Uniqueness of the surface wave speed: a proof that is independent of the Stroh formalism

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Abstract

It is well-known in surface-wave theory that the secular equation for the surface-wave speed v can be written as $\det M = 0$ in terms of the surface impedance matrix M. It has recently been shown by the present authors that M satisfies a simple algebraic Riccati equation. It is shown in the present paper that a purely matrix algebraic analysis of this equation suffices to prove that whenever a surface wave exists it is unique.

Keywords: Stroh formalism; surface waves; elastic half-space; surface impedance matrix.

1 Introduction

A surface wave is a travelling wave that propagates along the surface of a half-space and decays exponentially away from the surface. The transmitting medium may be for instance elastic, viscoelastic or piezoelectric. Surface waves were first studied by Rayleigh (1885) in the context of an isotropic elastic half-space and hence nowadays surface waves are also referred to as Rayleigh waves.

The extension from an isotropic elastic half-space to a generally anisotropic elastic half-space is non-trivial, and propagation of surface waves in the latter medium has been one of the most exciting research areas in linear elasticity. It seems that surface waves in an anisotropic elastic half-space were first studied by Synge (1956), but after deriving a complex secular equation without further reductions, he concluded incorrectly that only for certain discrete directions would the complex secular equation have a real solution for the wave speed. Making use of a formalism first established in Stroh (1958) (now known as the Stroh formalism), Stroh (1962) showed that Synge's (1956) complex secular equation can always be reduced to a purely real expression. An alternative proof of the reality of the secular equation was given by Currie (1974). Under the framework of the Stroh formalism, the uniqueness of surface waves in a generally anisotropic elastic half-space was first proved by Barnett *et al* (1973). The existence proof was first given by Barnett and Lothe (1974), and an alternative proof was subsequently given by Lothe and Barnett (1976) in terms of the surface impedance matrix defined first by Ingebrigtsen and Tonning (1969). Barnett and Lothe's existence argument was later made more precise by Chadwick and Smith (1977) with the aid of a careful classification of transonic states. The research on the existence and uniqueness of surface waves in a generally anisotropic elastic half-space culminated with the paper by Barnett and Lothe (1985) who, apart from sharpening Chadwick and Smith's (1977) existence results, presented a concise theory, more refined than that of Lothe and Barnett (1976), for the existence and uniqueness of surface waves with the aid of the impedance matrix. In this connection, mention should also be made of the paper by Taylor (1978) who attempted to give an existence proof with the aid of a matrix that is i times the surface impedance matrix, but it seems that his argument is in error.

Another thread of research on surface waves in a generally anisotropic elastic half-space is the computation of surface wave speed and derivation of explicit secular equations. An explicit secular equation was first derived by Currie (1979), although this paper seems to have been ignored by subsequent researchers. Mozhaev (1995) proposed a method based on first integrals of displacement components, and showed how an explicit secular equation could be obtained. Recently, motivated by Mozhaev's (1995) approach and ideas in the Stroh formalism, Destrade (2001a, b) developed another efficient method based on first integrals of traction components. Although both Mozhaev's (1995) and Destrade's (2001a, b) methods yield explicit secular equations that can be solved numerically for the wave speed, these secular equations also admit spurious roots that have to be carefully eliminated. Furthermore, as pointed out by Destrade (2001), both of these methods are only applicable to situations where the plane spanned by the direction of propagation and the normal to the free surface is a symmetry plane of the material. More recently, Ting (2002a, b) showed how a plethora of secular equations could be derived using a simple procedure. But again, the secular equations derived using Ting's method admit spurious roots and even some of the secular equations themselves are spurious.

If one's main concern is to compute the surface wave speed corresponding a given elastic half-space, the numerical method recently proposed by Fu and Mielke (2002) seems to be most practical and efficient. This method is based on the identity

$$(M - \mathrm{i}R)T^{-1}(M + \mathrm{i}R^{\mathrm{T}}) - Q + \rho v^2 I = 0, \tag{1.1}$$

where M is the surface impedance matrix, ρ is the material density, v is the surface wave speed and components of the matrices T, R, Q are defined in terms of the elastic stiffnesses C_{ijkl} by

$$T_{ik} = C_{i2k2}, \quad R_{ik} = C_{i1k2}, \quad Q_{ik} = C_{i1k1}. \tag{1.2}$$

In the most general case, the Hermitian matrix M involves three real components and three complex components so that the identity (1.1) yields nine real equations, which together with the secular equation $\det M = 0$, can easily be solved numerically for M and v. We note that it is advantageous to use the surface impedance matrix since $\det M = 0$ does not admit any spurious root.

We show in this paper that the identity (1.1) can also be used to prove the uniqueness of surface waves in a manner that is independent of the framework of the Stroh formalism. We note that the previous proof given by Barnett and Lothe (1985) is based on the following properties that hold in the subsonic interval $0 \leq v \leq \hat{v}$ where $\hat{v}$ is known as the limiting speed:

(i). The surface impedance matrix M is Hermitian;

(ii). The matrix $\mathrm{d}M/\mathrm{d}v$ is negative definite;

(iii). $\mathrm{tr}\, M \geq 0$, and $\mathbf{w} \cdot M\mathbf{w} \geq 0$ for all real vectors $\mathbf{w}$.

With the aid of these results, the uniqueness of surface waves is proved as follows. First, it can be shown using an energy argument (see e.g. Barnett and Lothe 1985 or Fu and Mielke 2002) that M must necessarily be positive definite at $v = 0$ so that all its eigenvalues are positive when $v = 0$. Property (ii) above implies that the eigenvalues of

M are monotone decreasing functions of v. Thus, a (subsonic) surface wave exists only if an eigenvalue of M, originally positive at $v = 0$, decreases to zero at $v = v_R < \hat{v}$. It can then be deduced that whenever such a v_R exists, it is unique (i.e., $\det M = 0$ has at most one solution $v \in (0, \hat{v})$). For if it is not unique, then two of the eigenvalues of M must be negative at $v = \hat{v}$ and any real vector $\mathbf{w}$ lying in the eigenspace of these two negative eigenvalues will violate result (iii) above; see Barnett and Lothe 1985, p. 145.

In the following section, we establish the three crucial properties above using an argument that is entirely free from the Stroh formalism. In fact, our derivation only involves an elementary matrix algebraic analysis of the Riccati equation (1.1) which also provides a formula for the desired solution M of (1.1) via two matrix-valued integrals whose integrands are given explicitly in terms of T, R and Q, see Theorem 2.7. In the concluding section we discuss connections with the Stroh formalism.

2 Main results

We consider a homogeneous, unstressed, generally anisotropic elastic half-space defined by

$$0 < x_2 < \infty, \quad -\infty < x_1, x_3 < \infty$$

relative to a rectangular coordinate system with coordinates (x_i). Free surface waves are governed by the equation of motion

$$C_{ijks} u_{k,sj} = \rho \ddot{u}_i, \quad 0 < x_2 < \infty, \tag{2.1}$$

the traction-free boundary condition

$$C_{i2ks} u_{k,s} = 0 \quad \text{on } \; x_2 = 0, \tag{2.2}$$

and the decay condition

$$u_k \to 0 \quad \text{as } x_2 \to \infty, \tag{2.3}$$

where (u_k) is the displacement, ρ the material density, a comma denotes differentiation with respect to spatial coordinates and a dot denotes material time derivative. The C_{ijks} are elastic stiffnesses and are assumed to satisfy the symmetry relations

$$C_{ijks} = C_{ksij} = C_{jiks}, \tag{2.4}$$

and the strong convexity condition

$$C_{ijks} \xi_{ij} \xi_{ks} > 0 \;\; \forall \text{ non-zero real symmetric tensors } \boldsymbol{\xi}. \tag{2.5}$$

The strong ellipticity condition is given by

$$C_{ijks} \eta_i \eta_k \gamma_j \gamma_s > 0 \;\; \forall \text{ non-zero real vectors } \boldsymbol{\eta} \text{ and } \boldsymbol{\gamma}, \tag{2.6}$$

and is implied by the strong convexity condition (2.5).

Without loss of generality, we may assume that the surface wave is propagating along the x_1-direction and that

$$\boldsymbol{u} = \boldsymbol{z}(m x_2) \mathrm{e}^{\mathrm{i} m (x_1 - vt)} + \text{C.C.}, \tag{2.7}$$

where $\mathbf{u} = (u_k)$, $\mathrm{i} = \sqrt{-1}$, m is a positive wave number, v is the propagation speed and C.C. denotes the complex conjugate of the preceding term. The elastic half-space is said to support a surface wave if we can find a real positive value v and a non-trivial vector function $\boldsymbol{z}(y)$ such that (2.1)–(2.3) are satisfied.

On substituting (2.7) into (2.1) and (2.2), we obtain

$$T\mathbf{z}''(y) + \mathrm{i}(R + R^{\mathrm{T}})\mathbf{z}'(y) - (Q - \rho v^2 I)\mathbf{z}(y) = \mathbf{0}, \quad 0 < y < \infty, \tag{2.8}$$

$$T\mathbf{z}' + \mathrm{i}R^{\mathrm{T}}\mathbf{z} = \mathbf{0}, \quad \text{on } y = 0, \tag{2.9}$$

where a prime signifies differentiation with respect to y $(= mx_2)$ and the matrices T, R, Q are defined by (1.2). We note that satisfaction of the strong ellipticity condition (2.6) ensures that T and Q are both positive definite and hence they are invertible.

We may view (2.8) as an initial value problem and look for a solution of the form

$$\mathbf{z} = \mathrm{e}^{-yE}\mathbf{z}(0), \tag{2.10}$$

where E is a 3×3 real matrix to be determined. On substituting (2.10) into (2.8) and (2.9), we obtain

$$TE^2 - \mathrm{i}(R + R^{\mathrm{T}})E - Q + \rho v^2 I = 0, \quad \left(-TE + \mathrm{i}R^{\mathrm{T}}\right)\mathbf{z}(0) = 0. \tag{2.11}$$

Equation $(2.11)_2$ motivates the introduction of M through $-TE + \mathrm{i}R^{\mathrm{T}} = -M$, or equivalently,

$$E = T^{-1}(M + \mathrm{i}R^{\mathrm{T}}). \tag{2.12}$$

The matrix M is known as the *surface impedance matrix*. Substituting (2.12) into $(2.11)_1$, we obtain the simple Riccati equation (1.1) which was first given by Mielke and Sprenger (1998) for $v = 0$ and by Fu and Mielke (2002) in the case of general v. Equation $(2.11)_2$ can be written $M\mathbf{z}(0) = 0$. Thus, a surface wave exists only if (1.1) has a solution M such that the eigenvalues of the matrix E computed according to (2.12) all have positive real part (in order to satisfy the decay condition) and that the following secular equation is satisfied:

$$\det M = 0. \tag{2.13}$$

It is known that the eigenvalues of any solution of $(2.11)_1$ will all have a non-zero real part if $0 \le v < \hat{v}$, where $\hat{v}$ is called the *limiting speed* (Chadwick and Smith 1977, p. 335). A surface wave with speed less than $\hat{v}$ is said to be *subsonic*.

Proposition 2.1 *The matrix problem*

$$TE^2 - \mathrm{i}(R + R^{\mathrm{T}})E - Q + \rho v^2 I = 0, \qquad \mathrm{Re\ spec}\ E > 0, \tag{2.14}$$

where "Re spec E" means the "real parts of the spectra of E", has a unique solution for E.

Proof. Let E be a solution of (2.14). Let λ be an eigenvalue of E and $\mathbf{a}$ an associated eigenvector (so that $E\mathbf{a} = \lambda\mathbf{a}$). It follows from $(2.14)_1$ that λ and $\mathbf{a}$ must satisfy the eigenvalue problem

$$\left\{\lambda^2 T - \mathrm{i}\lambda(R + R^{\mathrm{T}}) - Q + \rho v^2 I\right\}\mathbf{a} = \mathbf{0},$$

and so λ is a root of the characteristic equation $\det\left\{\lambda^2 T - \mathrm{i}\lambda(R + R^{\mathrm{T}}) - Q + \rho v^2 I\right\} = 0$. It can be seen that if σ is a root of this characteristic equation then so is $-\overline{\sigma}$, where a bar denotes complex conjugation. Thus, this characteristic equation has exactly three roots with positive real part. Collecting the corresponding generalized eigenspaces defines E QED

Theorem 2.2 *If E solves (2.14), then M obtained from (2.12) is Hermitian.*

Proof. Note that M and $\overline{M}^{\mathrm{T}}$ both solve (1.1). Subtracting these two equations we find

$$(M-\overline{M}^{\mathrm{T}})E+\overline{E}^{\mathrm{T}}(M-\overline{M}^{\mathrm{T}})=0, \tag{2.15}$$

which is a Liapunov matrix equation

$$XE+\overline{E}^{\mathrm{T}}X=B, \tag{2.16}$$

for the unknown $X=M-\overline{M}^{\mathrm{T}}$ with inhomogeneity $B=0$. The standard theory of Liapunov's matrix equation (see, e.g., Barnett 1992, pp. 307, 246) tells us that provided there are no eigenvalues λ_i, λ_j of E such that $\lambda_i+\lambda_j=0$, equation (2.16) has a unique solution X. For Re spec $(E)>0$ it is given by

$$X=\int_0^\infty \mathrm{e}^{-t\overline{E}^{\mathrm{T}}}B\,\mathrm{e}^{-tE}\,\mathrm{d}t. \tag{2.17}$$

Since we have $B=0$ the unique solution of (2.15) is $X=0$ and hence $M=\overline{M}^{\mathrm{T}}$. QED

Theorem 2.3 *Let M and E be the same as in Theorem 2.2. Then the matrix $\mathrm{d}M/\mathrm{d}v$ is negative definite.*

Proof. On differentiating (1.1) with respect to v, we obtain

$$M'E+\overline{E}^{\mathrm{T}}M'=-2\rho vI, \tag{2.18}$$

where $\mathrm{d}M/\mathrm{d}v$. Equation (2.18) is recognized as another Liapunov matrix equation, and so it has a unique solution for M' given by (see (2.17))

$$M'=\int_0^\infty \mathrm{e}^{-t\overline{E}^{\mathrm{T}}}(-2\rho vI)\mathrm{e}^{-tE}\,\mathrm{d}t=-2\rho v\int_0^\infty \mathrm{e}^{-t\overline{E}^{T}}\,\mathrm{e}^{-tE}\,\mathrm{d}t. \tag{2.19}$$

Thus, for arbitrary non-zero complex vectors $\boldsymbol{\xi}$ we have

$$\overline{\boldsymbol{\xi}}\cdot M'\boldsymbol{\xi}=-2\rho v\int_0^\infty \overline{\boldsymbol{\eta}}(t)\cdot\boldsymbol{\eta}(t)dt, \quad \text{where } \boldsymbol{\eta}(t)=\mathrm{e}^{-tE}\boldsymbol{\xi}. \tag{2.20}$$

Since $\boldsymbol{\eta}(0)=\boldsymbol{\xi}\neq\mathbf{0}$ and $\boldsymbol{\eta}(t)$ is continuous at $t=0$ (so that $\boldsymbol{\eta}(t)$ is non-zero at least in a small but finite interval), we have $\overline{\boldsymbol{\xi}}\cdot M'\boldsymbol{\xi}<0$ and hence M' is negative definite. QED

We now proceed to establish the result (iii) listed in the previous section. From now on, we write $Q-\rho v^2I$ simply as Q. We first define matrices T_θ, R_θ and Q_θ by

$$\begin{aligned}
T_\theta &= \cos\theta^2 T-\sin\theta\cos\theta(R+R^{\mathrm{T}})+\sin\theta^2 Q,\\
R_\theta &= \cos\theta^2 R-\sin\theta^2 R^{\mathrm{T}}+\sin\theta\cos\theta(T-Q),\\
Q_\theta &= \cos\theta^2 Q+\sin\theta\cos\theta(R+R^{\mathrm{T}})+\sin\theta^2 T,
\end{aligned} \tag{2.21}$$

where θ is an arbitrary angle. We note that the ik-component of Q_θ is $C_{ijkl}n_jn_l-\rho v^2\cos^2\theta\delta_{ik}$ and $T_\theta=Q_{\theta+\frac{\pi}{2}}$, where $n_1=\cos\theta, n_2=\sin\theta$ and the summation over repeated subscripts ranges from 1 to 2. Thus, by the definition of the limiting speed $\hat{v}$, both T_θ and Q_θ are positive definite for $0\le v<\hat{v}$ and all θ, but at the limiting speed T_θ and Q_θ may be either positive definite or positive semi-definite depending on θ (there exists at least one θ at which T_θ has an eigenvalue 0, and likewise for Q_θ).

We shall use E exclusively to denote the unique solution of (2.14), and likewise, we define E_θ to be the unique solution of the matrix problem

$$T_\theta E_\theta^2-\mathrm{i}(R_\theta+{R_\theta}^{\mathrm{T}})E_\theta-Q_\theta=0, \qquad \text{Re spec } E_\theta>0. \tag{2.22}$$

We define M_θ according to

$$E_\theta = {T_\theta}^{-1}(M_\theta + \mathrm{i}{R_\theta}^{\mathrm{T}}) \tag{2.23}$$

(cf. (2.12)), so that it is Hermitian and satisfies the Riccati equation

$$(M_\theta - \mathrm{i}R_\theta){T_\theta}^{-1}(M_\theta + \mathrm{i}{R_\theta}^{\mathrm{T}}) - Q_\theta = 0. \tag{2.24}$$

We have the following result.

Proposition 2.4 *The Hermitian matrix M_θ defined above is independent of θ.*

Proof. On differentiating (2.21) with respect to θ, we obtain

$${T_\theta}' = -R_\theta - {R_\theta}^{\mathrm{T}}, \quad {R_\theta}' = T_\theta - Q_\theta, \quad {Q_\theta}' = R_\theta + {R_\theta}^{\mathrm{T}}, \tag{2.25}$$

where again a prime denotes differentiation with respect to θ. Differentiating $T_\theta {T_\theta}^{-1} = I$, we also obtain

$$({T_\theta}^{-1})' = -{T_\theta}^{-1}{T_\theta}'{T_\theta}^{-1}. \tag{2.26}$$

To prove the Proposition, we first differentiate (2.24) with respect to θ and use (2.26) to obtain

$$({M_\theta}' - \mathrm{i}{R_\theta}')E_\theta + \overline{E}_\theta^{\mathrm{T}}({M_\theta}' + \mathrm{i}({R_\theta}')^{\mathrm{T}}) - (M_\theta - \mathrm{i}R_\theta){T_\theta}^{-1}{T_\theta}'{T_\theta}^{-1}(M + \mathrm{i}R_\theta) - {Q_\theta}' = 0, \tag{2.27}$$

where E_θ is calculated according to (2.23). On substituting (2.25) into (2.27), we obtain

$${M_\theta}'E_\theta + \overline{E}_\theta^{\mathrm{T}}{M_\theta}' - \mathrm{i}(T_\theta - Q_\theta)E_\theta + \mathrm{i}\overline{E}_\theta^{\mathrm{T}}(T_\theta - Q_\theta) + \overline{E}_\theta^{\mathrm{T}}(R_\theta + {R_\theta}^{\mathrm{T}})E_\theta - R_\theta - {R_\theta}^{\mathrm{T}} = 0. \tag{2.28}$$

On replacing the first Q_θ in the above equation by $\overline{E}_\theta^{\mathrm{T}}(M_\theta + \mathrm{i}{R_\theta}^{\mathrm{T}})$ and the second Q_θ by $(M_\theta - \mathrm{i}R_\theta)E_\theta$, both of which can be obtained from (2.24), we obtain after simplying

$${M_\theta}'E_\theta + \overline{E}_\theta^{\mathrm{T}}{M_\theta}' = 0, \tag{2.29}$$

which is another homogeneous Liapunov matrix equation. It then follows ${M_\theta}' = 0$, and so M_θ is independent of θ. QED

Since E_θ reduces to the E defined in Theorem 2.2 when $\theta = 0$, we have $M_\theta \equiv M$, where M is the corresponding M defined in Theorem 2.2. Thus,

$$E_\theta = {T_\theta}^{-1}(M + \mathrm{i}{R_\theta}^{\mathrm{T}}). \tag{2.30}$$

On differentiating this relation with respect to θ and in turn making use of (2.26), (2.25), (2.30) and (2.22), we obtain

$$\begin{aligned}
{E_\theta}' &= -{T_\theta}^{-1}{T_\theta}'{T_\theta}^{-1}(M + \mathrm{i}{R_\theta}^{T}) + \mathrm{i}{T_\theta}^{-1}(T_\theta - Q_\theta) \\
&= {T_\theta}^{-1}(R_\theta + {R_\theta}^{T}){T_\theta}^{-1}(M + \mathrm{i}{R_\theta}^{T}) + \mathrm{i}(I - {T_\theta}^{-1}Q_\theta) \\
&= {T_\theta}^{-1}(R_\theta + {R_\theta}^{T})E_\theta + \mathrm{i}(I - {T_\theta}^{-1}Q_\theta) \\
&= \mathrm{i}{T_\theta}^{-1}Q_\theta - \mathrm{i}{E_\theta}^2 + \mathrm{i}I - \mathrm{i}{T_\theta}^{-1}Q_\theta \\
&= \mathrm{i}(I - {E_\theta}^2).
\end{aligned} \tag{2.31}$$

On integrating this matrix differential equation subject to the conditions $E_0 = E$, we obtain

Proposition 2.5 *We have*

$$E_\theta = (\cos\theta I + \mathrm{i}\sin\theta E)^{-1}(\cos\theta E + \mathrm{i}\sin\theta I). \tag{2.32}$$

Proof. We first note that for any matrix U that is dependent on θ, we have $(U^{-1})' = -U^{-1}U'U^{-1}$. It then follows that $(U^{-1}U')' = -(U^{-1}U')^2 + U^{-1}U''$, or equivalently, $(-\mathrm{i}U^{-1}U')' = -\mathrm{i}(-\mathrm{i}U^{-1}U')^2 - \mathrm{i}U^{-1}U''$. This suggests a transformation $E_\theta = -\mathrm{i}U^{-1}U'$. On substituting this relation into (2.31), we obtain $U'' + U = 0$. The general solution of the latter equation is $U = \cos\theta K_1 + \sin\theta K_2$, where K_1, K_2 are arbitrary constant matrices. From the condition $E_0 = E$ we obtain $E = -\mathrm{i}K_1^{-1}K_2$. Thus $U = K_1(\cos\theta I + \mathrm{i}\sin\theta E)$ and

$$E_\theta = -\mathrm{i}U^{-1}U' = (\cos\theta I + \mathrm{i}\sin\theta E)^{-1}(\cos\theta E + \mathrm{i}\sin\theta I). \tag{2.33}$$

QED

If λ is an eigenvalue of E, then by (2.32) the corresponding eigenvalue of E_θ is

$$\lambda_\theta = \Psi(\lambda,\theta), \quad \text{where} \ \ \Psi(\lambda,\theta) = \frac{\lambda\cos\theta + \mathrm{i}\sin\theta}{\cos\theta + \mathrm{i}\lambda\sin\theta} = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\theta}\ln(\cos\theta + \mathrm{i}\lambda\sin\theta). \tag{2.34}$$

The real part of λ_θ is indeed positive, as required in (2.22). It then follows that

$$\int_0^\pi \lambda_\theta \,\mathrm{d}\theta = \pi. \tag{2.35}$$

This leads us to

Proposition 2.6 *We have*

$$\int_0^\pi E_\theta \,\mathrm{d}\theta = \pi I. \tag{2.36}$$

Proof. We use the spectral calculus for matrices. Choose any closed curve Γ in the complex plane surrounding all the eigenvalues of E with positive real part and lying in the open half plane $\mathrm{Re}\,\lambda > 0$. Then, by Proposition 2.5 we have

$$\begin{aligned}\int_0^\pi E_\theta \,\mathrm{d}\theta &= \int_0^\pi \frac{1}{2\pi\mathrm{i}}\oint_{\lambda\in\Gamma}\Psi(\lambda,\theta)(\lambda I - E)^{-1}\,\mathrm{d}\lambda\,\mathrm{d}\theta \\ &= \frac{1}{2\pi\mathrm{i}}\oint_{\lambda\in\Gamma}\int_0^\pi \Psi(\lambda,\theta)\,\mathrm{d}\theta\,(\lambda I - E)^{-1}\,\mathrm{d}\lambda \\ &= \frac{1}{2\pi\mathrm{i}}\oint_{\lambda\in\Gamma}\pi(\lambda I - E)^{-1}\,\mathrm{d}\lambda = \pi I,\end{aligned}$$

where we have used (2.35) in the second step. QED

On integrating (2.30) and making use of (2.36), we obtain

Theorem 2.7 *The unique solution of the algebraic Riccati equation (1.1) that satisfies* $\mathrm{Re}\,\mathrm{spec}\,(T^{-1}(M + \mathrm{i}R^{\mathrm{T}})) > 0$ *is given explicitly by*

$$M = \left(\int_0^\pi {T_\theta}^{-1}\,\mathrm{d}\theta\right)^{-1}\left(\pi I - \mathrm{i}\int_0^\pi {T_\theta}^{-1}{R_\theta}^{\mathrm{T}}\,\mathrm{d}\theta\right). \tag{2.37}$$

Define H and S through

$$H = \frac{1}{\pi}\int_0^\pi {T_\theta}^{-1}\,\mathrm{d}\theta, \quad S = -\frac{1}{\pi}\int_0^\pi {T_\theta}^{-1}{R_\theta}^{\mathrm{T}}\,\mathrm{d}\theta. \tag{2.38}$$

Equation (2.37) may then be written as

$$M = H^{-1} + \mathrm{i}H^{-1}S. \tag{2.39}$$

We note that since Q has been used to denote $Q - \rho v^2 I$ in the second half of this section, H, S and M all depend on the wave speed v. We also note that since at the limiting speed there exists at least one angle in $[0, \pi]$ at which T_θ has a zero eigenvalue, the matrix H is singular as $v \to \hat{v}$. But H^{-1} is well-defined for all $0 \le v \le \hat{v}$ and at $v = \hat{v}$ at least one of its eigenvalues must vanish.

It is obvious that H and S are both real matrices and H is symmetric. Since M must necessarily be Hermitian, it follows that $H^{-1}S$ is skew-symmetric, a result that has previously been obtained with the aid of the Stroh formalism. The property (iii) listed in the previous section then follows from the facts that $\mathrm{tr}\, M = \mathrm{tr}\, H^{-1}$, $\mathbf{w} \cdot M\mathbf{w} = \mathbf{w} \cdot H^{-1}\mathbf{w}$ for any real vector $\mathbf{w}$, and that H^{-1} is positive semi-definite for $0 \le v \le \hat{v}$.

3 Connection with the Stroh formalism

The expressions (2.38) and (2.39) are well-known in the Stroh formalism, but were derived using a different procedure (see Barnett and Lothe 1974, Lothe and Barnett 1976). The essence of the Stroh formalism (Stroh 1958, 1962) is to write (2.8) and (2.9) in the following form:

$$\frac{\mathrm{d}}{\mathrm{d}\tilde{y}}\boldsymbol{\xi} = N\boldsymbol{\xi}, \tag{3.1}$$

where $\tilde{y} = \mathrm{i}y$,

$$\boldsymbol{\xi} = \begin{pmatrix} \boldsymbol{z} \\ \boldsymbol{t} \end{pmatrix}, \quad \boldsymbol{t} = T\frac{\mathrm{d}}{\mathrm{d}\tilde{y}}\boldsymbol{z} + R^T\boldsymbol{z}, \quad N = \begin{pmatrix} -T^{-1}R^T & T^{-1} \\ RT^{-1}R^T - Q & -RT^{-1} \end{pmatrix}. \tag{3.2}$$

The vector $\boldsymbol{t}$ defined above is seen to be the traction; see (2.9). We recall that starting from equation (2.21) we have been using Q for $Q - \rho v^2 I$. The Stroh formalism is essentially a Hamiltonian formulation of the travelling wave problem with the spatial variable x_2 viewed as a time-like variable. The matrix N defined above is recognized as a *Hamiltonian matrix* and is much studied in control theory; see, e.g., Knobloch *et al.* (1993, Appendix A).

Our matrix E does not feature in the Stroh formalism, but its importance can be seen from the relation

$$N\begin{pmatrix} I \\ \mathrm{i}M \end{pmatrix} = \mathrm{i}\begin{pmatrix} I \\ \mathrm{i}M \end{pmatrix}E, \tag{3.3}$$

which shows that if λ is an eigenvalue of E then $\mathrm{i}\lambda$ is an eigenvalue of N.

Our differential equation (2.31) is analogous to Barnett and Lothe's (1976) equation (2.13), namely

$$\frac{\mathrm{d}}{\mathrm{d}\theta}N_\theta = -(I + N_\theta^2), \tag{3.4}$$

where N_θ is obtained from the expression $(3.2)_3$ for N by replacing T, R, Q by $T_\theta, R_\theta, Q_\theta$, respectively. Following the same line of argument as in the proof of Proposition 2.5, we may deduce that the unique solution of (3.4) that satisfies the initial condition $N_0 = N$ is

$$N_\theta = (\cos\theta I + \sin\theta N)^{-1}(-\sin\theta I + \cos\theta N). \tag{3.5}$$

Surprisingly, this explicit representation of N_θ does not seem to have previously been noted in the literature. We now show that some well-known results in the Stroh formalism can easily be deduced with the aid of this explicit representation.

Let $p^{(1)}, p^{(2)}, p^{(3)}$ be the three eigenvalues of N that have positive imaginary parts and

$$\boldsymbol{\xi}^{(1)} = \begin{pmatrix} \mathbf{a}^{(1)} \\ \mathbf{b}^{(1)} \end{pmatrix}, \quad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} \mathbf{a}^{(2)} \\ \mathbf{b}^{(2)} \end{pmatrix}, \quad \boldsymbol{\xi}^{(3)} = \begin{pmatrix} \mathbf{a}^{(3)} \\ \mathbf{b}^{(3)} \end{pmatrix} \tag{3.6}$$

be a set of associated eigenvectors (for simplicity we assume that N is diagonalizable). It follows from (3.5) that the corresponding eigenvalues of N_θ are

$$p_\theta^{(k)} = \frac{p^{(k)}\cos\theta - \sin\theta}{\cos\theta + p^{(k)}\sin\theta} = \frac{\mathrm{d}}{\mathrm{d}\theta}\ln(\cos\theta + p^{(k)}\sin\theta), \quad k = 1, 2, 3, \tag{3.7}$$

and the associated eigenvectors are the same as those of N. Thus, $N_\theta\boldsymbol{\xi}^{(k)} = p_\theta^{(k)}\boldsymbol{\xi}^{(k)}, \; k = 1, 2, 3$, and so

$$\bar{N}\boldsymbol{\xi}^{(k)} = \mathrm{i}\boldsymbol{\xi}^{(k)}, \quad k = 1, 2, 3, \tag{3.8}$$

where

$$\bar{N} = \frac{1}{\pi}\int_0^\pi N_\theta \, \mathrm{d}\theta = \begin{pmatrix} S & H \\ * & S^T \end{pmatrix}, \tag{3.9}$$

and we have made use of the property that

$$\int_0^\pi p_\theta^{(k)} \, \mathrm{d}\theta = \mathrm{i}\pi, \quad k = 1, 2, 3,$$

which is deduced from (3.7). The S and H in (3.9) are given by (2.38) and the element represented by $*$ is not written out to avoid introducing extra notation. It follows from (3.8) that $\bar{N}(\boldsymbol{\xi}^{(1)}, \boldsymbol{\xi}^{(2)}, \boldsymbol{\xi}^{(3)}) = \mathrm{i}(\boldsymbol{\xi}^{(1)}, \boldsymbol{\xi}^{(2)}, \boldsymbol{\xi}^{(3)})$. Thus, $SA + HB = \mathrm{i}A$ and so

$$M = -\mathrm{i}BA^{-1} = H^{-1} + \mathrm{i}H^{-1}S, \tag{3.10}$$

where

$$A = (\boldsymbol{a}^{(1)}, \boldsymbol{a}^{(2)}, \boldsymbol{a}^{(3)}), \quad B = (\boldsymbol{b}^{(1)}, \boldsymbol{b}^{(2)}, \boldsymbol{b}^{(3)}).$$

It was shown by Lothe and Barnett (1976) that (3.10) is also valid when N is not diagonalizable.

The above derivation of (3.10) under the Stroh formalism makes use of the eigenvectors of N and the nonsemisimple case has to be considered separately. Our derivation of (3.10) with the use of E_θ does not use such eigenvectors and seems to be more natural and more straightforward. In addition, our proof of the three properties of M listed in Section 1 only involves a matrix algebraic analysis of properties of M and E_θ alone.

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