tau) = \mathcal{G}_\tau(A((n{-}1)\tau_*)) \text{ for all } \tau \in [0, \tau_*), \\ &\|A(n\tau_*{-}0) - \mathcal{G}_{\tau_*}(A((n{-}1)\tau_*))\|_Z \le \delta \end{aligned}$$

hold, where $A(\tau{-}0) = \lim_{\widetilde{\tau} \nearrow \tau} A(\widetilde{\tau})$.

The following result shows that each true orbit of (6.21) can be shadowed after a transient period by a pseudo–orbit of the associated cGL (6.22). The essential feature here is to show that the jumps in the pseudo–orbits are small in terms of ε. See [Schn94c, Thm.3] for a proof.

Theorem 6.4
Under the assumption of Theorem 6.2 for all $\tau_* > 0$ and $C_* > 0$ there exist positive constants ε_0 and C such that for all $\varepsilon \in (0, \varepsilon_0]$ the following is true:

For all initial conditions u_0 with $\|u_0\|_Y \leq C_*\varepsilon$ the solution $u(t) = \mathcal{S}_t(u_0)$ exist for all time and there is a $(\tau_*, C\varepsilon^{1/4})$–pseudo–orbit A for (6.22) which satisfies $\|A(0)\|_Z \leq C$ and approximates $u(t)$ as follows:

$$\|u(t + \tau_*/\varepsilon^2) - U_A(t)\|_Y \leq C\varepsilon^{5/4} \quad \text{for all } t \geq 0.$$

This shadowing technique can be used to show that in fact all solutions starting in a small but ε–independent neighborhood $\mathcal{U}$ of 0 in Y exist for all time and are finally absorbed into a ball of radius $C\varepsilon$.

Theorem 6.5
Under the assumptions of Theorem 6.2 there exist positive constants ε_0, δ_0, and C such that for all $\varepsilon \in (0, \varepsilon_0]$ all solutions $u(t) = \mathcal{S}_t(u_0)$ of (6.21) with $\|u_0\|_Y \leq \delta_0$ exists for all $t > 0$ and satisfy $\limsup_{t\to\infty} \|u(t)\|_Y \leq C\varepsilon$.

It is possible to prove global existence for the solutions $u(t) = \mathcal{S}_t^\varepsilon(u_0)$ of (6.21) by using the methods in [MiS95]. However, using energy estimates the best one can obtain is an absorbing ball of radius $C\varepsilon^{1/2}$. The present result constitutes a new way to construct absorbing sets, namely by studying the dynamics rather than applying energy estimates.

6.5 Comparison of attractors

According to Section 5 the GLe has a global attractor $\mathcal{A}_{\mathrm{GL}} \subset Z$. Similarly there exist local attractors $\mathcal{A}^\varepsilon$ for the semigroups $\mathcal{S}_t^\varepsilon$. Local attractor means here that there is a neighborhood $\mathcal{U}^\varepsilon$ of 0 in Y, which is positively invariant with respect to the semiflow $\mathcal{S}_t^\varepsilon$. According to Theorem 6.5 the sets $\mathcal{U}^\varepsilon$ can be chosen such that they contain a ball of radius $\delta_0 > 0$. Now every bounded $B \subset \mathcal{U}^\varepsilon$ is attracted to $\mathcal{A}^\varepsilon$ in the weighted norm in Y:

$$\mathrm{dist}_{Y_w}(\mathcal{S}_t^\varepsilon(B), \mathcal{A}^\varepsilon) \to 0 \quad \text{for } t \to \infty.$$

The question is now how well the attractors $\mathcal{A}^\varepsilon$ can be described by the attractor $\mathcal{A}_{\mathrm{GL}}$. The following result is given in [MiS95] and generalizations are given in [PeS99, Schn99b].

Theorem 6.6
Under the assumptions of Theorem 6.2, for every $\delta > 0$ there exist $C > 0$ and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$ the estimates

$$\mathrm{dist}_{Z_w}(M_\varepsilon \mathcal{A}^\varepsilon, \mathcal{A}_{\mathrm{GL}}) \leq \sigma \quad \text{and } \mathrm{dist}_{Y_w}(E_{\mathrm{s}}\mathcal{A}^\varepsilon, \{0\}) \leq C\varepsilon^{5/4}$$

hold.

This result means that $\mathcal{A}^\varepsilon$ is upper semicontinuous towards the attractor $\mathcal{A}_{\mathrm{GL}}$ of the Ginzburg–Landau equation in the sense that

$$\mathrm{dist}_{Z_w}(M_\varepsilon \mathcal{A}^\varepsilon, \mathcal{A}_{\mathrm{GL}}) + \frac{1}{\varepsilon}\mathrm{dist}_{Y_w}(E_{\mathrm{s}}\mathcal{A}^\varepsilon, \{0\}) \to 0 \quad \text{for } \varepsilon \to 0.$$

Thus, the solutions u in the attractor $\mathcal{A}^\varepsilon$ have relatively small stable parts $u_{\mathrm{s}} = E_{\mathrm{s}} u$ and the critical part u_{c} is given by a modulated pattern $K_\varepsilon A \in Y$ with $A \in \mathcal{A}_{\mathrm{GL}} \subset Z$. In this way we have obtained an upper bound on the richness of the attractor $\mathcal{A}^\varepsilon$. In fact the upper semicontinuity result can be sharpened by combining it with the theory of pseudo–orbits as given in Section 6.4. Each orbit in $\mathcal{A}^\varepsilon$ can be shadowed by a $(\tau^*, C\varepsilon^{1/4})$–pseudo orbit A of cGL, which lies completely in $\mathcal{A}_{\mathrm{GL}}$, see [Schn99b].

It is an unsolved problem to show the opposite direction, namely that $\mathcal{A}^\varepsilon$ is also as rich as the attractor $\mathcal{A}_{\mathrm{GL}}$. This would provide a lower bound on the complexity of the attractor. In mathematical terms this amounts to lower semicontinuity in the sense $\mathrm{dist}_{Z_w}(\mathcal{A}_{\mathrm{GL}}, M_\varepsilon \mathcal{A}^\varepsilon) \to 0$ for $\varepsilon \to 0$.

The definition of the attractors $M_\varepsilon \mathcal{A}^\varepsilon$ and $\mathcal{A}_{\mathrm{GL}}$ involves the limit of time $\tau \to \infty$. (Note that $t \to \infty$ without $\varepsilon^2 t \to \infty$ is not interesting.) Thus, comparing the attractors means that we have chosen to consider first the limit $\tau \to \infty$ and then we consider the $\varepsilon \to 0$. However, in pratical cases one always deals with finite time τ and finite $\varepsilon > 0$, and it is not always clear whether we are closer to $\tau \approx \infty$ or to $\varepsilon \approx 0$ in a given setting. Thus, it appears reasonable to study the double limit $(\tau, \varepsilon) \to (\infty, 0)$ where no specific order is supposed. We follow here the ideas of [HaR93], which were already used in Theorem 5.3 to study the attractor on very large domains.

We define a generalized limit attractor $\mathcal{A}^*$ in the following way

$$\mathcal{A}^* = \Big\{ A \in Z \ : \ \exists\, \varepsilon_n \to 0 \ \exists\, \tau_n \to \infty \ \exists\, u_n \text{ with } \sup_{n\in\mathbb{N}} \|u_n\|_Y < \infty \\ \text{such that } \|M_{\varepsilon_n} \mathcal{S}^{\varepsilon_n}_{\tau_n/\varepsilon_n^2}(u_n) - A\|_Z \to 0 \Big\}.$$

Here the time sequence $t_n = \tau_n/\varepsilon^2$ is chosen such that it tends to infinity in the Ginzburg–Landau scaling. In contrast to the Theorem 5.3 we can take here the uniformly local convergence (in $Y = \mathrm{H}^1_{\mathrm{ul}}(\mathbb{R}, \mathbb{C})$) rather than the localized convergence in Y_w.

The following result relies on a slight generalization of the approximation estimate in Section 6.2 which follows by combining the approximation and attractivity results:

For each $R > 0$ and $\tau_0 > 0$ there exist $\varepsilon_0 > 0$ and $C_1 > 0$ such that for all $A \in Z$ with $\|A\|_Z \le R$ and all $\varepsilon \in (0, \varepsilon_0)$ the following estimate holds:

$$\|M_\varepsilon \mathcal{S}^\varepsilon_{\tau/\varepsilon^2}(K_\varepsilon A_0) - \mathcal{G}_\tau(A_0)\|_Z \le C_1(\tau_0)\varepsilon^{1/4} \quad \text{for } \tau \in [0, \tau_0]. \tag{6.25}$$

Theorem 6.7
Under the above assumptions we have $\mathcal{A}^* = \mathcal{A}_{\mathrm{GL}}$.

Proof: We first show $\mathcal{A}_{\mathrm{GL}} \subset \mathcal{A}^*$. Take $A \in \mathcal{A}_{\mathrm{GL}}$, $\tau_n = n$ and, by the invariance of $\mathcal{A}_{\mathrm{GL}}$, choose $B_n \in \mathcal{A}_{\mathrm{GL}}$ with $\mathcal{G}_{\tau_n}(B_n) = A$. Defining $u_n = K_{\varepsilon_n} B_n$ we obtain $\|u_n\|_Y \le C\varepsilon_n^{1/4} \le r_0$. Here we still have the sequence (ε_n) at our disposal. With (6.25) and $t_n = \tau_n/\varepsilon^2$ we obtain

$$\|M_{\varepsilon_n} \mathcal{S}^{\varepsilon_n}_{t_n}(u_n) - A\|_Z \le \|M_{\varepsilon_n} \mathcal{S}^{\varepsilon_n}_{t_n}(K_{\varepsilon_n} B_n) - \mathcal{G}_{\tau_n}(B_n)\|_Z \le C_1(\tau_n)\varepsilon_n^{1/4}.$$

Since the sequence $\tau_n = n$ was already fixed, we can now choose ε_n such that the right–hand side in the last estimate converges to 0 for $n \to \infty$, which implies $A \in \mathcal{A}^*$.

To show the inclusion $\mathcal{A}^* \subset \mathcal{A}_{\mathrm{GL}}$ we take $A \in \mathcal{A}^*$ and $A_n = M_{\varepsilon_n}\mathcal{S}^{\varepsilon_n}_{t_n}(u_n)$ with $\|A - A_n\|_Z, \varepsilon_n \to 0$, $t_n = \tau_n/\varepsilon_n^2$ and $\|u_n\|_Y \le r_0$. Choose $\widehat{\tau} \ge 1$ and restrict $n \in \mathbb{N}$ such that $\tau_n > \widehat{\tau}$. Let $v_n = \mathcal{S}^{\varepsilon_n}_{(t_n-\widehat{\tau})/\varepsilon_n^2}(u_n)$ and $B_{n,\widehat{\tau}} = M_{\varepsilon_n} v_n$. Then, by the attractivity theorem 6.2 we have $\|B_{n,\widehat{\tau}}\|_Z \le C$ with C independent of n and $\widehat{\tau}$. Moreover,

$$\mathrm{dist}_{Z_w}(A, \mathcal{A}_{\mathrm{GL}}) \le \|A - A_n\|_Z + \|A_n - \mathcal{G}_{\widehat{\tau}}(B_{n,\widehat{\tau}})\|_Z + \mathrm{dist}_{Z_w}(\mathcal{G}_{\widehat{\tau}}(B_{n,\widehat{\tau}}), \mathcal{A}_{\mathrm{GL}}).$$

For given $\delta > 0$, we can choose $\widehat{\tau}$ so large, that the third term is less than $\delta/3$ for all n. Clearly the first term is less than $\delta/3$ for sufficiently large n. While keeping $\widehat{\tau}$ fixed we can estimate the second term as

$$\begin{aligned} &= \|M_{\varepsilon_n}\mathcal{S}^{\varepsilon_n}_{\widehat{\tau}/\varepsilon^2}(v_n) - \mathcal{G}_{\widehat{\tau}}(B_{n,\widehat{\tau}})\|_Z \\ &\le \|M_{\varepsilon_n}[\mathcal{S}^{\varepsilon_n}_{\widehat{\tau}/\varepsilon^2}(v_n) - \mathcal{S}^{\varepsilon_n}_{\widehat{\tau}/\varepsilon^2}(K_{\varepsilon_n}B_{n,\widehat{\tau}})]\|_Z + \|M_{\varepsilon_n}\mathcal{S}^{\varepsilon_n}_{\widehat{\tau}/\varepsilon^2}(K_{\varepsilon_n}B_{n,\widehat{\tau}}) - \mathcal{G}_{\widehat{\tau}}(B_{n,\widehat{\tau}})\|_Z \\ &\le C_0(\widehat{\tau})\varepsilon_n^{1/4} + C_1(\widehat{\tau})\varepsilon_n^{1/4}, \end{aligned}$$

where we have used the first approximation theorem 6.1 and $\|v_n - K_{\varepsilon_n}B_{n,\widehat{\tau}}\|_Y \le C\varepsilon_n^{5/4}$ for the first term and (6.25) for the second.

By choosing n sufficiently large we have thus proved $\mathrm{dist}_{Z_w}(A, \mathcal{A}_{\mathrm{GL}}) \le \delta$ where $\delta > 0$ was arbitrary. Hence we conclude $A \in \mathcal{A}_{\mathrm{GL}}$. $\square$

6.6 Comparison of inertial manifolds

Another way to compare the dynamics between the original system and the associated GLe is obtained by restricting the analysis to periodic boundary conditions, where the period of the original system has to be large in oder to obtain the GLe on a fixed domain independent of ε. At the moment the results are rather limited, and we only report on the work in [MSZ99] where a general method is given to compare the inertial manifolds of two different parabolic problems which may have a different number of components, a different order, or are even posed on different domains.

We present the result only for the Swift–Hohenberg equation (SHe) $u_t = -(1+\partial_x^2)^2u + \varepsilon^2 u - u^3$, which takes after the rescaling $\widehat{u}(\tau,\xi) = \varepsilon^{-1}\widehat{u}(\varepsilon^2\tau, \varepsilon\xi)$ the form

$$\partial_\tau \widehat{u} = -\varepsilon^{-2}(1+\varepsilon^2\partial_\xi^2)^2\widehat{u} + \widehat{u} - \widehat{u}^3. \tag{6.26}$$

The associated modulation ansatz is

$$\widehat{u}(\tau,\xi) = A(\tau,\xi)\mathrm{e}^{\mathrm{i}\xi/\varepsilon} + \mathrm{c.c.} + \mathcal{O}(\varepsilon) \tag{6.27}$$

which leads to the associated real GLe

$$\partial_\tau A = 4\partial_\xi^2 A + A - 3|A|^2A. \tag{6.28}$$

We complement (6.26) by the periodicity assumption $\widehat{u}(\tau,\xi+\ell) = \widehat{u}(\tau,\xi)$ for all $\tau \ge 0$ and $\xi \in \mathbb{R}$. The period $\ell > 0$ is considered to be fixed. In particular, we choose the appropriate phase space

$$X_{\ell,\mathrm{per}} = \{\, \widehat{u} \in \mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}) \,:\, \widehat{u}(\xi+\ell) = \widehat{u}(\xi) \text{ for all } \xi \in \mathbb{R} \,\}.$$

With the modulation ansatz (6.27) this gives the quasiperiodic boundary condition $A(\xi+\ell) = e^{i\gamma}A(\xi)$ with $\gamma = (\ell/\varepsilon) \bmod 2\pi$. Thus, the associated phase space for the GLe is

$$Z_{\ell,\gamma} = \{ A \in \mathrm{H}^1_{\mathrm{loc}}(\mathbb{R},\mathbb{C}) \;:\; A(\xi+\ell) = e^{i\gamma}A(\xi) \text{ for all } \xi \in \mathbb{R} \}.$$

Because of the new boundary conditions it is a slight variation of a classical result that the rGL (6.28) restricted to the phase space $Z_{\ell,\gamma}$ has an inertial manifold, see e.g. [DG*88, Tem88]. Particularly, the eigenvalues of the linear part are given by $\lambda_n = 1 - 4(\gamma+2\pi n)^2/\ell^2$, $n \in \mathbb{Z}$, and thus there are sufficiently large spectral gaps.

To compare the two quite different systems (6.26) and (6.28) we have to decompose both phase spaces into three parts:

$$X_{\ell,\mathrm{per}} = X_1 \oplus X_2 \oplus X_3, \qquad Z_{\ell,\gamma} = Z_1 \oplus Z_2 \oplus Z_3.$$

These decompositions are associated with the spectral parts of the linear operators such that X_1, Z_1 contain the largest eigenvalues, X_2, Z_2 the intermediate ones, and X_3, Z_3 the smallest ones (going to $-\infty$). The important point is that there are isomorphisms $T_j : X_j \to Z_j$ for $j = 1, 2$ which make the two systems comparable. However, we do not need to compare X_3 with Z_3, as the contributions of these subspaces on the flow reduced to the inertial manifold will be small.

For the rGL and assuming $\gamma \in (-\pi, \pi]$ we define the subspaces

$$Z_j = \mathrm{span}\{ e^{i(2\pi k+\gamma)\xi/\ell}, ie^{i(2\pi k+\gamma)\xi/\ell} \;:\; k \in J_j(n_{\mathrm{GL}}, \gamma, \varepsilon) \}$$

where the index sets are given by

$$\begin{aligned} J_1(n_{\mathrm{GL}}, \gamma, \varepsilon) &= \{ k \in \mathbb{Z} \;:\; |k+\gamma/(2\pi)| < n_{\mathrm{GL}} + 3/4 \} \\ J_2(n_{\mathrm{GL}}, \gamma, \varepsilon) &= \{ k \in \mathbb{Z} \;:\; k \notin J_1(n_{\mathrm{GL}}, \gamma, \varepsilon),\ |k| < \varepsilon^{-\delta} \} \end{aligned}$$

Here the parameters $n_{\mathrm{GL}} \in \mathbb{N}$ and $\delta \in (1/2, 1)$ are chosen later. In particular, n_{GL} will be chosen in such a way that certain spectral gap conditions are satisfied: the spectral gap between the eigenvalues associated to Z_1 and Z_2 is of approximate size $32\pi^2 n_{\mathrm{GL}}/\ell^2$.

The space Z_3 is then simply the closure of the span of all the remaining eigenfunctions (exponentials of the type in Z_j). The spaces X_1 and X_2 are constructed by simply applying the modulation transformation (6.27) to the functions in Z_1 and Z_2, respectively. For sufficiently small ε, these mappings are isomorphisms by the construction of the index sets J_1 and J_2 and the assumption $\delta < 1$. We use the notation $T_j : X_j \to Z_j$ to indicate these isomorphisms.

It is shown in [MSZ99] that both systems have an inertial manifold which, after modifying the systems outside an absorbing ball, take the form

$$\begin{aligned} \mathcal{M}_X(\varepsilon) &= \mathrm{graph}(R_X) = \{ \widehat{u}_1 + R_X(\widehat{u}_1) \in X_{\ell,\mathrm{per}} \;:\; \widehat{u}_1 \in X_1 \}, \\ \mathcal{M}_Z(\gamma) &= \mathrm{graph}(R_Z) = \{ A_1 + R_Z(A_1) \in Z_{\ell,\gamma} \;:\; A_1 \in Z_1 \}, \end{aligned}$$

where the reduction functions $R_X : X_1 \to X_2 \oplus X_3$ and $R_Z : Z_1 \to Z_2 \oplus Z_3$ are globally Lipschitz continuous mappings. The sets $\mathcal{M}_X$ and $\mathcal{M}_Z$ are called inertial manifolds for (6.26) and (6.28), respectively, as these manifolds are invariant under the flow and they attract all solutions exponentially for positive times.

Inserting the reduction functions in the full infinite dimensional problems the system can be reduced to the finite–dimensional flow in the inertial manifold itself. Using $\widehat{u}_1 \in X_1$ and $A_1 \in Z_1$ as coordinates the reduced flows take the form

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}\tau}\widehat{u}_1 &= F_{\mathrm{red}}(\varepsilon, \widehat{u}_1) &&= -\varepsilon^{-2}(1+\varepsilon^2\partial_\xi^2)^2\widehat{u}_1 + \widehat{u}_1 - P_1[(\widehat{u}_1 + R_X(\widehat{u}_1))^3], \\ \tfrac{\mathrm{d}}{\mathrm{d}\tau}A_1 &= G_{\mathrm{red}}(\gamma, A_1) &&= 4\partial_\xi^2 A_1 + A_1 - 3Q_1[|A|^2A] \text{ where } A = A_1 + R_Z(A_1), \end{aligned}$$

where $P_1 : X \to X_1$ and $Q_1 : Z \to Z_1$ are the associated L^2–orthogonal projections.

It is now possible to compare the reduced flows of the two systems, since they are defined on finite dimensional spaces X_1 and Z_1 which are isomorphic. The following result holds, see [MSZ99].

Theorem 6.8
For given $\delta \in (1/2, 1)$ there exists $C > 0$ such that for each $\ell \geq 1$ there exists $n_{\mathrm{GL}} \in \mathbb{N}$ with $n_{\mathrm{GL}} \leq C\ell^3$ and a $q_0 \in \mathbb{N}$ such that for all $\gamma \in (-\pi, \pi)$ the following holds:

The rGL (6.28) posed in $Z_{\ell,\gamma}$ has an inertial manifold $\mathcal{M}_Z(\gamma)$ with dimension less than $4n_{\mathrm{GL}}+4$ and the reduced vector field $G_{\mathrm{red}}(\gamma, \cdot) : Z_1 \to Z_1$. Moreover, for all $q \in \mathbb{N}$ with $q > q_0$ the Swift–Hohenberg equation (6.26) posed in $X_{\ell,\mathrm{per}}$ and with $\varepsilon = \ell/(2\pi q - \gamma)$ has an inertial manifold $\mathcal{M}_X(\varepsilon)$ of the same dimension as $\mathcal{M}_Z(\gamma)$. The reduced vector field $F_{\mathrm{red}}(\varepsilon, \cdot) : X_1 \to X_1$ on $\mathcal{M}_X(\varepsilon)$ satisfies

$$\|G_{\mathrm{red}}(\gamma, A_1) - T_1 F_{\mathrm{red}}(\varepsilon, T_1^{-1}A_1)\|_Z \leq C(\varepsilon\|A_1\|_Z + \varepsilon^\delta).$$

It should be noted that the reduced vector field $F_{\mathrm{red}}(\varepsilon, \cdot)$ does not converge to a limit for $\varepsilon \to 0$ but it has to approach a family of vector field parametrized by $\gamma \in \mathbb{S}^1$. It is simple to check (e.g., by considering the steady states) that the vector fields $G_{\mathrm{red}}(\gamma, \cdot)$ are genuinely different.

We now point out some differences between the limit flow $G_{\mathrm{red}}(\gamma, \cdot)$ and the reduced flow $F_{\mathrm{red}}(\varepsilon, \cdot)$ of SHe. Both, the SHe and the rGL are gradient systems, namely with respect to the functionals

$$V(\widehat{u}) = \int_0^\ell \Big[\frac{1}{2\varepsilon^2}(\widehat{u}+\varepsilon^2\partial_\xi^2\widehat{u})^2 - \frac{1}{2}\widehat{u}^2 + \frac{1}{4}\widehat{u}^4\Big]\mathrm{d}\xi, \quad E(A) = \int_0^\ell \Big[2|\partial_\xi A|^2 - \frac{1}{2}|A|^2 + \frac{3}{4}|A|^4\Big]\mathrm{d}\xi.$$

Thus, the flows on the inertial manifolds $\mathcal{M}_X$ and $\mathcal{M}_Z$ are gradient–like systems in the sense that they have a strictly decreasing Liapunov function. In particular, this implies that all solutions in the global attractors have the α– and ω–limit sets in the set of equilibria.

Note, however, that there is a major difference between the flow of SHe and rGL. Both systems are translationally invariant but only rGL has an additional phase invariance $A \mapsto \mathrm{e}^{\mathrm{i}\alpha}A$. In fact, the set of equilibria in rGL was studied explicitly in Section 4.2.2. The rotating plane waves $A(x) = r\,\mathrm{e}^{\mathrm{i}k\xi}$ form a circle of equilibria, since the action of space translation and phase multiplication coincides. Moreover, there are steady states in $Z_{\ell,\gamma}$ for which $|A(\cdot)|$ is periodic but nonconstant. Such solutions occur always in a two–dimensional continuum of equilibria which has the topological type of a torus, see e.g. [BrR94, DGJ95].

If we now consider generic ℓ and γ then the set of equilibria for $G_{\mathrm{red}}(\gamma, \cdot)$ consists of finitely many circles and tori as described above. Moreover, we can assume that all these manifolds of equilibria are normally hyperbolic. If we now consider the flow of $F_{\rm red}(\varepsilon,\cdot)$ with $\ell/\varepsilon = \gamma \bmod 2\pi$ then the flow is a small perturbation of $G_{\rm red}(\gamma,\cdot)$ and we conclude that all the circles and tori persist as normally hyperbolic invariant manifolds. The flow on the circles will still be trivial as the translation symmetry of SHe forces equilibria to occur in circles. However, the flow on the tori will no longer be trivial: generically one expects that only a finite number of circles of equilibria will persist which are connected by heteroclinic orbits inside the torus. Thus, we see that the inertial manifold provides a tool to study the breakup of the normal form symmetry which is present in rGL, cf. [Mel98].

6.7 Results on the Couette–Taylor problem

We finally return to hydrodynamical problems to emphasize that the whole theory developed above is applicable to general partial differential equations, in particular to the three–dimensional Navier–Stokes equations. The Rayleigh–Bénard problem was studied in [Schn94b] and the Poiseuille problem in [Bol96]. Here we restrict ourselves to the Couette–Taylor problem, give a rough sketch of the relevant results and refer to [Schn99a, PeS99, Schn99b] for the details.

We study the Navier–Stokes equation exactly in the setting described in Section 3.2 and write the velocity field $v(t,\cdot) : \Omega = \mathbb{R}\times\Sigma \to \mathbb{R}^3$ as $V_{\rm Cou}+u$, where u is the perturbation of the Couette flow $V_{\rm Cou}$. The equations for u are

$$\left.\begin{array}{rcl} \partial_t u &=& \Delta u - \nabla p \\ && -\mathcal{R}[(V_{\rm Cou}\cdot\nabla)u+(u\cdot\nabla)V_{\rm Cou}+(u\cdot\nabla)u], \\ 0 &=& \operatorname{div} u, \end{array}\right\} \quad \text{in} \quad \Omega, \qquad (6.29)$$
$$\begin{array}{rcl} u &=& 0 \quad \text{on} \quad \partial\Omega, \\ \int_\Sigma u_1(t,x,y)\,\mathrm{d}y &=& 0 \quad \text{for} \quad x\in\mathbb{R}. \end{array}$$

Here the Reynolds number $\mathcal{R}$ is to be slightly above the threshold, namely $\mathcal{R} = \mathcal{R}_0+\varepsilon^2$. Moreover, we have incorporated the condition that the mass flux through each cross section $\{x\}\times\Sigma$ is 0 (here $u_1 \in \mathbb{R}$ denotes the axial component of $u \in \mathbb{R}^3$).

We assume that the other parameters are chosen such that the Ginzburg–Landau formalism is applicable. Then, the associated GLe is automatically one–dimensional and real:

$$\partial_\tau A = \gamma_1 \partial_\xi^2 A + \gamma_2 A - \gamma_3 |A|^2 A. \qquad (6.30)$$

By numerical calculations it can be shown that the coefficients γ_j are all positive.

The mathematical setup of (6.29) can be formulated in the spaces

$$X = \{\, u \in \mathrm{H}^2_{\rm ul}(\Omega,\mathbb{R}^3) \ :\ \operatorname{div} u \equiv 0,\ u|_{\partial\Omega} = 0,\ \int_\Sigma u_1\,\mathrm{d}y = 0 \,\}.$$

We also use the space X_w when the uniformly local norm is replaced by a weighted norm, where the weight w is chosen to be independent of $y\in\Sigma$, e.g., $w(x,y) = 1/\cosh(x)$ or $1/(1+x^2)$.

Problem (6.29) defines a local semiflow $(\mathcal{S}^\varepsilon_t)_{t\geq 0}$ on the space X. In general the question of global existence for the three–dimensional Navier–Stokes equation is not yet solved. However, using the Ginzburg–Landau formalism it is possible to obtain new results in that direction. Denote by $(\mathcal{G}_\tau)_{\tau\geq 0}$ the global semiflow of the rGL (6.30) on the space $Z = \mathrm{H}^2(\mathbb{R},\mathbb{C})$. For a solution $A : [0,\infty) \to Z$ the modulation ansatz is given by

$$U_A(t,x,y) = \varepsilon\, A(\varepsilon^2 t, \varepsilon x)\mathrm{e}^{\mathrm{i}k_0 x}\Phi_{0,k_0}(y) + \mathrm{c.c.}$$

The operators $K_\varepsilon : Z \to X$ and their approximate inverse $M_\varepsilon : X \to Z$ are defined exactly as in (6.19) and (6.20), respectively.

Using these specifications, the approximation theorem 6.1, the attractivity result in Theorem 6.2 as well as the shadowing by pseudo–orbits (Theorem 6.4) can be established. We will not repeat these results as stated theorems once again. However, we formulate two results which are derived by using the Ginzburg–Landau formalism but can be formulated independently of that theory.

As a parallel to Theorem 6.5 we obtain the following result (see [Schn99a, Thm.1.4]) which proves global existence for certain interesting solutions of the three–dimensional Navier–Stokes equations.

Theorem 6.9
There exist small positive numbers r_0 and ε_0 and a large positive C such that for all $\varepsilon \in (0,\varepsilon_0)$ the following holds. If $u_0 \in Z$ satisfies $\|u_0\|_X \leq r_0$, then the solution $u(t) = \mathcal{S}_t^\varepsilon(u_0)$ exists for all $t \geq 0$ and satisfies

$$\|u(t)\|_X \leq C r_0 \text{ for all } t \geq 0, \qquad \limsup_{t\to\infty} \|u(t)\|_X \leq C\varepsilon.$$

In particular, the set $B_\varepsilon = \cup_{t\geq 0}\mathcal{S}_t^\varepsilon(\{u : \|u\|_X \leq r_0\})$ is positively invariant and contains the ball of radius r_0 in X. Moreover, this set is absorbed into the much smaller ball, namely that of radius $2C\varepsilon$ in finite time.

Note that the small solutions $u \in Z$ might have arbitrary large or even infinite kinetic energy $\int_\Omega |u(t,x,y)|^2\,\mathrm{d}(x,y)$ in the classical sense of the Navier–Stokes equation. However, this energy must be spread over the cylinder so that in any finite subregion $(x_1,x_2)\times\Sigma$ the kinetic energy is bounded by $C(x_2-x_1)$.

In analogy to Section 5 it is possible to define a local Couette–Taylor attractor $\mathcal{A}_{\mathrm{CT}}^\varepsilon$ with respect to the positively invariant set B_ε, see [PeS99, Schn99b]. Clearly, $\mathcal{A}_{\mathrm{CT}}^\varepsilon$ will lie in the smaller ball of radius $C\varepsilon$. It is an (X,X_w)–attractor with the following properties:

a) $\mathcal{A}_{\mathrm{CT}}^\varepsilon$ is nonempty, closed and bounded in X and compact in X_w;

b) $\mathcal{A}_{\mathrm{CT}}^\varepsilon$ is invariant under the flow of $(\mathcal{S}_t^\varepsilon)_{t\geq 0}$;

c) $\mathcal{A}_{\mathrm{CT}}^\varepsilon$ attracts all subsets B of $B_\varepsilon \subset X$ in the X_w–metric, viz.,

$$\mathrm{dist}_{X_w}(\mathcal{S}_t^\varepsilon(B), \mathcal{A}_{\mathrm{CT}}^\varepsilon) \to 0 \text{ for } t \to \infty.$$

d) $\mathcal{A}_{\mathrm{CT}}^\varepsilon$ is invariant under spatial translations $\xi \mapsto \xi+\xi_0$ and under azimuthal rotations $\phi \mapsto \phi+\phi_0$ (cylinder coordinates, see Section 3.2).

In fact in [Schn99b, Thm.6.1] it is shown more: The Couette–Taylor attractor $\mathcal{A}_{\mathrm{CT}}^\varepsilon$ is also upper semicontinuous towards the Ginzburg–Landau attractor $\mathcal{A}_{\mathrm{GL}}$ in the sense of Theorem 6.6. Moreover, the whole attractor lies in the subspace $X_{\mathrm{ri}} \subset X$ of rotationally invariant functions. This can be explained by the fact that in the regime under consideration the instability occurs in the rotationally invariant subspace. Perturbations which have a nonzero azimuthal wavenumber are still uniformly damped. (If the first instability arises by functions which are not rotationally invariant then the associated eigenvalues cannot be simple and our basic spectral assumptions (2.3) are not satisfied.) We decompose $u \in X$ into $u = u_{\rm ri} + w$ where $u_{\rm ri} \in X_{\rm ri}$ is rotationally invariant whereas w has mean 0 when integrated over ϕ: $\int_{\mathbb{S}^1} w(t,x,r,\phi)\,{\rm d}\phi = 0$. Denote by $P : X \to X_{\rm ri}$ the associated projection, then we have

$$\mathrm{dist}_X((I-P)\mathcal{S}_t^\varepsilon(B_\varepsilon), \{0\}) \leq C\mathrm{e}^{-\alpha t} \text{ for } t \geq 0,$$

where $C, \alpha > 0$ are independent of $\varepsilon \in (0, \varepsilon_0)$.

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