# Evolution of rate-independent systems *

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# 1 Introduction

Rate-independent systems occur as limit problems in many physical and mechanical problems if the interesting time scales are much longer than the intrinsic time scales in the system. Rate-independent systems are sometimes also called quasi-static systems, however, the term "quasistatic" is often used in a more general sense, namely if the inertial terms in a system are neglected but viscous effects might still be present.

This survey only considers systems which satisfy the following exact definition of rate independence. The definition is formulated in terms of input functions $\ell : [0, \infty) \to \mathcal{X}$ and output functions $y : [0, \infty) \to \mathcal{Y}$. The usage of input and output functions is necessary, since rate-independent systems have no own dynamics, they rather respond to changes in the input.

**Definition 1.1** *A system $\mathcal{H}$ is called a **rate-independent system** with input data $y_0 \in \mathcal{Y}$ and $\ell \in \mathrm{C}^0([t_1, t_2], \mathcal{X})$ if the set $\mathcal{O}([t_1, t_2], y_0, \ell) \subset \mathrm{C}^0([t_1, t_2], \mathcal{Y}) \cap \{y(t_1) = y_0\}$ of possible outputs satisfies for all strictly monotone time reparametrizations $\alpha : [t_1, t_2] \to [t_1^*, t_2^*]$ with $\alpha(t_1) = t_1^*$ and $\alpha(t_2) = t_2^*$ the relation*

$$y \in \mathcal{O}([t_1, t_2], y_0, \ell) \quad \Longleftrightarrow \quad y \circ \alpha \in \mathcal{O}([t_1^*, t_2^*], y_0, \ell \circ \alpha).$$

*We call the system a **multi-valued evolutionary system** if the following additional conditions hold:*

$$\begin{array}{ll} \text{Concatenation:} & \widehat{y} \in \mathcal{O}([t_1, t_2], y_1, \ell),\ \widetilde{y} \in \mathcal{O}([t_2, t_3], y_2, \ell),\ \widehat{y}(t_2) = \widetilde{y}(t_2) = y_2 \\ & \Longrightarrow \quad y \in \mathcal{O}([t_1, t_3], y_1, \ell) \text{ where } y(t) = \begin{cases} \widehat{y}(t) & \text{for } t \in [t_1, t_2], \\ \widetilde{y}(t) & \text{for } t \in [t_2, t_3], \end{cases} \\ \text{restriction:} & t_1 < t_2 < t_3 < t_4 \text{ and } y \in \mathcal{O}([t_1, t_4], y_1, \ell) \\ & \Longrightarrow \quad y|_{[t_2, t_3]} \in \mathcal{O}([t_2, t_3], y(t_2), \ell). \end{array}$$

Note that the definition is such that the system may have several solutions for a given initial value $y(t_1)$ and a given input function $\ell$. Since rate-independent systems occur as limit problems, it is to be expected that the solutions are not unique without strong further assumptions.

Rate-independent systems occur on the level of ordinary differential equations as well as for partial differential equations. The simplest systems of this type arise in the limit $\varepsilon = 0$ in the following systems

$$\varepsilon M(t, y(t))\ddot{y}(t) + D(t, y(t))\dot{y}(t) = G(t, y(t))\dot{\ell}(t) \qquad \text{or} \qquad \varepsilon\dot{y}(t) = -\mathrm{D}\mathcal{U}(y(t)) + \ell(t),$$

which appears in the slow-time limit of rigid-body dynamics. However, such smooth systems are in some sense trivial, since $y(t)$ can be obtained as a function of $\ell(t)$ without any dynamical effects. Interesting problems occur only if nonsmoothness comes into play, like in dry friction, where the frictional force $R$ is a multi-valued, nonsmooth function of the velocity $v$, namely $R = \mathrm{Sign}(v)$ where Sign is the multi-valued signum function. Thus, replacing $\varepsilon\dot{y}(t)$ above by $R(\dot{y}) = \mathrm{Sign}(\dot{y})$, the corresponding system takes the form

$$0 \in \mathrm{Sign}(\dot{y}(t)) + \mathrm{D}\mathcal{U}(y(t)) - \ell(t), \quad y(0) = y_1.$$

Since $\mathrm{Sign}(\gamma v) = \mathrm{Sign}(v)$ for all $\gamma > 0$ and $v$, it is easy to see that the problem is rate independent.

Applications in partial differential equations arise naturally in the theory of elastoplasticity or if an elastic body like a rubber is drawn slowly over a rough surface such that dry friction acts but inertia does not matter. In fact, the driving problems in the theory of rate-independent hysteresis have been the theory of elastoplasticity on the one hand and hysteresis effects in magnetism on the other hand. While in the former theory the aspect of partial differential equations was always a focus of attention, in magnetism a proper theory for the field equations was attacked only recently. Instead of this, highly complex scalar-valued hysteresis operators like the Preisach and the Prandtl-Ishlinskii operators were developed. In the latter case the ordering properties of $\mathbb{R}^1$ are essential whereas in the former theory convexity methods in Hilbert spaces are the main tool and thus vector-valued and tensor-valued generalizations of the Preisach and Prandtl-Ishlinskii operator can be treated, see [KP89, Vis94, BS96, Kre96]. We will survey the scalar-valued theory only little, since our focus is on methods for problems in continuum mechanics, where complex hysteretic behavior occurs through spatial variations of the the internal variables.

This survey brings together different aspects of rate-independent models or hysteresis operators in the context of continuum mechanics. In fact, there are several areas in these fields, which have evolved quite independently and have developed their own languages and notations. Here we try to compare these different approaches by translating them into one language and thus hope to provide a useful overview of the different methods in the field. We will not try to survey the whole theory of hysteresis operators and rate-independent models which started on the mechanical side more than 100 years ago but had major mathematical achievements only in the mid 1970s ([Mor74, Mor76, Joh76]). The theory was formulated on the level of research monographs only 15 years later starting with [KP89]. Afterwards, several books [Mon93, Vis94, BS96, Kre96, Kre99] appeared, which cover a variety of different aspects. We also refer to these works for the historical background. Note that most of these books treat also a lot of models which are not rate-independent in the sense we have defined above, but they usually involve a rate-independent operator which is embedded into a larger system which is rate-independent. Nevertheless, the guiding theme of these works are the common difficulties one has in treating hysteretic behavior which is intrinsically nonsmooth. In [Vis94, Alb98] the emphasis on applications in continuum mechanics is quite similar to ours, but we restrict ourselves to pure rate independence.

The unified approach in this survey will be a new energetic approach developed within the last five years in [MT99, MTL02, MT04]. It combines in a natural way several different approaches. Classically, rate-independent systems are either written as an evolutionary (quasi-) variational inequality

$$\langle \mathrm{D}\mathcal{E}(t, y(t)), v - \dot{y}(t) \rangle + \Psi(y(t), v) - \Psi(y(t), \dot{y}(t)) \geq 0 \text{ for all } v \in Y,$$

where $Y$ is a Banach space with dual pairing $\langle \cdot, \cdot \rangle$, $\mathcal{E} : [0, T] \times Y \to \mathbb{R}$ is an energy-storage potential with Gateaux derivative $\mathrm{D}\mathcal{E}(t, y) \in Y^*$ and $\Psi : Y \times Y \to [0, \infty)$ is the dissipation (pseudo-) potential. Rate independence is implemented through the assumption that $\Psi(y, \cdot)$ is homogeneous of degree 1, i.e.,

$$\Psi(y,\gamma v)=\gamma\Psi(y,v) \text{ for } \gamma\geq 0, \qquad \text{and } \Psi(y,v)=\sup\{\,\langle\sigma,v\rangle \mid \sigma\in C_*(y)\,\},$$

where $C_*(y)\subset Y^*$ is often called the elastic domain. The equivalent formulation using subdifferentials is

$$0\in\partial_v\Psi(y,\dot y)+\mathrm{D}\mathcal{E}(t,y)\subset Y^*,$$

which is a slight generalization of the doubly nonlinear form studied in [CV90]. We continue to use $\partial_v\Psi(y,\dot y)$ to indicate the subdifferential of $v\mapsto\Psi(y,v)$ at the point $v=\dot y$, i.e., only with respect to the second variable.

Using the Legendre transform $\mathcal{L}$, such that $\Psi(y,\cdot)=\mathcal{L}I_{C_*(y)}$, one arrives at the following differential inclusion, also called generalized sweeping process,

$$\dot y(t)\in\partial I_{C_*(y(t))}(-\mathrm{D}\mathcal{E}(t,y(t))).$$

If the potential $\mathcal{E}(t,\cdot)$ is convex and if $\Psi$ does not depend on $y\in Y$, then the above equations are equivalent to the following energetic formulation:

Find $y:[0,T]\to Y$ with $y(0)=y_0$ such that for all $t\in[0,T]$
the stability (S) and the energy balance (E) hold:
(S) $\mathcal{E}(t,y(t))\leq\mathcal{E}(t,\widehat y)+\Psi(\widehat y-y(t))$ for all $\widehat y\in\mathcal{Y}$;
(E) $\mathcal{E}(t,y(t))+\int_0^t\Psi(\dot y(s))\,\mathrm{d}s=\mathcal{E}(0,y_0)+\int_0^t\partial_s\mathcal{E}(s,y(s))\,\mathrm{d}s.$

For general potentials $\mathcal{E}$, the energetic formulation may be considered as a weak form of the variational inequality, since it is derivative free for the solution $y$ as well as for the functionals $\mathcal{E}$ and $\Psi$. However, the smoothness of the loadings has to be a little higher since the power of the external forces, given via $t\mapsto\partial_t\mathcal{E}(t,y)$, must be well defined.

In Section 2 we study these systems in the standard case with a quadratic energy $\mathcal{E}(t,y)=\frac12\langle Ay,y\rangle-\langle\ell(t),y\rangle$ on a Hilbert space $Y$. We compare several equivalent formulations, address their basic properties and explain the typical approaches to prove existence and uniqueness of solutions. Thus, the hysteresis operator $\mathcal{H}$ with $y=\mathcal{H}(y_0,\ell)$ can be defined and in Section 2.4 we discuss the mapping properties of $\mathcal{H}$ in different Banach spaces.

However, the main emphasis of this work will not be the continuity properties of the solution operator. We focus mainly on the question of solvability in general nonconvex problems where uniqueness does not hold and where even existence of solutions is questionable. Thus, we will mostly make simple assumptions on the temporal behavior of the loading function which often can be generalized. Instead we want to be most general in terms of the behavior of the energy $\mathcal{E}(t,y)$ on the state variable $y$, such that we are able to deal with generally nonconvex problems like finite-strain elastoplasticity. Also the dissipation law has to be understood in a more general setting. In particular, the dissipation potential is replaced by a more general dissipation distance $\mathcal{D}:\mathcal{Y}\times\mathcal{Y}\to[0,\infty]$ which generalizes $\Psi$ via $\mathcal{D}(y_0,y_1)=\Psi(y_1-y_0)$. The main emphasis will be on the topological and analytical properties of the functionals

$$\mathcal{E}:[0,T]\times\mathcal{Y}\to\mathbb{R}_\infty:=\mathbb{R}\cup\{\infty\} \text{ and } \mathcal{D}:\mathcal{Y}\times\mathcal{Y}\to[0,\infty].$$

In Section 3 we set up the abstract formulation and show how first a priori estimates can be used to estimate possible solutions. Moreover, we introduce a time-incremental minimization problems (IP) which will be the basis of most of our existence proofs, namely

(IP) For a given $y_0$ and a partition $0 = t_0 < t_1 < \cdots < t_N = T$ find $y_1, y_2, \ldots, y_N \in \mathcal{Y}$ such that $y_k \in \operatorname{Arg\,min}\{\, \mathcal{E}(t_k, y) + \mathcal{D}(y_{k-1}, y) \mid y \in \mathcal{Y}\,\}$.

Under natural conditions it is possible to derive a priori estimates for the solutions $(y_k)_{k=1,\ldots,N}$ of (IP) in the form

$$\mathcal{E}(t_k, y_k) \leq E_* \quad \text{and} \quad \textstyle\sum_{k=1}^N \mathcal{D}(y_{k-1}, y_k) \leq E^*,$$

i.e., they are independent of the partitions. Moreover, uniform convexity of $\mathcal{E}(t,\cdot)+\mathcal{D}(y_0,\cdot)$ will provide a Lipschitz bound.

Thus, we see that the problem is governed by three different topologies (or function spaces). For instance in a Banach space setting, the energy-storage functional $\mathcal{E}$ will be coercive with respect to a Banach space $Y_1$. Moreover, it might be uniformly convex with respect to a norm of a larger Banach space $Y_2$. Finally, the dissipation distance $\mathcal{D}$ might be bounded from below by a norm of a Banach space $Y_3$. Then, for solutions of (S) & (E) or for piecewise constant interpolants of solutions to (IP) we can expect to obtain the following, typical a priori estimates

$$\|y(t)\|_{Y_1} \leq C_1, \quad \|y(t){-}y(s)\|_{Y_2} \leq C_2|t{-}s|, \quad \mathrm{Var}_{Y_3}(y;[0,T]) \leq c_3 \mathrm{Diss}_{\mathcal{D}}(z;[0,T]) \leq C_3,$$

for $t, s \in [0,T]$. Here, the total dissipation is defined as

$$\mathrm{Diss}_{\mathcal{D}}(y;[r,s]) = \sup\{\, \textstyle\sum_{j=1}^n \mathcal{D}(y(t_{j-1}), y(t_j)) \mid n \in \mathbb{N},\ r \leq t_0 < t_1 \cdots < t_n \leq s\,\}$$

and $\mathrm{Var}_{Y_3}(y;[r,s])$ is obtained similarly by replacing $\mathcal{D}(y(t_{j-1}), y(t_j))$ via $\|y(t_j){-}y(t_{j-1})\|_{Y_3}$. Thus, there will be two quite different approaches to show existence of solutions. The first one (see Section 4) is based on convexity, uses the norm in $Y_2$ and yields solutions in $\mathrm{C}^{\mathrm{Lip}}([0,T], Y_2)$. The second approach (see Section 5) works without convexity, relies on the dissipation estimate and provides solutions in $\mathrm{BV}([0,T], Y_3)$.

In Section 4 we study the convex cases in more detail. Under suitable additional smoothness assumptions it is then possible to prove existence and uniqueness of solutions. In this part no compactness arguments are needed to establish convergence; in fact, the error between the incremental solutions and the true solution can be estimated in terms of the fineness of the time discretization. This part is based on work in [MT04, BKS04, MR04b]. Section 4.4 shows that in the best case the solutions $y$ have derivatives of bounded variations, i.e., $\dot{y} \in \mathrm{BV}([0,T], Y)$. Adapting the proofs in [HR95, AC00] we provide a convergence of the incremental solutions which is linear in the fineness of the partition.

In Section 5 we study general nonconvex and nonsmooth systems, where uniqueness is not to be expected. The basic existence result relies on compactness assumptions and is based on work in [MT99, MTL02, MM04a, DFT04, FM04]. In terms of the above-mentioned Banach spaces $Y_1$ and $Y_3$ the compactness assumption roughly means, that $Y_1$ is compactly embedded in $Y_3$.

It can be seen easily that the solutions of (S) & (E) may have jumps if $\mathcal{E}$ is nonconvex. Thus, in Section 5.4 we explain how rate-independent limits of viscous problems have been obtained in [EM04b]. They turn jumps into suitable continuous pathes in state space. In Section 5.5 we study situations where the state space $\mathcal{Y}$ may depend on time. Finally, Section 5.6 addresses the question of relaxations of rate-independent problems, since many applications in continuum mechanics lead to systems in which the incremental problem (IP) does not have solutions due to formation of microstructure.

Section 6 is devoted to dissipation laws which are non-associated, i.e., they can not be derived from a principle of maximal dissipation. In particular, the energetic formulation is no longer available, since the set of frictional forces $\mathcal{R}(t, y, \dot{y})$ is no longer given by the subdifferential $\partial_v \Psi(y, \dot{y})$. In this area much less is known, but it is of great importance in queueing theory, in plasticity models in soil mechanics and in the area of Coulomb friction of sliding elastic bodies or structures. The last application was a major stimulant for the theory of non-associated flow rules over the last 15 years, [MO87, And91, AK97, MPS02].

The final Section 7 presents a selection of applications in continuum mechanics which are meant to illustrate the abstract theory developed in the previous sections. In Section 7.1 we recall the classical theory of linearized elastoplasticity, which was the main driving forces in the early mathematical developments, and in Section 7.2 mention some result in finite-strain elastoplasticity. We also discuss some models for shape-memory alloys (cf. Section 7.3) and for ferromagnetic materials (cf. Section 7.4). Finally, we show that certain damage problems can be also put into the energetic framework, namely a delamination problem (cf. Section 7.5) as well as a problem of rate-independent crack growth in brittle materials (cf. Section 7.6). The latter application is especially interesting, as it provides a true reason to the abstract formulation of the energetic problem in Section 5. In the crack problem the state space $\mathcal{Y}$ is far from being a subset of a Banach space, since $y = (u, \Gamma) \in \mathcal{Y}$ consists of subsets $\Gamma$ of the body $\overline{\Omega} \subset \mathbb{R}^d$ and a deformation $u \in \mathrm{W}^{1,p}(\Omega \backslash \Gamma; \mathbb{R}^d)$.

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# 2 The simple case with a quadratic energy

In this section we survey the classical results on evolutionary variational inequalities which can be formulated in several equivalent ways. We collect these formulations for later reference, since each of the formulations has advantages when generalizations have to be done. In the following subsection we shortly address the existence theory via monotone operators and via time-incremental methods. Finally, we will review some results on the continuity of the solution operator in different function spaces.

## 2.1 Equivalent formulations

We start with a Hilbert space $Y$ with dual $Y^*$ and dual pairing $\langle\cdot,\cdot\rangle : Y^*\times Y \to \mathbb{R}$ and a positive definite operator $A \in \mathrm{Lin}(Y, Y^*)$, i.e., $A = A^*$ and there exists a constant $\alpha > 0$ such that $\langle Ay, y\rangle \geq \alpha\|y\|^2$ for all $y \in Y$. Often, $Y^*$ and $Y$ are identified, $A$ is taken to be the identity $\mathbf{1}$ and instead of the dual pairing the scalar product is used. However, as is common practice in mechanics, we prefer to distinguish the space and its dual.

For a function $\ell \in \mathrm{C}^1([0,T], Y^*)$ we define the energy functional

$$\mathcal{E}(t,y) = \tfrac{1}{2}\langle Ay, y\rangle - \langle \ell(t), y\rangle.$$

Here $\ell$ serves as input datum and is called external loading in mechanics. We will use $\Sigma = -\mathrm{D}_y\mathcal{E}(t,y) = \ell(t) - Ay$ to denote the force generated by the potential.

Moreover, let a dissipation functional $\Psi : Y \to [0,\infty]$ be given which is convex, lower semi-continuous and positively homogeneous of degree 1, i.e.,

$$\Psi(\gamma v) = \gamma\Psi(v) \quad \text{for all } \gamma \geq 0 \text{ and } v \in Y.$$

(Throughout we assume that “convex” already means that the function is also “proper”, i.e., not identically $+\infty$.) Its subdifferential is given via $\partial\Psi(v) = \{\, \sigma \in Y^* \mid \forall w \in Y : \Psi(w) \geq \Psi(v) + \langle \sigma, w - v\rangle \,\}$ and we set $C_* = \partial\Psi(0) \subset Y^*$, which is convex and closed. Duality theory shows that $\Psi$ is the Legendre transform of the characteristic function $I_{C_*} : Y^* \to [0,\infty]$, i.e., $\Psi(v) = \sup\{\, \langle \sigma, v\rangle \mid \sigma \in C_* \,\}$.

The ***subdifferential formulation*** (SF) of the rate-independent hysteresis problem associated with $\mathcal{E}$ and $\Psi$ reads

$$\text{(SF)} \qquad 0 \in \partial\Psi(\dot y(t)) + \mathrm{D}_y\mathcal{E}(t, y(t)) = \partial\Psi(\dot y(t)) + Ay - \ell(t) \subset Y^*. \tag{2.1}$$

In mechanics, (SF) is a force balance which may be written as $\Sigma \in \partial\Psi(\dot y)$.

Using the definition of the subdifferential $\partial\Psi(\dot y)$ leads to the ***variational inequality***

$$\text{(VI)} \qquad \forall v \in Y : \ \langle Ay - \ell(t), v - \dot y\rangle + \Psi(v) - \Psi(\dot y) \geq 0. \tag{2.2}$$

This formulation is called the ***primal form***, since $y \in Y$ is the primal variable while $\Sigma \in Y^*$ is the dual variable.

Using the Legendre transform $\Psi = \mathcal{L}(I_{C_*})$ we can rewrite (2.1) (which reads in short form $\Sigma \in \partial\Psi(\dot y)$) as the ***differential inclusion***:

$$\text{(DI)} \qquad \dot y(t) \in \partial I_{C_*}(\Sigma) = \partial I_{C_*}(-\mathrm{D}\mathcal{E}(t, y(t))) = \mathrm{N}_{C_*}(\Sigma(t)) \subset Y, \tag{2.3}$$

where we used the standard result that for closed convex sets $C_*$ the subdifferential $\partial I_{C_*}(\sigma)$ equals the outward normal cone $\mathrm{N}_{C_*}(\sigma) = \{\, v \in Y \mid \forall \widehat\sigma \in C_* : \langle \widehat\sigma - \sigma, v\rangle \leq 0 \,\}$.

Introducing the variable $u = -Ay \in Y^*$ and the moving sets $C_*(t) = -\ell(t) + C_* \subset Y^*$, we arrive at the ***sweeping-process*** formulation

$$\text{(SW)} \qquad -\dot u(t) \in A\mathrm{N}_{C_*(t)}(u(t)), \tag{2.4}$$

which is used in [Mon93] with $A = \mathbf{1}$ since $Y = Y^*$ is assumed.

Using the definition of the subdifferential $\partial I_{C_*}$ in (2.3) we see that (VI) is equivalent to the ***dual variational inequality***

$$\text{(DVI)} \qquad \Sigma = \ell - Ay \in C_* \ \text{ and } \ \langle \Sigma - \widehat{\sigma}, \dot{y} \rangle \geq 0 \text{ for all } \widehat{\sigma} \in C_*. \tag{2.5}$$

Integration over $[0,T]$ leads to a weakened form which allows $y$ to lie in $\mathrm{BV}([0,T],X)$ by employing a suitable Stieltjes integral:

$$\Sigma = \ell - Ay \in C_* \ \text{ and } \ \int_0^T \langle \Sigma(t) - \widetilde{\sigma}(t), \mathrm{d}y(t) \rangle \geq 0 \text{ for all } \widetilde{\sigma} \in \mathrm{C}^0([0,T], C_*). \tag{2.6}$$

In (DVI) we may also eliminate completely the primal variable $y$ by using $\dot{y} = A^{-1}(\dot{\ell} - \dot{\Sigma})$:

$$\Sigma \in C_* \ \text{ and } \ \forall \widehat{\sigma} \in C_* : \langle \Sigma - \widehat{\sigma}, A^{-1}(\dot{\Sigma} - \dot{\ell}) \rangle \geq 0. \tag{2.7}$$

Finally, we derive the energetic formulation which is the basis for the more recent approach to general nonconvex problems. Whereas the equivalence of the above problems is well-known, see, e.g., [DL76, HR99]), the equivalence to the energetic formulation is less known. Thus, we explain it in more detail.

It can be easily seen that (VI) is equivalent to the following two local conditions

$$\begin{aligned} &\text{(S)}_{\mathrm{loc}} \quad \forall \widehat{v} \in Y: \ \langle \mathrm{D}\mathcal{E}(t, y(t)), \widehat{v} \rangle + \Psi(\widehat{v}) \geq 0; \\ &\text{(E)}_{\mathrm{loc}} \quad \langle \mathrm{D}\mathcal{E}(t, y(t)), \dot{y}(t) \rangle + \Psi(\dot{y}(t)) \leq 0. \end{aligned} \tag{2.8}$$

For $(\mathrm{S})_{\mathrm{loc}}$ simply let $v = \alpha \widehat{v}$ with $\alpha \to \infty$ in (VI), and for $(\mathrm{E})_{\mathrm{loc}}$ let $v = 0$. However, since $\mathcal{E}(t, \cdot)$ and $\Psi$ are convex, we conclude that $y(t)$ is a global minimizer of $\widehat{y} \mapsto \mathcal{E}(t, \widehat{y}) + \Psi(\widehat{y} - y(t))$ by letting $\widehat{y} = y(t) + \widehat{v}$. Moreover, $(\mathrm{S})_{\mathrm{loc}}$ and $(\mathrm{E})_{\mathrm{loc}}$ together imply $\langle \mathrm{D}\mathcal{E}(t, y(t)), \dot{y}(t) \rangle + \Psi(\dot{y}(t)) = 0$, which leads to

$$\tfrac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}(t, y(t)) = \partial_t \mathcal{E}(t, y(t)) - \Psi(\dot{y}(t)) = -\langle \dot{\ell}(t), y(t) \rangle - \Psi(\dot{y}(t)).$$

This leads to the ***energetic formulation*** which is based on the ***global stability condition*** (S) and the ***global energy balance*** (E), which is obtained by integration over $t \in [0,T]$:

$$\begin{aligned} &\text{(S)} \quad \forall \widehat{y} \in Y: \ \mathcal{E}(t, y(t)) \leq \mathcal{E}(t, \widehat{y}) + \Psi(\widehat{y} - y(t)); \\ &\text{(E)} \quad \mathcal{E}(t, y(t)) + \mathrm{Diss}_\Psi(y; [0,t]) = \mathcal{E}(0, y(0)) + \int_0^t \partial_s \mathcal{E}(s, y(s))\,\mathrm{d}s, \end{aligned} \tag{2.9}$$

where $\mathrm{Diss}_\Psi(y; [r,s]) = \int_r^s \Psi(\dot{y}(t))\,\mathrm{d}t$ and $\int_0^t \partial_s \mathcal{E}(s, y(s))\,\mathrm{d}s = -\int_0^t \langle \dot{\ell}(s), y(s) \rangle\,\mathrm{d}s$.

The stability condition can be formulated in terms of the ***sets of stable states***

$$\begin{aligned} \mathcal{S}(t) &= \{\, y \in Y \mid \forall \widehat{y} \in Y: \ \mathcal{E}(t, y) \leq \mathcal{E}(t, \widehat{y}) + \Psi(\widehat{y} - y) \,\} \subset Y, \\ \mathcal{S}_{[0,T]} &= \textstyle\bigcup_{t \in [0,T]} (t, \mathcal{S}(t)) \subset [0,T] \times Y. \end{aligned}$$

Now, (S) just means $y(t) \in \mathcal{S}(t)$. The major simplification in the theory of quadratic energies arises from the fact that $\mathcal{S}(t)$ can be given explicitly in the form

$$\mathcal{S}(t) = \{\, y \in Y \mid \ell(t) - Ay \in C_* \,\} = A^{-1}(\ell(t) - C_*).$$

Hence, $\mathcal{S}(t)$ is a closed convex set and thus it is weakly closed.

A typical situation in continuum mechanical problems is that the state variables $y \in Y$ consist of two components, namely an elastic (or non-dissipative) component $u \in U$ and an internal (or dissipative) component $z \in Z$. The splitting is such that $\Psi : Y \to [0,\infty]$ depends only on $\dot{z}$ but not on $\dot{y}$. In particular we have

$$y = (u,z) \in U{\times}Z = Y \quad \text{and} \quad \Psi(\dot{y}) = \Psi((\dot{u},\dot{z})) = \widetilde{\Psi}(\dot{z}), \tag{2.10}$$

where $\widetilde{\Psi}$ satisfies $\widetilde{\Psi}(\dot{z}) > 0$ for $\dot{z} \neq 0$. The linear operator $A$ takes the form $\left(\begin{smallmatrix} A_{UU} & A_{UZ} \\ A_{ZU} & A_{ZZ} \end{smallmatrix}\right)$ $\in \mathrm{Lin}(U{\times}Z, U^*{\times}Z^*)$, $\ell(t) = (\ell_U(t), \ell_Z(t)) \in U^*{\times}Z^*$ and for the subdifferential $\partial\Psi$ we have

$$\partial\Psi((\dot{u},\dot{z})) = \{0\}{\times}\partial\widetilde{\Psi}(\dot{z}) \in U^*{\times}Z^*.$$

With these definitions the subdifferential formulation (SF), see (2.1), takes the form

$$0 = A_{UU}u + A_{UZ}z - \ell_U(t) \in U^* \quad \text{and} \quad 0 \in \partial\widetilde{\Psi}(\dot{z}) + A_{ZU}u + A_{ZZ}z - \ell_Z(t) \subset Z^*. \tag{2.11}$$

The first equation is then called the elastic equilibrium equation, and for given $z$ and $\ell_U$ it can be solved uniquely for $u \in U$, since $A_{UU}$ is again positive definite. The second relation is the flow rule for the internal variable $z$.

## 2.2 Basic a priori estimates and uniqueness

We first provide a few a priori estimates for the solutions of the above formulations. In particular, we will obtain uniqueness of solutions as well as continuous dependence on the data. For this, we make the following assumption on the dissipation functional:

$$\exists\, c_1 > 0\ \forall\, v \in Y:\ \Psi(v) \geq c_1 \|v\|_X,$$

where $\|\cdot\|_X$ denotes a semi-norm. Note that in the case $X = Y$ this implies that $C_* = \partial\Psi(0)$ satisfies $\{\, \sigma \in Y^* \mid \|\sigma\|_* \leq c_1 \,\} \subset C_*$. Similarly, if $\Psi$ is bounded from above by $c_2\|\cdot\|_Y$, then $C_*$ is contained inside a ball of radius $c_2$.

The assumptions on $A$ and on $\ell$ imply

$$\mathcal{E}(t,y) \geq \tfrac{\alpha}{2}\|y\|^2 - \|\ell(t)\|_*\|y\| \geq \Lambda\|y\| - \tfrac{1}{2\alpha}(\Lambda{+}\|\ell(t)\|_*)^2, \tag{2.12}$$

for any $\Lambda \geq 0$. Thus, with $\Lambda_0 = \|\ell\|_{\mathrm{L}^\infty}$ and $\Lambda_1 = \|\dot{\ell}\|_{\mathrm{L}^\infty}$ we obtain

$$|\partial_t\mathcal{E}(t,y)| \leq |\langle\dot{\ell}(t),y\rangle| \leq \Lambda_1\|y\| \leq \tfrac{\Lambda_1}{\Lambda}\Big(\mathcal{E}(t,y) + \tfrac{(\Lambda+\Lambda_0)^2}{2\alpha}\Big), \tag{2.13}$$

where $\Lambda > 0$ is still arbitrary. Below we choose $\Lambda = \Lambda_0$.

Since any solution $y : [0,T] \to Y$ satisfies the energy balance (E) we find, using $\mathrm{Diss}_\Psi \geq 0$, the estimate $\mathcal{E}(t,y(t)) \leq \mathcal{E}(0,y(0)) + \int_0^t \frac{\Lambda_1}{\Lambda_0}\big(\mathcal{E}(s,y(s)) + \frac{2\Lambda_0^2}{\alpha}\big)\,\mathrm{d}s$. Applying Gronwall's estimate to $\mathcal{E}(t,y(t)) + 2\Lambda_0^2/\alpha$ we find

$$\forall\, t \in [0,T]:\ \mathcal{E}(t,y(t)) \leq \mathrm{e}^{t\Lambda_1/\Lambda_0}\Big(\mathcal{E}(0,y_0) + 2\Lambda_0^2/\alpha\Big) - 2\Lambda_0^2/\alpha. \tag{2.14}$$

Inserting this into (E) once again, we obtain the second estimate

$$\forall\, t \in [0,T]: \; c_1 \int_0^t \|\dot y(s)\|_X \,\mathrm{d}s \le \mathrm{Diss}_\Psi(y;[0,t]) \le \mathrm{e}^{t\Lambda_1/\Lambda_0}\Big(\mathcal{E}(0,y_0) + 2\Lambda_0^2/\alpha\Big). \tag{2.15}$$

It turns out that these two purely ***energetic estimates*** apply in very general situations as long as (2.13) holds. They imply a priori estimates for $\|y(t)\|$ via the coercivity of the energy, cf. (2.12), as well as some control on the derivative $\dot y(t)$ via (2.15).

However, in the case of a quadratic energy (or in general convex situations, see Section 3.5) we may also derive Lipschitz bounds using the uniform convexity due to $A$. Using (i) $(\mathrm{E})_{\mathrm{loc}}$ for $y(s)$, (ii) $\Psi(y(t)-y(s)) \le \mathrm{Diss}_\Psi(y,[s,t])$ and (iii) (E) we obtain for $0 \le s < t \le T$:

$$\begin{aligned}
&\tfrac{\alpha}{2}\|y(t)-y(s)\|^2 \;\le\; \|y(t)-y(s)\|_A^2 \;=\; \mathcal{E}(s,y(t)) - \mathcal{E}(s,y(s)) - \langle \mathrm{D}\mathcal{E}(s,y(s)), y(t)-y(s)\rangle \\
&\le_{(\mathrm{i})}\; \mathcal{E}(s,y(t)) - \mathcal{E}(s,y(s)) + \Psi(y(t)-y(s)) \;\le_{(\mathrm{ii})}\; \mathcal{E}(s,y(t)) - \mathcal{E}(s,y(s)) + \mathrm{Diss}_\Psi(y,[s,t]) \\
&=_{(\mathrm{iii})}\; \mathcal{E}(s,y(t)) - \mathcal{E}(t,y(t)) + \textstyle\int_s^t \partial_\tau\mathcal{E}(\tau,y(\tau))\,\mathrm{d}\tau \;=\; \int_s^t [\partial_\tau\mathcal{E}(\tau,y(\tau)) - \partial_\tau\mathcal{E}(\tau,y(t))]\,\mathrm{d}\tau \\
&\le\; \Lambda_1 \textstyle\int_s^t \|y(t)-y(\tau)\|\,\mathrm{d}\tau.
\end{aligned}$$

From this, we easily derive the Lipschitz bound (cf. Theorem 3.4)

$$\forall\, s,t \in [0,T]: \; \|y(t)-y(s)\| \le \Lambda_1/\alpha\,|t-s| \qquad \text{or} \qquad \|\dot y\|_{\mathrm{L}^\infty((0,T),Y)} \le \Lambda_1/\alpha.$$

Similar estimates give the continuity with respect to the data. Let $y_1$ and $y_2$ be two solutions with data $(y_j^0, \ell_j)$. Then, with (VI) we find

$$\begin{aligned}
&\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\langle A(y_1-y_2), y_1-y_2\rangle \;=\; \langle Ay_1, \dot y_1-\dot y_2\rangle + \langle Ay_2, \dot y_2-\dot y_1\rangle \\
&= -\langle Ay_1-\ell_1, \dot y_2-\dot y_1\rangle - \langle Ay_2-\ell_2, \dot y_1-\dot y_2\rangle + \langle \ell_1-\ell_2, \dot y_1-\dot y_2\rangle \\
&\le\; \Psi(\dot y_2)-\Psi(\dot y_1)+\Psi(\dot y_1)-\Psi(\dot y_2) + \|\ell_1-\ell_2\|_*\|\dot y_1-\dot y_2\| \;\le\; \|\ell_1-\ell_2\|_*(\|\dot y_1(s)\|+\|\dot y_2(s)\|)/\alpha.
\end{aligned}$$

Thus, for all $t \in [0,T]$ we have the estimate

$$\alpha\|y_1(t)-y_2(t)\|^2 \le \|y_1(t)-y_2(t)\|_A^2 \le \|y_1^0-y_2^0\|_A^2 + \tfrac{2\Lambda_1}{\alpha}\int_0^t \|\ell_1(s)-\ell_2(s)\|_*\,\mathrm{d}s. \tag{2.16}$$

Alternatively, the last estimate can be written in the form

$$\|y_1(t)-y_2(t)\|_A^2 \le \|y_1^0-y_2^0\|_A^2 + 2\|\ell_1-\ell_2\|_{\mathrm{L}^\infty([0,T],Y^*)}\int_0^t \big(\|\dot y_1(s)\|+\|\dot y_2(s)\|\big)\,\mathrm{d}s. \tag{2.17}$$

## 2.3 Basic existence theory

There are essentially two different approaches to the existence theory. The first approach uses time discretization and solves a (static) variational inequality or a minimization problem in each time step. This method will be the main focus in this work as it generalizes to complicated nonsmooth and nonconvex situations. However, the method is restricted to symmetric operators in the variational inequality or, what is the same, to associated flow laws for the rate-independent problem.

The second approach is based on the theory of monotone or accretive operators, which is somehow more restrictive as it heavily uses the Hilbert or Banach space structure. However, it allows more general flow laws, such as non-associated ones, see Section 6. Here, we show how in the present situation both methods can be applied and finally mention also the so-called Yosida regularization which is often used to treat nonsmooth problems.

### 2.3.1 Time-incremental minimization

This approach will be discussed in full detail in later sections. Here we give a simplified version of the existence proof which shows the same features as in Section 5.1, where a much more general situation is treated. Here we use the simplifying structures of the quadratic energy.

We choose a sequence of partition $0 = t_0^N < t_1^N < \cdots < t_{N-1}^N < t_N^N = T$ of the interval $[0, T]$ such that the fineness $f_N = \max\{\, t_j^N - t_{j-1}^N \mid j = 1, \ldots, N \,\}$ tends to 0. For given initial value $y_0 \in \mathcal{S}(0)$ we solve iteratively

$$y_k \in \operatorname{Arg\,min}\{\, \mathcal{E}(t_k, y) + \Psi(y - y_k) \mid y \in Y \,\}.$$

By convexity of $\mathcal{E}(t, \cdot)$, this minimization problem is equivalent to the static variational inequality

$$\text{Find } y_k \in Y \text{ such that } \forall\, \widehat{v} \in Y: \ \langle Ay_k - \ell(t_k), \widehat{v} - (y_k - y_{k-1})\rangle - \Psi(y_k - y_{k-1}) + \Psi(\widehat{v}) \geq 0.$$

In fact, most work on evolutionary variational inequalities uses this form of the incremental problem, considers the piecewise linear interpolant of their solutions and shows that their limit exists and satisfies the corresponding variational inequality (VI), given in (2.2). We will instead stay with the minimization formulation and show that the limit function satisfies the energetic formulation (S) & (E), see (2.9). Thus, we also provide a simplified version of the general proof of Theorem 5.2.

**Theorem 2.1** *Let $\ell \in \mathrm{C}^1([0, T], Y^*)$ and $y_0 \in \mathcal{S}(0) = A^{-1}(\ell(0) - C_*)$. Then, the energetic problem (S) & (E) (cf.* (2.9)*) and hence also (VI) have a unique solution $y \in \mathrm{C}^{\mathrm{Lip}}([0, T], Y)$.*

**Proof:** Except for the uniqueness part, we follow the six steps of the proof of Theorem 5.2. We use the norm $\|\cdot\|_A$ on $Y$ and the dual norm on $Y^*$ and indicate this fact by writing $Y_A$ and $Y_A^*$, respectively.

*Step 0: Uniqueness.* For any two solutions estimate (2.16) with $\ell_1 = \ell_2 = \ell$ gives the estimate $\|y_1(t) - y_2(t)\|_A \leq \|y_1(0) - y_2(0)\|_A$, which proves uniqueness.

*Step 1: A priori estimates.* Let $\Pi = \{\, t_k \,|\, k = 0, \ldots, N \,\}$ be any partition. Since the functional $y \mapsto \mathcal{E}(t_k, y) + \Psi(y - y_{k-1})$ is strictly convex, it has a unique minimizer $y_k$ in each step and we have

$$\forall\, y \in Y: \ \mathcal{E}(t_k, y) + \Psi(y - y_{k-1}) \geq \tfrac{1}{2}\|y - y_k\|_A^2 + \mathcal{E}(t_k, y_k) + \Psi(y_k - y_{k-1}). \tag{2.18}$$

Inserting $y = y_{k-1}$ and $y = y_{k+1}$, respectively, we obtain

$$\begin{aligned} \forall\, k \in \{1, \ldots, N\}: \ & \mathcal{E}(t_k, y_k) + \Psi(y_k - y_{k-1}) \\ & \leq \ \mathcal{E}(t_k, y_{k-1}) = \mathcal{E}(t_{k-1}, y_{k-1}) + \textstyle\int_{t_{k-1}}^{t_k} \partial_s \mathcal{E}(s, y_{k-1})\,\mathrm{d}s; \end{aligned} \tag{2.19}$$

$$\forall\, k \in \{0, \ldots, N-1\}: \ \tfrac{1}{2}\|y_{k+1} - y_k\|_A^2 \ \leq \ \mathcal{E}(t_k, y_{k+1}) + \Psi(y_{k+1} - y_k) - \mathcal{E}(t_k, y_k), \tag{2.20}$$

where we have estimated $\Psi(y_k - y_{k-1}) \geq 0$. Note that the last estimate is claimed also for $k=0$, which follows from the assumption $y_0 \in \mathcal{S}(0)$. Thus, for $k = 0, \ldots, N-1$ we obtain

$$\begin{aligned}
\tfrac{1}{2}\|y_{k+1}-y_k\|_A^2 \;&\leq\; \mathcal{E}(t_k, y_{k+1}) + \Psi(y_{k+1}-y_k) - \mathcal{E}(t_k, y_k) \\
&\leq\; \mathcal{E}(t_{k+1}, y_{k+1}) - \textstyle\int_{t_k}^{t_{k+1}} \partial_s \mathcal{E}(s, y_{k+1})\,\mathrm{d}s + \Psi(y_{k+1}-y_k) - \mathcal{E}(t_k, y_k) \\
&\leq_{(2.19)}\; \mathcal{E}(t_k, y_k) - \textstyle\int_{t_k}^{t_{k+1}} \partial_s \mathcal{E}(s, y_{k+1})\,\mathrm{d}s - \mathcal{E}(t_k, y_k) \\
&=\; \textstyle\int_{t_k}^{t_{k+1}} [\partial_s \mathcal{E}(s, y_k) - \partial_s \mathcal{E}(s, y_{k+1})]\,\mathrm{d}s \;\leq\; \Lambda_{1,A}(t_{k+1}-t_k)\|y_{k+1}-y_k\|_A,
\end{aligned}$$

where $\Lambda_{1,A} = \|\dot{\ell}\|_{\mathrm{L}^\infty([0,T],Y_A^*)}$. Thus, the piecewise linear interpolant $\widehat{y}^\Pi : [0,T] \to Y$ and the piecewise constant interpolant $\overline{y}^\Pi : [0,T] \to Y$ with $y(t) = y_{k-1}$ for $t \in [t_{k-1}, t_k)$ satisfy the a priori bounds

$$\begin{aligned}
&\|\widehat{y}^\Pi\|_{\mathrm{C}^0([0,T],Y_A)} \leq \|y_0\|_A + 2\Lambda_{1,A}T, \quad \|\overline{y}^\Pi\|_{\mathrm{L}^\infty([0,T],Y_A)} \leq \|y_0\|_A + 2\Lambda_{1,A}T, \\
&\|\dot{\widehat{y}}^\Pi\|_{\mathrm{L}^\infty([0,T],Y_A)} \leq 2\Lambda_{1,A}, \quad \|\widehat{y}^\Pi - \overline{y}^\Pi\|_{\mathrm{L}^\infty([0,T],Y_A)} \leq 2\Lambda_{1,A} f(\Pi).
\end{aligned}$$

*Step 2: Selection of a subsequence.* Now we choose an arbitrary sequence $(\Pi^m)_{m\in\mathbb{N}}$ of partitions with $f(\Pi^m) \to 0$. Since the function $\widehat{y}^{\Pi^m}$ satisfies a uniform Lipschitz bound and since closed balls in the reflexive Banach space $Y$ are weakly compact, we can apply the Arzela-Ascoli theorem which provides a subsequence $(m_l)_{l\in\mathbb{N}}$ and a limit function $y : [0,T] \to Y$ such that $\widehat{y}^l = \widehat{y}^{\Pi^{m_l}}$ and $\overline{y}^l = \overline{y}^{\Pi^{m_l}}$ satisfy

$$\forall\, t \in [0,T]: \quad \widehat{y}^l(t) \rightharpoonup y(t), \quad \overline{y}^l(t) \rightharpoonup y(t) \quad \text{and} \quad \|\dot{y}\|_{\mathrm{L}^\infty([0,T],Y_A)} \leq 2\Lambda_{1,A}.$$

It remains to be shown that $y$ is a solution.

*Step 3: Stability of the limit function.* Estimate (2.18) and the triangle inequality for $\Psi$ imply, for all $y \in Y$,

$$\mathcal{E}(t_k, y_k) \leq \mathcal{E}(t_k, y) + \Psi(y - y_{k-1}) - \Psi(y_k - y_{k-1}) \leq \mathcal{E}(t_k, y) + \Psi(y - y_k),$$

which means $y_k \in \mathcal{S}(t_k)$. Hence, for the sequence $\widehat{y}^l$ we have $\widehat{y}^l(t) \in \mathcal{S}(t)$ for each $t \in \Pi^{m_l}$. As $f(\Pi^{m_l}) \to 0$ we find, for each $t_* \in [0,T]$, a sequence $t^l$ with $t^l \to t_*$. Using $\|\dot{\widehat{y}}^l\|_{\mathrm{L}^\infty} \leq 2\Lambda_{1,A}$ we obtain $\widehat{y}^l(t^l) \rightharpoonup y(t_*)$. Moreover, the graph set

$$\mathcal{S}_{[0,T]} = \textstyle\bigcup_{t\in[0,T]} (t, \mathcal{S}(t)) = \{\, (t,y) \mid y \in \mathcal{S}(t) \,\} = \{\, (t,y) \mid y \in A^{-1}(\ell(t) - C_*) \,\} \subset [0,T] \times Y,$$

is closed with respect to the weak topology of $Y$, since each $\mathcal{S}(t)$ is strongly closed and convex and since $\ell$ is strongly continuous. Thus, $(t^l, \widehat{y}^l(t^l)) \in \mathcal{S}_{[0,T]}$ implies $(t_*, y(t_*)) \in \mathcal{S}_{[0,T]}$, which means $y(t_*) \in \mathcal{S}(t_*)$ and the stability (S) is proved.

*Step 4: Upper energy estimate.* For $t \in (0,T]$ and $l \in \mathbb{N}$ let $j$ be the largest index with $t_j = \max\{\, t_n \in \Pi^{m_l} \mid t_n \leq t \,\}$. Then, adding (2.19) from $k=1$ to $j$ gives

$$\begin{aligned}
&\mathcal{E}(t, \overline{y}^l(t)) + \mathrm{Diss}_\Psi(\overline{y}^l; [0,t]) = \textstyle\int_{t_j}^t \partial_s \mathcal{E}(s, \overline{y}^l(t_j))\,\mathrm{d}s + \mathcal{E}(t_j, \overline{y}^l(t_j)) + \sum_{k=1}^j \Psi(\overline{y}^l(t_k) - \overline{y}^l(t_{k-1})) \\
&\leq \textstyle\int_{t_j}^t \partial_s \mathcal{E}(s, \overline{y}^l(s))\,\mathrm{d}s + \mathcal{E}(0, y_0) + \int_0^{t_j} \partial_s \mathcal{E}(s, \overline{y}^l(s))\,\mathrm{d}s = \mathcal{E}(0, y_0) - \int_0^t \langle \dot{\ell}(s), \overline{y}^l(s) \rangle\,\mathrm{d}s.
\end{aligned}$$

The right-hand side of this estimate converges to $\mathcal{E}(0,y_0) - \int_0^t \langle \dot{\ell}(s), y(s)\rangle \,\mathrm{d}s$ by the weak convergence and Lebesgue theorem on dominated convergence. The left-hand side is lower semi-continuous, since $\mathcal{E}(t,\cdot)$ and $\mathrm{Diss}_\Psi$ are convex and strongly continuous. Thus, for each $t \in [0,T]$ we conclude $\mathcal{E}(t,y(t))+\mathrm{Diss}_\Psi(y;[0,t]) \leq \mathcal{E}(0,y_0)- \int_0^t \langle \dot{\ell}(s), y(s)\rangle \,\mathrm{d}s$, which is the desired upper energy estimate.

*Step 5: Lower energy estimate.* The lower estimate is a consequence of stability of the limit function $y$ as proved in Step 3. Take an arbitrary partition $\mathcal{T} = \{ \tau_j \mid j = 0,\ldots,M \}$ of the interval $[0,t]$. Then, for $j \geq 1$ stability of $y(\tau_{j-1})$ gives

$$\begin{aligned}&\mathcal{E}(\tau_j,y(\tau_j))+\Psi(y(\tau_j)-y(\tau_{j-1})) = \int_{\tau_{j-1}}^{\tau_j} \partial_s\mathcal{E}(s,y(\tau_j))\,\mathrm{d}s+\mathcal{E}(\tau_{j-1},y(\tau_j))+\Psi(y(\tau_j)-y(\tau_{j-1}))\\ &\geq \int_{\tau_{j-1}}^{\tau_j} \partial_s\mathcal{E}(s,y(\tau_j))\,\mathrm{d}s + \mathcal{E}(\tau_{j-1},y(\tau_{j-1})).\end{aligned}$$

Adding these estimates from $j=1$ to $M$ and using the definition of $\mathrm{Diss}_\Psi$, we find

$$\begin{aligned}&\mathcal{E}(t,y(t))+\mathrm{Diss}_\Psi(y;[0,t]) \geq \mathcal{E}(t,y(t))+ \textstyle\sum_1^M \Psi(y(\tau_j)-y(\tau_{j-1}))\\ &\geq \mathcal{E}(0,y_0)- \textstyle\sum_1^M \int_{\tau_{j-1}}^{\tau_j} \langle \dot{\ell}(s), y(\tau_j)\rangle \,\mathrm{d}s \geq \mathcal{E}(0,y_0)- \int_0^t \langle \dot{\ell}(s), y(s)\rangle \,\mathrm{d}s - f(\mathcal{T})2\Lambda_{1,A}.\end{aligned}$$

Thus, by making the partition $\mathcal{T}$ as fine as we like, we obtain the lower energy estimate and together with Step 4 the energy balance (E) is established and $y$ is a solution.

*Step 6: Improved convergence.* By Step 0 we know that there exists at most one solution. We conclude that not only the subsequence $y^l$ converges weakly to $y$, but the whole sequence $\widehat{y}^{\Pi^m}$ converges weakly to $y$. In fact, in Section 4.1 it is shown that the convergence is strong with a convergence like $\sqrt{f(\Pi)}$, see Theorem 4.3. ∎

### 2.3.2 Monotone operators

The existence theory via monotone operators is based on the concept of multi-valued monotone operators on a Hilbert space. A mapping $\mathcal{M} : D(\mathcal{M}) \subset Y \to \mathcal{P}(Y)$, where $\mathcal{P}$ denotes the power set, is called monotone, if

$$\forall\, y_1, y_2 \in D(\mathcal{M})\; \forall\, w_1 \in \mathcal{M}(y_1),\; w_2 \in \mathcal{M}(y_2):\; \langle w_1{-}w_2 | y_1{-}y_2\rangle \geq 0,$$

where $\langle\cdot|\cdot\rangle$ denotes the scalar product in $Y$. A monotone operator $\mathcal{M}$ is called ***maximal monotone***, if its graph $G(\mathcal{M}) = \cup_{y\in D(\mathcal{M})}(y,\mathcal{M}(y)) \subset Y{\times}Y$ does not have a proper extension to a graph of another monotone operator. The following result is contained in many textbooks, see, e.g., [Zei85, Thm. 55A] or [Bre73].

**Theorem 2.2** *Let $Y$ be a real, separable Hilbert space and $\mathcal{M} : D(\mathcal{M}) \subset Y \to \mathcal{P}(Y)$ a maximal monotone operator. Then, for each $T > 0$ the Cauchy problem*

$$0 \in \dot{\widehat{y}}(t) + \mathcal{M}(\widehat{y}(t)) - b(t) \quad \textit{for almost every } t \in [0,T], \qquad \widehat{y}(0) = \widehat{y}^0, \tag{2.21}$$

*has, for each $\widehat{y}^0 \in D(\mathcal{M})$ and each $b \in \mathrm{W}^{1,1}((0,T),Y)$, a unique solution $\widehat{y} \in \mathrm{W}^{1,1}((0,T),Y)$. In fact, we have $\widehat{y} \in \mathrm{W}^{1,\infty}((0,T),Y)$.*

*Moreover, two solutions $\widehat{y}_1, \widehat{y}_2$ associated with data $(\widehat{y}_j^0, b_j)$ satisfy the estimate*

$$\|\widehat{y}_1(t)-\widehat{y}_2(t)\| \leq \|\widehat{y}_1^0-\widehat{y}_2^0\| + \int_0^t \|b_1(s)-b_2(s)\|\,\mathrm{d}s \quad \textit{for } t \in [0,T]. \tag{2.22}$$

In our rate-independent problems $b$ will be related to $\dot{\ell}$ and the values of $\mathcal{M}$ must be closed cones, as the values $\mathcal{M}(\widehat{y})$ of any maximal monotone operator are closed and convex and rate independence gives $\mathcal{M}(\widehat{y}) = \gamma\mathcal{M}(\widehat{y})$ for all $\gamma > 0$. To apply the above theorem to our problems of Section 2.1 we let $\widehat{y}(t) = y(t) - A^{-1}\ell(t)$ and $\mathcal{M}(\widehat{y}) = -\partial I_{C_*}(-A\widehat{y})$. Then (DI) takes the form

$$0 \in \dot{\widehat{y}}(t) + \mathcal{M}(\widehat{y}(t)) - A^{-1}\dot{\ell}(t), \quad \widehat{y}(0) = y_0 - A^{-1}\ell(0).$$

Clearly, $\widehat{y}(0) \in D(\mathcal{M}) = \{\, \widehat{y} \mid \mathcal{M}(\widehat{y}) \neq \emptyset \,\}$ means $-A\widehat{y}(0) \in C_*$ which is equivalent to the stability condition $y_0 \in \mathcal{S}(0) = A^{-1}(\ell(0) - C_*)$. Moreover, $\mathcal{M}$ is a maximal monotone operator, since it is the subdifferential of the lower semi-continuous, convex function $\varphi : Y \to \mathbb{R}_\infty; \widehat{y} \mapsto I_{C_*}(-A\widehat{y})$, i.e., $\varphi = I_{-A^{-1}C_*}$. For this choose the scalar product to be defined by $A$ as usual, namely $\langle y_1 | y_2 \rangle = \langle Ay_1, y_2 \rangle$; then it is easy to check that the Hilbert space subdifferential using the scalar product $\partial\varphi(\widehat{y}) = \{\, w \in Y \mid \forall\, y \in Y :\ \varphi(y) \geq \varphi(\widehat{y}) + \langle w | y-\widehat{y} \rangle \,\}$ is equal to $-\mathrm{N}_{C_*}(-A\widehat{y})$ as desired.

In principle this approach provides the desired existence result. However, it has the disadvantage that it strongly relies on the linearity of $y \mapsto \mathrm{D}\mathcal{E}(t,y) = Ay - \ell(t)$ and that it uses the time derivative $\dot{\ell}$.

Finally, let us mention that the restriction of $y$ being a Hilbert space can be avoided by using the theory of m-accretive operators on Banach spaces, see [Bar76] or [Vis94, Sect. XII.4] for a short survey.

### 2.3.3 Doubly nonlinear problems

An approach better adapted to our needs is the theory of doubly nonlinear equations developed in [CV90]. There, general equations of the type

$$0 \in \mathcal{R}(\dot{y}(t)) + \Sigma_0(y(t)) - \ell(t) \quad \text{for a.a. } t \in [0,T], \qquad y(0) = y_0, \tag{2.23}$$

are studied, where $\mathcal{R}$ and $\Sigma_0$ are (possibly multi-valued) maximal monotone operators on the Hilbert space $Y$. Theorem 2.1 treats the case $\Sigma_0 = \partial\mathcal{U}$ while $\mathcal{R}$ is general with linear growth. Theorem 2.2 is dedicated to the case $\mathcal{R} = \partial\Psi$ with strongly monotone $\Sigma_0$. Finally, Theorem 2.3 assumes $\mathcal{R} = \partial\Psi$ and $\Sigma_0 = \partial\mathcal{U}$ with minimal assumptions on the potentials $\Psi$ and $\mathcal{U}$. Theorem 2.2 is most suited for our purposes and we repeat it for the reader's convenience.

**Theorem 2.3** *The Hilbert space $Y$ is densely and compactly embedded into the Hilbert space $H$. The dissipation potential $\Psi : H \to \mathbb{R}_\infty$ is convex and lower semi-continuous and $\mathcal{R} = \partial\Psi$. The mapping $\Sigma_0 : Y \to Y^*$ is Lipschitz continuous and uniformly monotone, i.e.,*

$$\begin{aligned} \exists\, c_1, C_2 > 0\ \forall y_1, y_2 \in Y \quad : \quad & \|\Sigma_0(y_1)-\Sigma_0(y_2)\|_{Y^*} \leq C_2\|y_1-y_2\|_Y \ \textit{and} \\ & \langle \Sigma_0(y_1)-\Sigma_0(y_2), y_1-y_2 \rangle \geq c_1\|y_1-y_2\|_Y^2. \end{aligned}$$

*Then, for every $\ell \in \mathrm{H}^1((0,T),Y^*)$ and every $y_0$ with $\ell(0) - \Sigma_0(y_0) \in D(\mathcal{L}\Psi)$ there exists a solution $y \in \mathrm{H}^1((0,T),Y)$ of* (2.23).

Here $\mathcal{L}\Psi$ denotes the Legendre-Fenchel transform of the convex function $\Psi$. In the case of a $\Psi$ which is 1-homogeneous, we have $\mathcal{L}\Psi = I_{C_*}$ and thus $D(\mathcal{L}\Psi) = C_*$. Clearly, the above theorem provides the solvability of our subdifferential formulation (SF) given in (2.1) if we let $\Sigma_0(y) = Ay$. However, for this general version involving monotone operators $\Sigma_0$ instead of a linear and positive operator $A$ we have to pay by making an additional assumption on $C_* = \partial\Psi(0) \subset H = H^*$. Since $Y$ is compactly embedded in $H$, we find that $C_* \cap \{\, \sigma \in Y^* \mid \|\sigma\|_* \leq \rho \,\}$ is compact in $Y^*$ for $\rho > 0$. This relates to the conditions of the closedness of the stable set in the weak topology, which is central in the abstract Section 5.

It is interesting to note that the existence result for monotone operators is mostly obtained by using regularization techniques, which replace the nonsmooth, multi-valued problem by a classical Lipschitz continuous ordinary differential equation. The main technique is the ***Yosida regularization*** which works for all maximal monotone operators (even for m-accretive operators), see [Zei85, Sect. 55.2]. If $M : Y \to \mathcal{P}(Y)$ is maximal monotone, then for all $\varepsilon > 0$ the problem $y + \varepsilon M(y) = b$ has a unique solution, which we denote by $y = R_{\varepsilon,M}(b)$. The Yosida regularization of $M$ is then defined to be the operator

$$M_\varepsilon^{\mathrm{Y}} : Y \to Y;\ y \mapsto \tfrac{1}{\varepsilon}(y - R_{\varepsilon,M}(y)).$$

Defining $M_0$ via $M_0(y) = \arg\min\{\, \|w\| \mid w \in M(w) \,\}$ we obtain the following standard results: each $M_\varepsilon^{\mathrm{Y}}$ is Lipschitz continuous and maximal monotone, and for $y \in D(M)$ we have $M_\varepsilon(y)^{\mathrm{Y}} \to M_0(y)$ for $\varepsilon \to 0$ whereas $y \notin D(M)$ implies $\|M_\varepsilon^{\mathrm{Y}}(y)\| \to \infty$.

Instead of solving the Cauchy problem (2.21) one then solves the regularized problem $0 = \dot{y}(t) + M_\varepsilon^{\mathrm{Y}}(y(t)) - b(t)$ which by Lipschitz continuity of $M_\varepsilon^{\mathrm{Y}}$ has a unique solution $y_\varepsilon \in \mathrm{C}^{\mathrm{Lip}}([0,T],Y)$. Using the monotonicity properties it is then possible to show that the limit $y(t) = \lim_{\varepsilon\to 0} y_\varepsilon(t)$ exists and solves (2.21).

The point here is that in the case of our interest the operator $M$ is the subdifferential of the characteristic function $\varphi = I_C$ with $C = -A^{-1}C_* \subset Y$. It is easy to see that $R_{\varepsilon,M}$ is independent of $\varepsilon$ and is equal to the orthogonal projection $P_C : Y \to C \subset Y$, i.e, $P_C(y) = \arg\min\{\, \|\widehat{y} - y\|_Y \mid \widehat{y} \in C \,\}$. Thus, the Yosida regularization $M_\varepsilon^{\mathrm{Y}}$ takes the form

$$M_\varepsilon^{\mathrm{Y}}(y) = \tfrac{1}{\varepsilon}\big(y - P_C(y)\big) = \partial\mathcal{D}_\varepsilon(y) \quad \text{where } \mathcal{D}_\varepsilon(y) = \tfrac{1}{2\varepsilon}\mathrm{dist}(y,C)^2.$$

Thus, the regularized equation

$$0 = \dot{y} + M_\varepsilon^{\mathrm{Y}}(y) - b(t) = \dot{y} + \tfrac{1}{\varepsilon}\big(y - P_C(y)\big) - b(t)$$

corresponds to the traditional viscoplastic approximation to plasticity. This fact was first noted in [Ort81].

In [CV90] the doubly nonlinear problems (2.23) are regularized in a two-fold way. The auxiliary problem is

$$0 \in \varepsilon\dot{y}(t) + \mathcal{R}(\dot{y}(t)) + M_\varepsilon^{\mathrm{Y}}(y(t)) - \ell(t),$$

where the first term can be understood as a viscous friction term. To solve for $\dot{y}$ we use the operator $\mathbf{R}_\varepsilon : v \mapsto R_{1/\varepsilon,\mathcal{R}}(\tfrac{1}{\varepsilon}v)$, which leads to the ordinary differential equation $0 =$

$\dot{y} + \mathbf{R}_\varepsilon(M_\varepsilon(y(t)) - \ell(t))$, which is Lipschitz continuous. We will consider related viscous regularizations in Section 5.4. Such regularizations are important if $M$ is no longer strictly monotone and thus solutions can develop jumps.

Note that the viscous regularization $\mathcal{R}_\varepsilon^{\text{vis}} = \varepsilon\mathbf{1}+\mathcal{R}$ for $\mathcal{R} = \partial\Psi$ is adjusted to the fact that $\partial\Psi$ is 1-homogeneous. For $\mathcal{R}(v) = \operatorname{Sign}(v)$ the regularization $\varepsilon\mathbf{1}+\mathcal{R}$ has a Lipschitz continuous inverse, while the Yosida regularization $\mathcal{R}_\varepsilon^{\mathrm{Y}}$ is bounded and hence not invertible, namely $\mathcal{R}_\varepsilon^{\mathrm{Y}}(v) = \min\{\frac{1}{\varepsilon}, \frac{1}{\|v\|}\}v$.

### 2.3.4 Sweeping processes

Finally, we mention some special existence results related to the sweeping-process formulation taken from [Mon93, Ch. 2&5]. The origins stem from Moreau, see, e.g., [Mor74]. There, for $y \in \mathrm{BV}_+([0,T], Y)$, the BV functions which are continuous from the right, the differential measure $\mathrm{d}y$ and the associated Radon measure $|\mathrm{d}y|$ are defined such that $\mathrm{d}y([s,t]) = \int_s^t \mathrm{d}y = y(t)-y(s)$ and $\mathrm{Diss}_{\|\cdot\|}(y) = \int_{[s,t]} |\mathrm{d}y|$. The new point is that $y$ may have jumps, i.e., $t$ is a jump point, if $\lim_{r\searrow t} y(r) = y(t) \neq y_-(t) = \lim_{s\nearrow t} y(s)$. This implies $|\mathrm{d}y|(\{t\}) = \|y(t)-y_-(t)\|$ and $\mathrm{d}y(\{t\}) = y(t)-y_-(t)$. The important fact is that $\mathrm{d}y$ can be decomposed into a directional part $y'$ and the length $|\mathrm{d}y|$ such that $y'$ is defined $|\mathrm{d}y|$-almost everywhere and takes values in the unit sphere $\{\, y \in Y \mid \|y\| = 1\,\}$. One shortly writes $y' = \frac{\mathrm{d}y}{|\mathrm{d}y|}$.

For a given family $(C(t))_{[0,T]}$ of closed convex sets, we can now formulate the sweeping process using the normal cone $\mathrm{N}_{C(t)}(y) \subset Y$, which is defined via the scalar product on $Y$, as follows

$$
\begin{aligned}
y(0) = y_0 \in C(0), \qquad \forall\, t \in [0,T]:\ y(t) \in C(t),\\
-\tfrac{\mathrm{d}y}{|\mathrm{d}y|}(t) \in \mathrm{N}_{C(t)}(y(t)) \text{ for } |\mathrm{d}y|\text{-almost every } t \in [0,T].
\end{aligned}
$$

The existence of solutions in $\mathrm{BV}_+([0,T], Y)$ can now be shown using special incremental problems or the Yosida regularization if one of the following conditions hold:

(A) [Mon93, Ch. 1, Thm. 1.5]: $t \mapsto C(t)$ is right-continuous and of bounded variation with respect to the Hausdorff distance on closed sets.

(B) [Mon93, Ch. 2, Thm. 2.1]: $t \mapsto C(t)$ is Hausdorff continuous and each $C(t)$ has nonempty interior (then $y \in \mathrm{C}^0([0,T], Y) \cap \mathrm{BV}([0,T], Y)$), cf. Theorem 2.6.

(C) [Mon93, Ch. 2, Thm. 2.4]: $Y$ is finite dimensional, there exists $\rho > 0$ and $y_0 \in Y$ with $B_\rho(y_0) \subset C(t)$ and $t \mapsto C(t)$ is lower semi-continuous from the right (i.e., $\forall\, t_0 \in [0,T)\ \forall O \subset Y$ open with $O \cap C(t_0) \neq \emptyset\ \exists\, \varepsilon > 0 : O \cap C(t) \neq \emptyset$ for all $t \in [t_0, t_0{+}\varepsilon]$).

(D) [Mon93, Ch. 2, Thm. 2.4]: $Y$ is finite dimensional, each $C(t)$ has nonempty interior and $t \mapsto C(t)$ is lower semi-continuous (i.e., $\forall\, t_0 \in [0,T]\ \forall O \subset Y$ open with $O \cap C(t_0) \neq \emptyset\ \exists\, \varepsilon > 0 : O \cap C(t) \neq \emptyset$ for all $t \in [t_0{-}\varepsilon, t_0{+}\varepsilon] \cap [0,T]$).

## 2.4 Continuity properties of the solution operator

In Sections 2.2 and 2.3 we have seen that associated with the linear operator $A : Y \to Y^*$ and the dissipation functional $\Psi = \mathcal{L}I_{C_*}$ is a solution operator

$$\mathcal{H} : \left\{ \begin{array}{ccc} C_* \times \mathrm{C}^{\mathrm{Lip}}([0,T],Y^*) & \to & \mathrm{C}^{\mathrm{Lip}}([0,T],Y), \\ (\sigma_0,\ell) & \mapsto & y(\cdot); \end{array} \right. \qquad \text{with } y(0) = y_0 = A^{-1}(\ell(0) - \sigma_0),$$

where $y$ solves any of the equivalent formulations in Section 2.1. In fact, the estimates (2.16), (2.17) and (2.22) provided a first result on continuous-dependence for $\mathcal{H}$. Here and below we use the energy norm $\|y\|_A = \langle Ay, y\rangle^{1/2}$ on $Y$ and the dual norm on $Y^*$, i.e., $\|\sigma\|_* = \langle \sigma, A^{-1}\sigma\rangle^{1/2}$.

In particular, we have established that $\mathcal{H}(\cdot,\ell)$ defines a contraction semigroup, namely for $y_j = \mathcal{H}(\sigma_j,\ell)$ with fixed $\ell \in \mathrm{C}^{\mathrm{Lip}}$ we have

$$\|y_1(t) - y_2(t)\|_A = \|\mathcal{H}(\sigma_1,\ell)(t) - \mathcal{H}(\sigma_2,\ell)(t)\|_A \le \|\sigma_1 - \sigma_2\|_* = \|y_1(0) - y_2(0)\|_A \qquad (2.24)$$

for all $t \in [0,T]$.

So far, we have shown that $\mathcal{H}(\sigma_0,\cdot)$ maps $\mathrm{C}^{\mathrm{Lip}}([0,T],Y^*)$ into $\mathrm{C}^{\mathrm{Lip}}([0,T],Y)$. However, using the rate-independence, it can be easily seen that $\mathcal{H}(\sigma_0,\cdot)$ also maps $\mathrm{W}^{1,p}([0,T],Y^*)$ into $\mathrm{W}^{1,p}([0,T],Y)$ for any $p \in [1,\infty]$ (where $\mathrm{C}^{\mathrm{Lip}} = \mathrm{W}^{1,\infty}$ since $Y$ is a Hilbert space). For this, just note that $\ell \in \mathrm{W}^{1,p}([0,T],Y^*)$ allows to define the new time variable

$$\tau = \alpha(t) = t + \textstyle\int_0^t \|\dot{\ell}(s)\|\,\mathrm{d}s \in [0,T_\circ] \text{ with } T_\circ = \alpha(T).$$

If $\beta : [0,T_\circ] \to [0,T]$ is the inverse of $\alpha$, then $\ell_\circ = \ell \circ \beta \in \mathrm{C}^{\mathrm{Lip}}([0,T_\circ],Y^*)$, since $\ell_\circ'(\tau) = (1+\|\dot{\ell}(\beta(\tau))\|_*)^{-1}\dot{\ell}(\beta(\tau))$ has a norm less than 1 a.e. in $[0,T_\circ]$. With the corresponding solution $y_\circ = \mathcal{H}(\sigma_0,\ell_\circ) \in \mathrm{C}^{\mathrm{Lip}}([0,T_\circ],Y)$ we are then able to obtain the solution $y = \mathcal{H}(\sigma_0,\ell)$ in the form $y = y_\circ \circ \alpha$ and find

$$\begin{aligned} \|\dot{y}\|_{\mathrm{L}^p([0,T],Y)} &= \|y_\circ'(\alpha(\cdot))\dot{\alpha}(\cdot)\|_{\mathrm{L}^p([0,T],Y)} \le \|y_\circ'\|_{\mathrm{L}^\infty([0,T],Y)}\|\dot{\alpha}(\cdot)\|_{\mathrm{L}^p([0,T])} \\ &\le \|y_\circ'\|_{\mathrm{L}^\infty([0,T],Y)}\Big(T^{1/P} + \|\dot{\ell}\|_{\mathrm{L}^p([0,T],Y^*)}\Big), \end{aligned}$$

since $\dot{\alpha} = 1 + \|\dot{\ell}\|_*$.

In [Kre99, Thm. 3.6] (see also [BS96]) we find the following general result, where the last statement follows from (2.22).

**Proposition 2.4** *For each $p \in [1,\infty]$ the mapping $\mathcal{H} : C_* \times \mathrm{W}^{1,p}([0,T],Y^*) \to \mathrm{W}^{1,p}([0,T],Y)$ is continuous. Moreover, the mapping $\mathcal{H} : C_* \times \mathrm{W}^{1,1}([0,T],Y^*) \to \mathrm{C}^0([0,T],Y)$ is globally Lipschitz continuous.*

**Remark 2.5** *In [Kre99] the continuity results are formulated in terms of the play operator $\mathbf{P}$ and the stop operator $\mathbf{S}$ and written in the Hilbert space setting with $A = \mathbf{1}$ and $\langle y_1|y_2\rangle = \langle Ay_1, y_2\rangle$. In our setting the corresponding definitions are*

$$y = \mathbf{P}(\sigma_0,\ell) = \mathcal{H}(\sigma_0,\ell) \quad \text{and} \quad \sigma = \mathbf{S}(\sigma_0,\ell) = \ell - Ay = \ell - A\mathcal{H}(\sigma_0,\ell) \in \mathrm{C}^0([0,T],C_*),$$

*such that $\mathbf{S}(\sigma_0,\ell)(0) = \sigma_0$ and $\mathbf{S} + A\mathbf{P} = \mathbf{1}$.*

The next result states that $\mathcal{H}$ can be extended to an operator from $C_*\times\mathrm{C}^0([0,T],Y^*)$ into $\mathrm{BV}([0,T],Y)\cap\mathrm{C}^0([0,T],Y)$, if $0$ is an interior point of $C_*$, which is equivalent to a lower bound on $\Psi$:

$$B^*_\rho(0)=\{\,\sigma\in Y^*\mid \|\sigma\|_*\leq\rho\,\}\subset C_* \quad\Longleftrightarrow\quad \forall\, v\in Y:\ \Psi(v)\geq\rho\|v\|_A. \tag{2.25}$$

For the reader's convenience we give the main ingredients of the proof, which is a special case of [Mon93, Ch.2, Thm. 2.1]. The solutions $y=\mathcal{H}(\sigma_0,\ell)$ in the following result will solve the weakened form (2.6) of the dual variational inequality which reads

$$\begin{array}{c}\int_0^T\langle\ell(t){-}Ay(t)-\widetilde{\sigma}(t),\mathrm{d}y(t)\rangle\geq 0\quad\text{for all }\widetilde{\sigma}\in\mathrm{C}^0([0,T],C_*),\\ y(0)=A^{-1}(\ell(0){-}\sigma_0).\end{array} \tag{2.26}$$

See also [KL02, KV03] for generalizations to the space of regulated functions, i.e., to functions which may be not continuous but have limits from the left and right at each point.

**Theorem 2.6** *Let $C_*$ and $\Psi$ satisfy (2.25). Then, the operator $\mathcal{H}:C_*\times\mathrm{W}^{1,1}([0,T],Y^*)\to\mathrm{W}^{1,1}([0,T],Y)$ can be extended to a continuous operator from $C_*\times\mathrm{C}^0([0,T],Y^*)$ into $\mathrm{C}^0([0,T],Y)$. Moreover, $\mathcal{H}$ maps $C_*\times\mathrm{C}^0([0,T],Y^*)$ into $\mathrm{BV}([0,T],Y)\cap\mathrm{C}^0([0,T],Y)$ and around each $(\sigma_0,\ell)\in C_*\times\mathrm{C}^0([0,T],Y^*)$ there exists a neighborhood $\mathcal{B}$ such that $\mathcal{H}:\mathcal{B}\to\mathrm{C}^0([0,T],Y^*)$ is Hölder continuous with exponent $1/2$.*

**Proof:** By (2.24) it suffices to consider the case $\sigma_0=0$. Choose any $\widehat{\ell}\in\mathrm{C}^0([0,T],Y^*)$, then by uniform continuity there exists $N\in\mathbb{N}$ such that with $t_j=jT/N$, $j=0,\dots,N$, we have $\|\widehat{\ell}(t){-}\widehat{\ell}(t_j)\|_*\leq\rho/3$ for $t\in[t_j,t_{j+1}]$. Denote by $\mathcal{B}$ the set of all $\ell\in\mathrm{C}^0([0,T],Y^*)$ with $\|\widehat{\ell}{-}\ell\|_\infty\leq\rho/6$, then $\|\ell(t){-}\ell(t_j)\|_*\leq\rho/2$ for $t\in[t_j,t_{j+1}]$.

We show that for $\ell\in\mathcal{B}\cap\mathrm{W}^{1,1}([0,T],Y^*)$ the solutions $y=\mathcal{H}(0,\ell)$ have a uniformly bounded variation. For $t\in[t_j,t_{j+1}]$ we have, by using $(\mathrm{E})_{\mathrm{loc}}$ from (2.8) and (2.25),

$$\begin{aligned}&\tfrac12\tfrac{\mathrm{d}}{\mathrm{d}t}\|Ay(t){-}\ell(t_j)\|_*^2=\langle Ay{-}\ell(t_j),\dot y\rangle=\langle Ay{-}\ell(t),\dot y\rangle+\langle\ell(t){-}\ell(t_j),\dot y\rangle\\ &\leq\ -\Psi(\dot y)+\|\ell(t){-}\ell(t_j)\|_*\|\dot y\|_A\leq-\Psi(\dot y)+\tfrac{\rho}{2}\|\dot y\|_A\leq-\tfrac12\Psi(\dot y).\end{aligned}$$

Thus, we conclude $\|Ay(t_{j+1}){-}\ell(t_j)\|_*\leq\delta_j:=\|Ay(t_j){-}\ell(t_j)\|_*$ and

$$\int_{t_j}^{t_{j+1}}\Psi(\dot y(s))\,\mathrm{d}s\leq\|Ay(t_j){-}\ell(t_j)\|_*^2-\|Ay(t_{j+1}){-}\ell(t_j)\|_*^2\leq\delta_j^2. \tag{2.27}$$

This implies $\delta_{j+1}\leq\delta_j+\|\ell(t_{j+1}){-}\ell(t_j)\|_*\leq\delta_j+\rho/2$. With $\delta_0=\|Ay(0){-}\ell(0)\|_*=\|\sigma_0\|_*$ we find by induction $\delta_j\leq\|\sigma_0\|_*{+}j\rho/2$. Adding (2.27) over $j$ gives the a priori bound

$$\rho\int_0^T\|\dot y(t)\|_A\,\mathrm{d}t\leq\int_0^T\Psi(\dot y(t))\,\mathrm{d}t\leq N(\|\sigma_0\|_*{+}N\rho/2)^2=K_{\mathcal{B}}. \tag{2.28}$$

Now employ the estimate (2.17) to obtain for all $\ell_1,\ell_2\in\mathcal{B}\cap\mathrm{W}^{1,1}([0,T],Y^*)$ the estimate

$$\|y_1{-}y_2\|^2_{\mathrm{C}^0([0,T],Y_A)}\leq\|A^{-1}(\ell_1(0){-}\ell_2(0))\|_A^2+\tfrac{2K_{\mathcal{B}}}{\rho}\|\ell_1{-}\ell_2\|_{\mathrm{C}^0([0,T],Y^*)}.$$

Thus, by density there is a unique Hölder continuous extension of $\mathcal{H}$ to all of $\mathcal{B}$. ■

The a priori bound (2.28) can easily be extended to all small neighborhoods of compact subsets of $\mathrm{C}^0([0,T],Y^*)$. Uniform continuity results for $\mathcal{H}$ can be obtained by further assumptions.

**Theorem 2.7** *(A) [Kre99, Thm. 4.1] If $C_*$ is uniformly strictly convex (see [Kre99, Def. 2.13]), then there exists a monotone function $\beta : [0,\infty) \to [0,\infty)$ with $\beta(\delta) \to 0$ for $\delta \to 0$, such that the operator $\mathcal{H} : C_* \times \mathrm{C}^0([0,T], Y^*) \to \mathrm{C}^0([0,T], Y)$ satisfies*

$$\|\mathcal{H}(\sigma_1,\ell_1) - \mathcal{H}(\sigma_2,\ell_2)\|_\infty \le \max\Big\{ \|(\sigma_1 - \ell_1(0)) - (\sigma_2 - \ell_2(0))\|_*, \beta(\|\ell_1 - \ell_2\|_\infty) \Big\}.$$

*(B) [Kre99, Cor. 5.9], [Des98] Let $C_*$ satisfy $B^*_\rho(0) \subset C_* \subset B^*_R(0)$ and assume that for each point $\sigma \in \partial C_*$ there exists a unique outward normal vector $n(\sigma)$ of length 1 such that $n : \partial C_* \to Y$ is Lipschitz continuous. Then, for each ball $\mathcal{B}_R(0) = \{\, \ell \in \mathrm{W}^{1,1}([0,T], Y^*) \,|\, \|\ell\|_{\mathrm{W}^{1,1}} \le R \,\}$ there exists a Lipschitz constant $L_R$ such that*

$$\forall\, \sigma_1, \sigma_2 \in C_* \;\forall \ell_1, \ell_2 \in \mathcal{B}_R :\; \|\mathcal{H}(\sigma_1,\ell_1) - \mathcal{H}(\sigma_2,\ell_2)\|_{\mathrm{W}^{1,1}} \le L_R\Big( \|\sigma_1 - \sigma_2\|_* + \|\ell_1 - \ell_2\|_{\mathrm{W}^{1,1}} \Big).$$

See also [Kre99, Kre01] for certain counterexamples which show that without the given conditions the results may fail. In particular, it is not sufficient that $n : \partial C_* \to Y$ is $\mathrm{C}^1$, since global Lipschitz continuity implies the boundedness of the derivative.

In [KP89] hysteresis problems with ***polyhedral characteristics*** are studied, i.e., $C_*$ has the form

$$C_* = \{\, \sigma \in Y^* \mid \forall j = 1, \dots, K :\; \langle \sigma, n_j \rangle \le \beta_j \,\}, \tag{2.29}$$

where $n_j \in Y$ with $\|n_j\| = 1$ and $\beta_j > 0$ are given. Denoting $X = \mathrm{span}\{n_1, \dots, n_K\} \subset Y$ and $X^* = AX$ we may decompose $Y$ and $Y^*$ as follows:

$$Y = X \oplus X^\perp \text{ with } X^\perp = \{\, y \in Y \mid \forall j = 1, \dots, K :\; \langle A n_j, y \rangle = 0 \,\}$$
$$\text{and } Y^* = X^* \oplus X^\perp_* \text{ with } X^\perp_* = A X^\perp.$$

Thus, $C_*$ takes the form of a polyhedral cylinder $\widehat{C}_* + X^\perp_*$ with $\widehat{C}_* = C_* \cap X^*$ and the Legrendre transform gives $\Psi(\dot y) = \Psi(\dot x + \hat{\dot x}^\perp) = \widehat{\Psi}(\dot x)$. The important fact is that $0 \in X^*$ is now an interior point of $\widehat{C}_*$. Writing $A = \mathrm{diag}(A_X, A^\perp)$, $y = x + x^\perp$ and $\ell = \ell_X + \ell^\perp$, the Legrendre transform gives $\Psi(\dot x + \dot x^\perp) = \widehat{\Psi}(\dot x)$ and the subdifferential equation (SF) (2.1) decouples into

$$0 \in \partial\widehat{\Psi}(\dot x(t)) + A_X x(t) - \ell_X(t) \text{ in } X^* \quad \text{and} \quad 0 = A^\perp x^\perp(t) - \ell^\perp(t) \text{ in } X^\perp_*.$$

Hence, we find $x^\perp$ by solving a static problem, namely $x^\perp(t) = A^{-1}\ell^\perp(t)$. Only, $x : [0,T] \to X$ has to be found by solving a hysteresis problem.

**Theorem 2.8** [KP89, DT99] *If $C_*$ is polyhedral as defined in* (2.29)*, then there exist global Lipschitz constants $L_\mathrm{C}$ and $L_\mathrm{W}$ such that the following holds ($y_j = \mathcal{H}(\sigma_j, \ell_j)$):*

$$\begin{aligned}
&\forall\, \sigma_1, \sigma_2 \in \mathrm{C}_* \;\forall \ell_1, \ell_2 \in \mathrm{C}^0([0,T], Y^*) :\\
&\qquad \|y_1 - y_2\|_{\mathrm{C}^0([0,T],Y)} \le L_\mathrm{C}\big( \|\sigma_1 - \sigma_2\|_* + \|\ell_1 - \ell_2\|_{\mathrm{C}^0([0,T],Y^*)} \big),\\
&\forall\, \sigma_1, \sigma_2 \in \mathrm{C}_* \;\forall \ell_1, \ell_2 \in \mathrm{W}^{1,1}([0,T], Y^*) :\\
&\qquad \|y_1 - y_2\|_{\mathrm{W}^{1,1}([0,T],Y)} \le L_\mathrm{W}\big( \|\sigma_1 - \sigma_2\|_* + \|\ell_1 - \ell_2\|_{\mathrm{W}^{1,1}([0,T],Y^*)} \big).
\end{aligned}$$

Short and complete proofs of these results can be found in [Kre99, Sect. 6]. Further results concerning the continuity of $\mathcal{H}$ with respect to changes in the set $C_*$ can be found in [Kre96]. If $(C_*^m)_{m\in\mathbb{N}}$ is a sequence of closed convex sets containing $0 \in Y^*$ and converging in the sense of the Hausdorff distance to $C_*$, then, for each $p \in [1,\infty)$, the convergence of the data $(\sigma_m,\ell_m) \to (\sigma,\ell)$ in $Y^*\times\mathrm{W}^{1,p}([0,T],Y^*)$ implies the convergence of the solutions, namely $\mathcal{H}_{C_*^m}(\sigma_m,\ell_m) \to \mathcal{H}_{C_*}(\sigma,\ell)$ in $\mathrm{W}^{1,p}([0,T],Y)$. If additionally all $C_*^m$ are recession sets, then the result holds as well with $\mathrm{C}^0([0,T],Y^*)$ and $\mathrm{C}^0([0,T],Y)$, respectively. In fact, in the light of Theorem 2.6 it should be sufficient to assume that (2.25) holds uniformly for $(C_*^m)_{m\in\mathbb{N}}$ and that the property of recession sets is no longer needed.

# 3 Incremental problems and a priori estimates

For general nonlinear problems in discrete and continuum mechanics the assumption of convexity is not reasonable. First, the constitutive laws may be nonmonotone giving rise to nonconvex potentials and, second, even the underlying set of states may have no underlying structure any more. Such situations are usually modelled by working on suitable subsets of Banach spaces which are equipped either with the weak or the strong topology. Here we propose an abstract approach which does not need any Banach space theory and thus highlights the fundamental nature of the energy functionals $\mathcal{E}$ and $\mathcal{D}$ even more.

## 3.1 Abstract setup of the problem

To keep the connection with continuum mechanics we consider the set $\mathcal{Y} = \mathcal{F}\times\mathcal{Z}$ as the basic state space, cf. (2.10) for the same splitting in the linearized setting. Whenever possible we will write $y$ instead of $(\varphi, z)$ to shorten notation. Note that the splitting is done such that changes in $z$ involve dissipation whereas those of $\varphi$ do not. The state space $\mathcal{Y}$ is equipped with a Hausdorff topology $\mathcal{T} = \mathcal{T}_{\mathcal{F}}\times\mathcal{T}_{\mathcal{Z}}$ and we denote by $y_k \overset{\mathcal{Y}}{\to} y$, $\varphi_k \overset{\mathcal{F}}{\to} \varphi$ and $z_k \overset{\mathcal{Z}}{\to} z$ the corresponding convergence of sequences. Throughout it will be sufficient to consider sequential closedness, compactness and continuity.

The first ingredient of the energetic formulation is the ***dissipation distance*** $\mathcal{D} : \mathcal{Z}\times\mathcal{Z} \to [0,\infty]$, which is a semi-distance, which means

$$\begin{aligned} &\text{(i)} && \forall\, z_1, z_2 \in \mathcal{Z} :\ \mathcal{D}(z_1,z_2) = 0 \quad \iff \quad z_1 = z_2, \\ &\text{(ii)} && \forall\, z_1, z_2, z_3 \in \mathcal{Z} :\ \mathcal{D}(z_1,z_3) \le \mathcal{D}(z_1,z_2) + \mathcal{D}(z_2,z_3). \end{aligned} \tag{A1}$$

Here (i) is the classical positivity of a distance and (ii) the triangle inequality. Note that we allow the value $\infty$ and that we do not enforce symmetry, i.e., $\mathcal{D}(z_1,z_2) \neq \mathcal{D}(z_2,z_1)$ is allowed, as this is needed in many applications.

One major point of the theory is the interplay between the topology $\mathcal{T}_{\mathcal{Z}}$ and the dissipation distance. To have a typical nontrivial application in mind, one may consider $\mathcal{Z} = \{\, z \in \mathrm{L}^1(\Omega,\mathbb{R}^k) \mid \|z\|_{\mathrm{L}^\infty} \le 1 \,\}$ equipped with the weak $\mathrm{L}^1$-topology and the dissipation distance $\mathcal{D}(z_1,z_2) = \|z_1{-}z_2\|_{\mathrm{L}^1}$.

For a given curve $z : [0,T] \to \mathcal{Z}$ we define the total dissipation on $[s,t]$ via

$$\mathrm{Diss}_{\mathcal{D}}(z;[s,t]) = \sup\{\, \textstyle\sum_1^N \mathcal{D}(z(\tau_{j-1}),z(\tau_j)) \mid N{\in}\mathbb{N},\, s{=}\tau_0{<}\tau_1{<}\cdots{<}\tau_N{=}t \,\}. \tag{3.1}$$

Further we define the following set of functions:

$$\mathrm{BV}_{\mathcal{D}}([0,T],\mathcal{Z}) := \{\, z : [0,T] \to \mathcal{Z} \mid \mathrm{Diss}_{\mathcal{D}}(z;[0,T]) < \infty \,\}.$$

The functions are defined everywhere and changing them at one point may increase the dissipation. Moreover, the dissipation is additive:

$$\mathrm{Diss}_{\mathcal{D}}(z;[r,t]) = \mathrm{Diss}_{\mathcal{D}}(z;[r,s]) + \mathrm{Diss}_{\mathcal{D}}(z;[s,t]) \quad \text{for all } r < s < t.$$

Later on, we will sometimes use the notation $\mathcal{D}(y_0, y_1)$ instead of $\mathcal{D}(z_0, z_1)$ where $y_j = (\varphi_j, z_j)$. This slight abuse of notation will never lead to confusion, since $\mathcal{D}$ as a function on $\mathcal{Y} = \mathcal{F}\times\mathcal{Z}$ still satisfies all assumptions except of the positivity (A1)(i).

The second ingredient is the energy-storage functional $\mathcal{E} : [0,T]\times\mathcal{Y} \to \mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$. Here $t \in [0,T]$ plays the rôle of a (very slow) process time which changes the underlying system via changing loading conditions. We assume that for all $y$ with $\mathcal{E}(t, y_*) < \infty$ the function $t \mapsto \mathcal{E}(t, y_*)$ is differentiable, namely

$$\begin{aligned} &\text{There exist } c_E^{(1)}, c_E^{(0)} > 0 \text{ such that for all } y_* \in \mathcal{Y}: \\ &\mathcal{E}(t,y_*) < \infty \;\Rightarrow\; \partial_t\mathcal{E}(\cdot,y_*) : [0,T] \to \mathbb{R} \text{ is measurable} \\ &\qquad\qquad\qquad\quad \text{and } |\partial_t\mathcal{E}(t,y_*)| \le c_E^{(1)}(\mathcal{E}(t,y_*)+c_E^{(0)}). \end{aligned} \tag{A2}$$

From (A2) and Gronwall's inequality we easily derive

$$\mathcal{E}(t,y)+c_E^{(0)} \le (\mathcal{E}(s,y)+c_E^{(0)})\mathrm{e}^{c_E^{(1)}|t-s|}, \tag{3.2}$$

which implies the Lipschitz continuity of $t \mapsto \mathcal{E}(t,y)$. The notion of ***self-controlling models*** in [Che01, Che03] corresponds closely to our condition (A2).

**Definition 3.1** *A curve $y = (\varphi, z) : [0,T] \to \mathcal{Y} = \mathcal{F}\times\mathcal{Z}$ is called an **energetic solution** of the rate-independent system associated with $(\mathcal{D},\mathcal{E})$, if $t \mapsto \partial_t\mathcal{E}(t, y(t))$ is integrable and if the **global stability (S)** and the **energy equality (E)** hold for all $t \in [0,T]$:*

***(S)** For all $\widehat{y} = (\widehat{\varphi}, \widehat{z}) \in \mathcal{Y}$ we have $\mathcal{E}(t, y(t)) \le \mathcal{E}(t, \widehat{y}) + \mathcal{D}(z(t), \widehat{z})$.*

***(E)** $\mathcal{E}(t, y(t)) + \mathrm{Diss}_{\mathcal{D}}(z;[0,t]) = \mathcal{E}(0, y(0)) + \int_0^t \partial_t\mathcal{E}(\tau, y(\tau))\,\mathrm{d}\tau$.*

The stability condition (S) can be rephrased by defining the set $\mathcal{S}(t)$ of stable states at time $t$ via

$$\begin{aligned} \mathcal{S}(t) &:= \{\, y \in \mathcal{Y} \mid \mathcal{E}(t,y) < \infty,\ \mathcal{E}(t,y) \le \mathcal{E}(t,\widehat{y}) + \mathcal{D}(y,\widehat{y}) \text{ for all } \widehat{y} \in \mathcal{Y} \,\}, \\ \mathcal{S}_{[0,T]} &:= \{\, (t,y) \in [0,T]\times\mathcal{Y} \mid y \in \mathcal{S}(t) \,\} = \cup_{t\in[0,T]}(t, \mathcal{S}(t)). \end{aligned}$$

Then, (S) simply means that $y(t) \in \mathcal{S}(t)$ for all $t \in [0,T]$. The properties of the stable sets turn out to be crucial for deriving existence results.

The definition of solutions of (S) & (E) is such that it implies the two natural requirements for evolutionary problems, namely that ***restrictions*** and ***concatenations*** of solutions remain solutions. To be more precise, for any solution $y : [0,T] \to \mathcal{Y}$ and any subinterval $[s,t] \subset [0,T]$, the restriction $y|_{[s,t]}$ solves (S) & (E) with initial datum $y(s)$. Moreover, if $y_1 : [0,t] \to \mathcal{Y}$ and $y_2 : [t,T] \to \mathcal{Y}$ solve (S) & (E) on the respective intervals and if $y_1(t) = y_2(t)$, then the concatenation $y : [0,T] \to \mathcal{Y}$ solves (S) & (E) as well. Thus, the definition implies that if solvability can be shown for all $z_0 \in \mathcal{S}(0)$, then we have a multi-valued evolutionary semigroup as explained in Section 1, see also [Bal97].

Rate-independence manifests itself by the fact that the problem has no intrinsic time scale. It is easy to show that $y$ is a solution for $(\mathcal{D},\mathcal{E})$ if and only if the reparametrized curve $\widetilde{y} : t \mapsto y(\alpha(t))$, where $\dot{\alpha} > 0$, is a solution for $(\mathcal{D},\widetilde{\mathcal{E}})$ with $\widetilde{\mathcal{E}}(t,y) = \mathcal{E}(\alpha(t),y)$. In particular, the stability (S) is a static concept and the energy balance (E) is rate-independent, since the dissipation defined via (3.1) is scale invariant like the length of a curve.

Before discussing the question of existence of solutions we want to point out, that the concept of energetic solutions provides a priori bounds on the solutions. For the time-continuous problem these bounds are easy to derive and the main structure becomes more transparent. Of course, similar estimates will be crucial in the time-discrete setting. Using the assumption (A2) the energy balance (E) gives

$$\mathcal{E}(t,y(t)) + \mathrm{Diss}_{\mathcal{D}}(z,[0,t]) \leq \mathcal{E}(0,y(0)) + \textstyle\int_0^t c_E^{(1)}(\mathcal{E}(s,y(s)){+}c_E^{(0)})\,\mathrm{d}s. \tag{3.3}$$

Omitting the dissipation and adding $c_E^{(0)}$ on both sides allows for an application of Gronwall's inequality and we obtain

$$\mathcal{E}(t,y(t)) \leq (\mathcal{E}(0,y(0)){+}c_E^{(0)})\mathrm{e}^{c_E^{(1)}t} - c_E^{(0)}.$$

Inserting this again into (3.3) we can also estimate the dissipation via

$$\mathrm{Diss}_{\mathcal{D}}(z;[0,T]) \leq (\mathcal{E}(0,y(0)){+}c_E^{(0)})\mathrm{e}^{c_E^{(1)}T},$$

since $\mathcal{E}(t,y(t)) \geq -c_E^{(0)}$ by (A2).

Because of the rate-independence it is easily possible to generalize assumption (A2) to include absolutely continuous loadings instead of $\mathrm{C}^1$-loadings. We may replace (A2) by

$$\exists\, c_E^{(0)}, \lambda \in \mathrm{L}^1([0,T]) : \quad |\partial_t \mathcal{E}(t,y_*)| \leq \lambda(t)(\mathcal{E}(t,y_*){+}c_E^{(0)}).$$

We $\Lambda(t) = \int_0^t \lambda(\tau)\,\mathrm{d}\tau$ we easily find the estimates $\mathcal{E}(t,y){+}c_E^{(0)} \leq (\mathcal{E}(s,y){+}c_E^{(0)})\mathrm{e}^{|\Lambda(t)-\Lambda(s)|}$, $\mathcal{E}(t,y(t)) \leq (\mathcal{E}(0,y(0)){+}c_E^{(0)})\mathrm{e}^{\Lambda(t)} - c_E^{(0)}$ and $\mathrm{Diss}_{\mathcal{D}}(z;[0,T]) \leq (\mathcal{E}(0,y(0)){+}c_E^{(0)})\mathrm{e}^{\Lambda(T)}$.

## 3.2 The time-incremental problem

The most natural approach to solve (S) & (E) is via time discretization using the fact that incremental problems exist which are minimization problems. Using the classical approach for the ***direct method in the calculus of variations*** (cf. [Dac89]) it is possible to find solutions as minimizers of a lower semi-continuous functional on $\mathcal{Y}$. For this we make the following standard assumptions:

$$\begin{aligned}&\mathcal{E}(\cdot,\cdot) : \mathcal{Y}{\times}[0,T] \to \mathbb{R}_\infty \text{ has compact sublevels,}\\ &\mathcal{D} : \mathcal{Z}{\times}\mathcal{Z} \to [0,\infty] \text{ is lower semi-continuous.}\end{aligned} \tag{A3}$$

Here the sublevels $L_\alpha$ of $\mathcal{E}$ are defined via $L_\alpha := \{\, (t,y) \in [0,T]\times\mathcal{Y} \mid \mathcal{E}(t,y) \le \alpha \,\}$. Compactness of all $L_\alpha$ implies lower semi-continuity, in particular each $\mathcal{E}(t,\cdot) : \mathcal{Y}\times\mathbb{R}_\infty$ will be lower semi-continuous. In the standard case $\mathcal{Y}$ is a closed, convex and bounded subset of a reflexive Banach space (like the Sobolev space $\mathrm{W}^{1,p}(\Omega,\mathbb{R}^m)$ with $p \in (1,\infty)$) equipped with its weak topology $\mathcal{T}$. Then, lower semi-continuity of $\mathcal{E}$ and $\mathcal{D}$ in $(\mathcal{Y},\mathcal{T})$ is the same as the classical weak lower semi-continuity in the calculus of variations, see [Dac89].

For the time discretization we choose a partition $(t_k)_k \in \mathrm{Part}^N([0,T])$ and seek for a $y_k$ which approximates the solution $y$ at $t_k$, i.e., $y_k \approx y(t_k)$. Our energetic approach has the major advantage that the values $y_k$ can be found incrementally via minimization problems. Since the methods of the calculus of variations are especially suited for applications in material modeling this will allow for a rich field of applications.

In our general setting the ***incremental problem*** (IP) takes the following form:

$$\textbf{(IP)}\qquad \begin{array}{c}\text{For } y_0 \in \mathcal{S}(0) \subset \mathcal{Y} \text{ find } y_1,\dots,y_N \in \mathcal{Y} \text{ such that} \\ y_k \in \operatorname{Arg\,min}\{\, \mathcal{E}(t_k,y)+\mathcal{D}(y_{k-1},y) \mid y \in \mathcal{Y} \,\} \quad \text{for } k=1,\dots,N.\end{array} \tag{3.4}$$

Here "$\operatorname{Arg\,min}$" denotes the set of all minimizers. The following result shows that (IP) is intrinsically linked to (S) & (E).

## 3.3 Energetic a priori bounds

Without any smallness assumptions on the time steps, the solutions of (IP) satisfy properties which are closely related to (S) & (E).

**Theorem 3.2** *Let* (A1) *and* (A2) *hold. Any solution of the incremental problem* (3.4) *satisfies the following properties:*

*(i)* *For $k = 0,\dots,N$ we have that $y_k$ is stable at time $t_k$, i.e., $y_k \in \mathcal{S}(t_k)$;*

*(ii)* *For $k = 1,\dots,N$ we have*
$\int_{t_{k-1}}^{t_k} \partial_s\mathcal{E}(s,y_k)\,\mathrm{d}s \le e_k - e_{k-1} + \delta_k \le \int_{t_{k-1}}^{t_k} \partial_s\mathcal{E}(s,y_{k-1})\,\mathrm{d}s,$
*where $e_j = \mathcal{E}(t_j,y_j)$ and $\delta_k = \mathcal{D}(z_{k-1},z_k)$.*

*(iii) If* (A3) *holds, then solutions of (IP) exist.*

**Proof:** ad (i). The stability follows from minimization properties of the solutions and the triangle inequality. For all $\widehat{y} \in \mathcal{Y}$ and we have

$$\begin{aligned}\mathcal{E}(t_k,\widehat{y}) + \mathcal{D}(z_k,\widehat{z}) &= \mathcal{E}(t_k,\widehat{y}) + \mathcal{D}(z_{k-1},\widehat{z}) + \mathcal{D}(z_k,\widehat{z}) - \mathcal{D}(z_{k-1},\widehat{z}) \\ &\ge \mathcal{E}(t_k,y_k) + \mathcal{D}(z_{k-1},z_k) + \mathcal{D}(z_k,\widehat{z}) - \mathcal{D}(z_{k-1},\widehat{z}) \;\ge\; \mathcal{E}(t_k,y_k).\end{aligned}$$

ad (ii). The first estimate is deduced from $y_{k-1} \in \mathcal{S}(t_{k-1})$ as follows:

$$\begin{aligned}&\mathcal{E}(t_k,y_k) + \mathcal{D}(z_{k-1},z_k) - \mathcal{E}(t_{k-1},y_{k-1}) = \\ &\mathcal{E}(t_{k-1},y_k) + \int_{[t_{k-1},t_k]} \partial_s\mathcal{E}(s,y_k)\,\mathrm{d}s + \mathcal{D}(z_{k-1},z_k) - \mathcal{E}(t_{k-1},y_{k-1}) \ge \int_{[t_{k-1},t_k]} \partial_s\mathcal{E}(s,y_k)\,\mathrm{d}s.\end{aligned}$$

Since $y_k \in \operatorname{Arg\,min}\{\, \mathcal{E}(t_k, y) + \mathcal{D}(z_{k-1}, z) \mid y \in \mathcal{Y} \,\}$ the second estimate follows via

$$\begin{aligned}&\mathcal{E}(t_k, y_k) - \mathcal{E}(t_{k-1}, y_{k-1}) + \mathcal{D}(z_{k-1}, z_k)\\ &\leq \mathcal{E}(t_k, y_{k-1}) - \mathcal{E}(t_{k-1}, y_{k-1}) + \mathcal{D}(z_{k-1}, z_{k-1}) \;=\; \int_{[t_{k-1},t_k]} \partial_s \mathcal{E}(s, y_{k-1})\,\mathrm{d}s.\end{aligned} \tag{3.5}$$

ad (iii). The minimizers are constructed inductively. In the $k$-th step $y_{k-1}$ is known and any minimizer $y$ has to satisfy $\mathcal{J}_k(y) := \mathcal{E}(t_k, y) + \mathcal{D}(z_{k-1}, z) \leq \mathcal{E}(t_k, y_{k-1}) = \mathcal{J}_k(y_{k-1})$ since $y = y_{k-1}$ is a candidate. Using $\mathcal{D} \geq 0$ it suffices to minimize the lower semi-continuous functional $\mathcal{J}_k$ on the compact sublevel $\mathcal{E}(t_k, \cdot) \leq \mathcal{E}(t_k, y_{k-1})$. Hence, Weierstraß' extremum principle provides the existence of a minimizer $y_k$. ■

Now we use assumption (A2) to obtain a priori bounds on the energy and the dissipation for the solution of (IP). Combining (A2), (3.2) and the upper estimate in (ii) of Theorem 3.2 give

$$e_k + \delta_k \;\leq\; e_{k-1} + (c_E^{(0)} + e_{k-1})\big(\mathrm{e}^{c_E^{(1)}(t_k - t_{k-1})} - 1\big) \tag{3.6}$$

$$\;=\; (c_E^{(0)} + e_{k-1})\mathrm{e}^{c_E^{(1)}(t_k - t_{k-1})} - c_E^{(0)}. \tag{3.7}$$

Using $\delta_k \geq 0$ and (3.7), induction over $k$ leads to

$$c_E^{(0)} + e_k \leq (c_E^{(0)} + e_0) \prod_{j=1}^k \mathrm{e}^{c_E^{(1)}(t_j - t_{j-1})} = (c_E^{(0)} + e_0)\mathrm{e}^{c_E^{(1)} t_k} \text{ for } k = 1, \ldots, N. \tag{3.8}$$

For the dissipated energy we find the we can estimate

$$\begin{aligned}\textstyle\sum_{j=1}^k \delta_j \;&\leq_{(3.6)}\; e_0 - e_k + \textstyle\sum_{j=1}^k (c_E^{(0)} + e_{j-1})(\mathrm{e}^{c_E^{(1)}(t_j - t_{j-1})} - 1)\\ &\leq_{(3.8)}\; (c_E^{(0)} + e_0) - (c_E^{(0)} + e_k) + (c_E^{(0)} + e_0) \textstyle\sum_1^k (\mathrm{e}^{c_E^{(1)} t_j} - \mathrm{e}^{c_E^{(1)} t_{j-1}})\\ &\leq\; (c_E^{(0)} + e_0)\mathrm{e}^{c_E^{(1)} t_k},\end{aligned}$$

where $c_E^{(0)} + e_k \geq 0$ was used in the last step.

For each incremental solution $(y_k)_{k=1,\ldots,N}$ of (IP) associated with a partition $\Pi \in \mathrm{Part}^N([0, T])$ wedefine the piecewise constant interpolant $Y^\Pi$ with

$$Y^\Pi(T) = y_N \quad \text{and} \quad Y^\Pi(t) = y_{k-1} \text{ for } t \in [t_{k-1}, t_k), \text{ where } k = 1, \ldots, N. \tag{3.9}$$

**Corollary 3.3** *Assume that* (A1) *and* (A2) *hold and let* $\Pi \in \mathrm{Part}^N([0, T])$. *Then, for any solution* $(y_k)_{k=0,\ldots,N}$ *of (IP) the interpolant* $Y^\Pi = (\varphi^\Pi, z^\Pi) : [0, T] \to \mathcal{F} \times \mathcal{Z}$ *satisfies the following three properties*

*(1): (S)$_{\text{discr}}$ For* $t \in \Pi$ *we have* $Y^\Pi(t) \in \mathcal{S}(t)$*;*

*(2) (E)$_{\text{discr}}$ For* $s, t \in \Pi$ *with* $s < t$ *we have the energy estimate*

$$\mathcal{E}(t, Y^\Pi(t)) + \mathrm{Diss}_{\mathcal{D}}(z^\Pi; [s, t]) \leq \mathcal{E}(s, Y^\Pi(s)) + \int_s^t \partial_\tau \mathcal{E}(\tau, Y^\Pi(\tau))\,\mathrm{d}\tau;$$

*(3) For all* $t \in [0, T]$ *we have the a priori estimates (with* $E_0 = \mathcal{E}(0, y_0) + c_E^{(0)}$*)*

$$\mathcal{E}(t, Y^\Pi(t)) \leq E_0\, \mathrm{e}^{c_E^{(1)} t} - c_E^{(0)}, \;\; \textit{and} \;\; \mathrm{Diss}_{\mathcal{D}}(z^\Pi; [0, T]) \leq E_0\, \mathrm{e}^{c_E^{(1)} T}. \tag{3.10}$$

## 3.4 The condensed and reduced incremental problem

Recall the incremental problem in the form

$$(\varphi_k, z_k) \in \operatorname{Arg\,min}\{\, \mathcal{E}(t_k, \widehat{\varphi}, \widehat{z}) + \mathcal{D}(z_{k-1}, \widehat{z}) \mid (\widehat{\varphi}, \widehat{z}) \in \mathcal{F}\times\mathcal{Z} \,\}. \tag{3.11}$$

In many applications in continuum mechanics a specific feature occurs, namely that $\mathcal{E}(t, \varphi, z)$ and $\mathcal{D}(z_0, z)$ depend only locally on $z$, in the sense that at $x \in \Omega$ the integral over $\Omega$ uses $z$ only through its point value $z(x)$. Hence, $z$ can be eliminated pointwise. We define the ***condensed energy density*** $W^{\rm cond}$ and the ***update function*** $Z^{\rm update}$ for the internal variable via

$$\begin{aligned} W^{\rm cond}(z_{\rm old}; x, F) &:= \min\{\, W(x, F, z) + D(x, z_{\rm old}, z) \mid z \in Z \,\},\\ Z^{\rm update}(z_{\rm old}; x, F) &\in \operatorname{Arg\,min}\{\, W(x, F, z) + D(x, z_{\rm old}, z) \mid z \in Z \,\}. \end{aligned} \tag{3.12}$$

With this we obtain a functional $\mathcal{E}^{\rm cond}(z_{\rm old}; t, \varphi) = \int_\Omega W^{\rm cond}(z_{\rm old}; \mathrm{D}\varphi)\,\mathrm{d}x - \langle \ell_{\rm ext}(t), \varphi\rangle$ and the solution of (3.11) is equivalent to finding $\varphi_k \in \operatorname{Arg\,min}\{\, \mathcal{E}^{\rm cond}(z_{k-1}; t_k, \widehat{\varphi}) \mid \widehat{\varphi} \in \mathcal{F} \,\}$ and then letting $z_k = Z^{\rm update}(z_{k-1}; \mathrm{D}\varphi_k)$. For more details we refer to [Mie03a, Mie04b].

The above condensation is very useful for computational purposes and it also allows for an existence theory for (IP) in the case of finite-strain elastoplasticity, see [Mie04b]. However, for the mathematical theory associated with the time-continuous problem (S) & (E) it seems advantageous to reduce the problem to the $z$-variable alone. The major difficulty in considering the pair $y = (\varphi, z)$ is that $\varphi \in \mathcal{F}$ does not appear in the dissipation. Hence, by (S), $\varphi(t)$ will always be a global minimizer of $\mathcal{E}(t, \cdot, z(t))$. But otherwise we have no control over the temporal oscillations in the approximate functions $\varphi^N : [0, T] \to \mathcal{F}$.

A first possible approach to tackle this difficulty is to introduce the reduced energy functional

$$\mathcal{E}^{\rm red}(t, z) = \min\{\, \mathcal{E}(t, \varphi, z) \mid \varphi \in \mathcal{F} \,\}.$$

However, in general we will lose the exact control, since $\mathcal{E}^{\rm red}$ is no longer explicit. In particular, the differentiability of $t \mapsto \mathcal{E}^{\rm red}(t, z)$ is not guaranteed in general. At the moment, there is only one way out, which is not always acceptable: We simply restrict ourselves to problems where the minimizer $\varphi = \Phi(t, z)$ of $\mathcal{E}(t, \cdot, z)$ is unique and depends continuously on $(t, z)$. Then, $\mathcal{E}^{\rm red}(t, z) = \mathcal{E}(t, \Phi(t, z), z)$ and $\partial_t \mathcal{E}^{\rm red}(t, z) = \partial_t \mathcal{E}(t, \Phi(t, z), z)$.

The same assumption is needed if we keep the $\varphi$-variable. In this second approach the bottleneck is the assumption (A4) which states that the dissipation controls convergence in $\mathcal{Y} = \mathcal{F}\times\mathcal{Z}$. Of course, this has to be true only on $\mathcal{V}_{[0,T]} = \mathcal{R}_{[0,T]} \cap \mathcal{S}_{[0,T]}$. Note that $(\varphi, z) \in \mathcal{S}(t)$ already implies $\varphi = \Phi(t, z)$. Hence, (A4) can be satisfied by assuming that (A4) holds for $\widetilde{\mathcal{D}} : \mathcal{Z}\times\mathcal{Z} \to [0, \infty]$ and that $\Phi$ is continuous.

This uniqueness assumption will be used in Sections 7.3 and 7.5, see also [MTL02, MR03, KMR04]. However, in Section 7.6 this uniqueness can be dispensed with.

## 3.5 Lipschitz bounds via convexity

As we have seen in the quadratic case, we are able to obtain also Lipschitz bounds, if convexity of $\mathcal{E}(t, \cdot)$ is used. Classical convexity theory involves a Banach space $\mathcal{Y}$ as the basic state space.

Then, a function $f : \mathcal{Y} \to \mathbb{R}_\infty$ is called ***uniformly convex***, if there exists $\alpha > 0$ such that for all $y_0, y_1 \in \mathcal{Y}$ we have

$$f(y_\theta) \leq (1-\theta)f(y_0) + \theta f(y_1) - \tfrac{\alpha}{2}\theta(1-\theta)\|y_1-y_0\|^2, \quad \text{where } y_\theta = (1-\theta)y_0 + \theta y_1.$$

This notion can be generalized to arbitrary metric spaces by using so-called geodesical convexity. So let $(\mathcal{Y}, d)$ be a general metric space. For each $y_0, y_1 \in \mathcal{Y}$ we define the set

$$[y_0, y_1]_\theta = \{\, y \in \mathcal{Y} \mid d(y, y_0) = \theta d(y_0, y_1) \text{ and } d(y, y_1) = (1-\theta)d(y_0, y_1) \,\}$$

We define the following convexity notions for $f : \mathcal{Y} \to \mathbb{R}_\infty$:

$$\begin{array}{ll} f \text{ convex:} & \forall\, y_0, y_1 \in \mathcal{Y}\ \forall\, \theta \in (0,1)\ \exists\, y \in [y_0, y_1]_\theta : f(y) \leq (1-\theta)f(y_0) + \theta f(y_1); \\ f \text{ strictly convex:} & \forall\, y_0, y_1 \in \mathcal{Y}\ \forall\, \theta \in (0,1)\ \exists\, y \in [y_0, y_1]_\theta : f(y) < (1-\theta)f(y_0) + \theta f(y_1); \\ f\ \alpha\text{-convex:} & \forall\, y_0, y_1 \in \mathcal{Y}\ \forall\, \theta \in (0,1)\ \exists\, y \in [y_0, y_1]_\theta : \\ & \qquad f(y) \leq (1-\theta)f(y_0) + \theta f(y_1) - \tfrac{\alpha}{2}\theta(1-\theta)d(y_0, y_1)^2. \end{array}$$

For strictly convex Banach spaces the sets $[y_0, y_1]_\theta$ are singletons and the above notions coincide with the classical ones. If an $\alpha$-convex function $f$ is two-times differentiable, then $\mathrm{D}^2 f$ satisfies $\mathrm{D}^2 f(y)[v, v] \geq \alpha\|v\|^2$. We again use the notion uniformly convex, if we have $\alpha$-convexity for some $\alpha > 0$.

The major difficulty in general rate-independent problems is that the dissipation distance $\mathcal{D}$ is not convex, even if $\mathcal{Y}$ is a Banach space. For instance, we may consider a system $\widetilde{\mathcal{E}}$, $\widehat{D}$ on $\mathcal{Y}$ which is a Banach space and $\widehat{\mathcal{D}}$ has the form $\widehat{\mathcal{D}}(\widehat{y}_0, \widehat{y}_1) = \Psi(\widehat{y}_1 - \widehat{y}_0)$. Doing a coordinate transformation $\widehat{y} = \Phi(y)$ we arrive at the transformed dissipation distance $\mathcal{D} : (y_0, y_1) \mapsto \widehat{\mathcal{D}}(\Phi(y_0), \Phi(y_1))$, which is no longer convex on $\mathcal{Y} \times \mathcal{Y}$ if $\Phi$ is a nonlinear mapping. However, with respect to the transformed metric $d : (y_0, y_1) \mapsto \|\Phi(y_1) - \Phi(y_0)\|$ geodesic convexity is preserved.

It should be noted that our dissipation distance is in general different from the metric to be considered. For instance we want to allow for $\mathcal{D}$ assuming the value $+\infty$ and for unsymmetry. It is also important to note, that in general the function $d(y_*, \cdot) : y \mapsto d(y_*, y)$ is not geodesically convex on $(\mathcal{Y}, d)$, for instance the arc-length distance on $\mathbb{S}^1$. However, on $\mathbb{S}^1$ nonconstant convex functions exist, but they must attain the value $+\infty$.

**Theorem 3.4** *Assume that $\mathcal{E}$ and $\mathcal{D}$ are defined on the metric space $(\mathcal{Y}, d)$ and satisfy* (A2) *and the following two conditions:*

$$\exists\, \alpha > 0\ \forall\, (t, y) \in [0, T] \times \mathcal{Y} :\ \widehat{y} \mapsto \mathcal{E}(t, \widehat{y}) + \mathcal{D}(y, \widehat{y}) \text{ is } \alpha\text{-convex} \tag{3.13}$$

$$\forall\, t \in [0, T]\ \forall\, y_0, y_1 \in \mathcal{Y} : |\partial_t \mathcal{E}(t, y_0) - \partial_t \mathcal{E}(t, y_1)| \leq C_3 d(y_0, y_1). \tag{3.14}$$

*Then, any solution $y : [0, T] \to Y$ of (S) & (E) satisfies the estimate*

$$\forall\, s, t \in [0, T] :\ d(y(s), y(t)) \leq \tfrac{C_3}{\alpha}\,|t-s|.$$

*Moreover, the solution $(y_k)_{k=0,1,\ldots,N}$ of (IP) is unique and satisfies*

$$\forall\, k = 1, \ldots, N :\ d(y_{k-1}, y_k) \leq 2\,\tfrac{C_3}{\alpha}\,(t_k - t_{k-1}).$$

**Proof:** We use the fact that for the minimizer $y_*$ of an $\alpha$-convex function $f$ we always have $f(y) \geq f(y_*) + \frac{\alpha}{2} d(y, y_*)^2$, hence it is unique.

For the first assertion we use $y(s) \in \mathcal{S}(s)$, $\mathcal{D}(y(s), y(t)) \leq \mathrm{Diss}_{\mathcal{D}}(y; [s,t])$ and (E):

$$\begin{aligned} \tfrac{\alpha}{2} d(y(s), y(t))^2 &\leq \mathcal{E}(s, y(t)) + \mathcal{D}(y(s), y(t)) - \mathcal{E}(s, y(s)) \\ &\leq \mathcal{E}(t, y(t)) + \mathrm{Diss}_{\mathcal{D}}(y; [s,t]) - \mathcal{E}(s, y(s)) - \textstyle\int_s^t \partial_r \mathcal{E}(r, y(t))\,\mathrm{d}r \\ &\leq \textstyle\int_s^t [\partial_r \mathcal{E}(r, y(r)) - \partial_r \mathcal{E}(r, y(t))]\,\mathrm{d}r. \end{aligned}$$

For fixed $t > 0$ let $\delta(s) = d(y(s), y(t))$, then $\delta(s)^2 \leq \frac{2C_3}{\alpha} \int_s^t \delta(r)\,\mathrm{d}r$ for $s \in [0,t]$. Now, let $h(\tau) = \int_{t-\tau}^t \delta(r)\,\mathrm{d}r$ for $\tau \geq 0$, then $h(0) = 0$ and $h'(\tau) \leq (2C_3 h(\tau)/\alpha)^{1/2}$. This implies $h(\tau) \leq C_3 \tau^2/(2\alpha)$ and hence $\delta(t-\tau) \leq (C_3\tau/\alpha)^2$, which is the desired result.

The second assertion follows in the same way by using Thm. 3.2(i) and (ii):

$$\tfrac{\alpha}{2} d(y_k, y_{k-1})^2 \leq \textstyle\int_{t_{k-1}}^{t_k} [\partial_r \mathcal{E}(r, y_{k-1}) - \mathcal{E}(r, y_k)]\,\mathrm{d}r \leq C_3 (t_k - t_{k-1}) d(y_k, y_{k-1}),$$

which proves the claim. ■

With this result we supply Lipschitz continuity results for the solution of (S) & (E) and of (IP). They will be useful in the construction of solutions.

**Example 3.5** *We illustrate the concept with a simple example on $\mathcal{Y} = \mathbb{R}$. Take the energy functional $\mathcal{E}(t,y) = (y - \ell(t))^2/2$ and the dissipation metric*

$$\Psi(y, \dot{y}) = h(y)|\dot{y}|, \ \text{where } h(z) = \begin{cases} 1+a - a\,\mathrm{sign}(z) & \text{for } |z| \geq 1, \\ 1+a - az & \text{for } |z| \leq 1, \end{cases}$$

*where $a > 0$. It is easy to see that the associated dissipation distance is given by $\mathcal{D}(y_0, y_1) = |H(y_1) - H(y_0)|$ with $H(y) = \int_0^y h(s)\,\mathrm{d}s$.*

*Thus, using the classical distance $d(y_0, y_1) = |y_1 - y_0|$ on $\mathbb{R}$, we see that $y \mapsto \mathcal{E}(t,y) + \mathcal{D}(y_0, y)$ is (geodesically) $\alpha$-convex, for $\alpha = \inf\{\, \mathcal{E}''(t,y) + \partial_y^2 \mathcal{D}(y_0, y) \mid y \in \mathbb{R}\,\}$. Thus, for $y_0 \geq 1$ we obtain $\alpha = 1$ and for $y_0 < 1$ we have $\alpha = 1 - a$. In particular, $y_0 < 1$ and $a > 1$ imply that convexity is lost and uniqueness of minimizers and Lipschitz bounds are no longer valid.*

## 3.6 A simplified incremental problem

If $\mathcal{Y}$ is a Banach space $Y$ and the dissipation distance $\mathcal{D}$ is implicitly defined through a dissipation potential $\Psi : Y \times Y \to [0, \infty]$ via

$$\mathcal{D}(y_0, y_1) := \inf\{\, \textstyle\int_0^1 \Psi(y(t), \dot{y}(t))\,\mathrm{d}t \mid y \in \mathrm{C}^1([0,1], Y), y(0) = y_0, y(1) = y_1 \,\},$$

then it is desirable to approach the differential inclusion

$$0 \in \partial_v \Psi(y, \dot{y}) + \mathrm{D}\mathcal{E}(t,y) \quad \subset Y^*$$

via an incremental problem which avoids $\mathcal{D}$ and uses $\Psi$ instead. Under suitable conditions we have

$$\mathcal{D}(y_0, y_0 + \varepsilon v) = \varepsilon \Psi(y_0, v) + \tfrac{\varepsilon^2}{2} \mathrm{D}_z \Psi(y_0, v)[v] + o(\varepsilon^2).$$

Thus, if some convexity is available then we may hope that the increments are small and thus it should suffice to approximate $\mathcal{D}(y_0, y_1)$ by $\Psi(y_0, y_1{-}y_0)$. This leads to the following incremental problem:

$$
(\widetilde{\mathbf{IP}}) \qquad \begin{array}{c} \text{For } y_0 \in \mathcal{S}(0) \text{ find } y_1, \ldots, y_N \in \mathcal{Y} \text{ such that} \\ y_k \in \operatorname{Arg\,min}\{\, \mathcal{E}(t_k, y) + \Psi(y_{k-1}, y{-}y_{k-1}) \mid y \in Y \,\}. \end{array} \tag{3.15}
$$

The function $\Psi(y, \cdot)$ is always convex, hence the solutions $y_k$ are unique as soon as $\mathcal{E}(t_k, \cdot)$ is strictly convex. Thus, $(\widetilde{\text{IP}})$ is solvable, but we find no counterpart to Thm. 3.2 concerning stability and energy inequalities. The problem is that we do not have a counterpart to the triangle inequality. To obtain useful bounds we need the dependence of $\Psi(z, v)$ on $z$ to be sufficiently mild.

In general we need the following smallness assumption:

$$
\begin{array}{l} \exists\, \psi_* > 0\ \exists\, \alpha > \psi_* \text{ such that } \mathcal{E}(t, \cdot) \text{ is } \alpha\text{-convex} \\ \text{and } \forall\, v, y_1, y_2 \in Y:\ |\Psi(y_1, v){-}\Psi(y_2, v)| \leq \psi_* d(y_1, y_2)\|v\|. \end{array} \tag{3.16}
$$

Then the incremental solutions as well as time-continuous solutions satisfy an a priori Lipschitz bound. From $\widetilde{(\text{IP})}$ and $\alpha$-convexity we obtain

$$
\forall\, y \in Y:\ \mathcal{E}(t_j, y_j) + \Psi(y_{j-1}, y_j{-}y_{j-1}) + \tfrac{\alpha}{2} d(y, y_j)^2 \leq \mathcal{E}(t_j, y) + \Psi(y_{j-1}, y{-}y_{j-1}).
$$

As shorthand we introduce $\delta_j = d(y_j, y_{j-1})$. Fixing $k$ we use the above estimate with $j = k$ and $y = y_{k-1}$ and add this to the estimate obtained with $j = k{-}1$ and $y = y_k$. This gives (a):

$$
\begin{aligned}
\alpha\delta_k^2 \ \leq_{(a)}\ & \mathcal{E}(t_k, y_{k-1}) - \mathcal{E}(t_k, y_k) + \mathcal{E}(t_{k-1}, y_k) - \mathcal{E}(t_{k-1}, y_{k-1}) \\
& + 0 - \Psi(y_{k-1}, y_k{-}y_{k-1}) + \Psi(y_{k-2}, y_k{-}y_{k-2}) - \Psi(y_{k-2}, y_{k-1}{-}y_{k-2}) \\
\leq_{(b)}\ & \textstyle\int_{t_{k-1}}^{t_k} \partial_r \mathcal{E}(r, y_{k-1}){-}\partial_r \mathcal{E}(r, y_k)\,\mathrm{d}r - \Psi(y_{k-1}, y_k{-}y_{k-1}) + \Psi(y_{k-2}, y_k{-}y_{k-1}) \\
\leq_{(c)}\ & C_3(t_k{-}t_{k-1})\delta_k + \psi_*\delta_{k-1}\delta_k.
\end{aligned}
$$

For (b) we use the fact that $\Psi(y_{k-2}, \cdot)$ satisfies the triangle inequality and (c) follows from (3.14) and (3.16). After dividing by $\alpha\delta_k$ we find the recurrence relation

$$
\delta_k \leq \tfrac{C_3}{\alpha}(t_k{-}t_{k-1}) + \tfrac{\psi_*}{\alpha}\delta_{k-1},
$$

which provides the desired a priori bound on $\delta_k$, since $\psi_*/\alpha < 1$.

In Example 3.5 the condition $\psi_*/\alpha < 1$ takes the from $\psi_* = \|h'\|_\infty = |a| < 1$ since $\alpha = 1$.

# 4 Convex energies

In this section we treat the case of general, uniformly convex energies $\mathcal{E}$ on a reflexive Banach space $Y$ (which in most cases will, in fact, be a Hilbert space). Since in this section the distinction between the elastic variable $\varphi \in \mathcal{F}$ and the dissipative variable $z \in \mathcal{Z}$ is not of importance, we will use exclusively the variable $y$ and write $\mathcal{D}(y_0, y_1)$ instead of $\mathcal{D}(z_0, z_1)$. The point is, of course, that $\mathcal{D}(y_0, y_1) = 0$ does not imply $y_0 = y_1$. Similarly, $\Psi(y, v) = 0$ will not imply $v = 0$.

In most parts we will follow [MT04] and use the classical convexity with respect to the linear Banach space structure, since the general theory of geodesic convexity is not developed enough. However, in Section 4.3 we will address the question of more general concepts of convexity. By strict convexity on a Banach space $Y$ the energetic formulation (S) & (E) is equivalent to the variational inequality (VI) or to the subdifferential formulation (SF):

$$\begin{aligned}
&\text{(VI)} \qquad \forall\, v \in Y:\ \langle \mathrm{D}\mathcal{E}(t,y(t)), y-\dot y(t)\rangle + \Psi(y(t),v) - \Psi(y(t),\dot y(t)) \geq 0,\\
&\text{(SF)} \qquad 0 \in \partial_v \Psi(y(t),\dot y(t)) + \mathrm{D}\mathcal{E}(t,y(t))\ \subset Y^*.
\end{aligned}$$

In fact, in (SF) it suffices to use the subdifferential $\partial\mathcal{E}(t,y(t))$ in the case of a nondifferentiable, convex energy.

However, in this section we will always assume smoothness, namely

$$\begin{gathered}
\mathcal{E} \in \mathrm{C}^3([0,T]{\times}Y,\mathbb{R}) \text{ satisfies (A1) and}\\
\text{for each } E_0>0 \text{ there exist constants } C_{ty}, C_{yyy}, C_{tyy}>0, \text{ such that}\\
\mathcal{E}(0,y)\leq E_0 \quad \Longrightarrow \quad \left\{ \begin{array}{c} \|\mathrm{D}\mathcal{E}(t,y)\|, \|\mathrm{D}^2\mathcal{E}(t,y)\|, \|\partial_t \mathrm{D}\mathcal{E}(t,y)\| \leq C_{ty}\\ \|\mathrm{D}^3\mathcal{E}(t,y)\| \leq C_{yyy}, \|\partial_t\mathrm{D}^2\mathcal{E}(t,y)\| \leq C_{tyy}. \end{array}\right.
\end{gathered} \tag{C1}$$

In the quadratic case of Section 2 both constants $C_{tyy}$ and $C_{yyy}$ are equal to $0$.

The second major assumption is of course uniform convexity, i.e.,

$$\exists\, \alpha>0\ \forall\, v,y\in Y:\ \langle \mathrm{D}^2\mathcal{E}(t,y)v,v\rangle \geq \alpha\|v\|^2. \tag{C2}$$

Because of $\mathrm{C}^2$-regularity this condition is equivalent to $\alpha$-convexity as defined in Section 3.5, namely

$$\mathcal{E}(t,(1{-}\theta)y_0{+}\theta y_1) \leq (1{-}\theta)\mathcal{E}(t,y_0) + \theta\mathcal{E}(t,y_1) - \tfrac{\alpha}{2}(1{-}\theta)\theta\|y_1{-}y_0\|^2.$$

In Section 4.1 we will treat the case of a translation invariant dissipation metric, i.e., $\mathrm{D}_y\Psi \equiv 0$, and in Section 4.3 we will consider partial results in cases where $\Psi$ really depends on $y$. In any case $\Psi: Y{\times}Y \to [0,\infty]$ will always be such that $\Psi(z,\cdot): Y\to[0,\infty]$ is 1-homogeneous, convex and lower semi-continuous. The value $+\infty$ is allowed as well as $\Psi(z,v)=0$ for $v\neq 0$. Recall that typical applications in continuum mechanics relate to spaces $Y=\mathrm{L}^2(\Omega)$ and $\Psi(y,v)=\|v\|_{\mathrm{L}^1(\Omega)}=\int_\Omega |v(x)|\,\mathrm{d}x$. Thus, we do not need any of the assumptions $\Psi(z,v)\geq \rho_1\|v\|$ or $\Psi(y,v)\leq\rho_2\|v\|$ with $\rho_j>0$ (which would mean equivalently for $C_*(y)=\partial\Psi(z,0)$ the inclusions $B^*_{\rho_1}(0)\subset C_*(y)$ and $C_*(z)\subset B^*_{\rho_2}(0)$, respectively).

## 4.1 Translation invariant dissipation distances

Like in [MT04] we consider the case $\mathrm{D}_y\Psi\equiv 0$, i.e., $\Psi(y,v)=\Psi(v)$:

$$\Psi: Y\to[0,\infty] \quad \text{is convex, 1-homogeneous and lower semi-continuous.} \tag{C3}$$

As a consequence the dissipation distance $\mathcal{D}: Y{\times}Y\to[0,\infty]$ has a simple form and the closed, convex sets $C_*(y)=\partial\Psi(y,0)$ are constant:

$$\mathcal{D}(y_0,y_1)=\Psi(y_1{-}y_0) \qquad \text{and} \qquad C_*=\partial\Psi(0)\subset Y^*.$$

By smoothness and convexity of $\mathcal{E}$ the stable sets can be characterized as

$$\mathcal{S}(t) = \{\, y \in Y \mid -\mathrm{D}\mathcal{E}(t,y) \in C_* \,\}.$$

Note that in general these sets are not convex unless $\mathcal{E}(t,\cdot)$ is quadratic, see Example 5.12. In particular, $\mathcal{S}(t)$ will in general be strongly closed but not weakly closed.

The major advantage in assuming that $\Psi$ is translation invariant is that in the variational inequality (VI) we can compare two solutions. For instance, if $y_1$ and $y_2$ are solutions of

$$\text{(VI)} \qquad \forall\, v \in Y:\ \langle \mathrm{D}\mathcal{E}(t,y(t)), v - \dot y(t)\rangle + \Psi(v) - \Psi(\dot y(t)) \geq 0, \tag{4.1}$$

then we may test with $v_j = \dot y_{3-j}(t)$ and add the two inequalities to obtain

$$\langle \mathrm{D}\mathcal{E}(t,y_1(t)) - \mathrm{D}\mathcal{E}(t,y_2(t)), \dot y_1(t) - \dot y_2(t)\rangle \leq 0, \tag{4.2}$$

which generalizes the basic monotonicity estimate employed in Section 2.2 to show that in the quadratic case the hysteresis operator defines a contraction semigroup, see (2.16).

We will first use this estimate to prove an existence result and then show that similar methods allow us to establish existence via proving strong convergence of the solutions obtained by the time-incremental method.

**Proposition 4.1** *If the assumption* (C1) *to* (C3) *hold, then the variational inequality* (4.1) *has for each $y_0 \in \mathcal{S}(0)$ at most one solution with $y(0) = y_0$.*

**Proof:** Let $y_1$ and $y_2$ be two solutions. By Theorem 3.4 we know that each solution must be Lipschitz continuous, hence $\dot y_j(t)$ exist a.e. in $[0,T]$ and satisfies $\|\dot y_j(t)\| \leq K$. With $\mathrm{D}\mathcal{E}_j = \mathrm{D}\mathcal{E}(t,y_j(t))$ define

$$\gamma(t) = \langle \mathrm{D}\mathcal{E}_1 - \mathrm{D}\mathcal{E}_2, y_1(t) - y_2(t)\rangle \geq \alpha \|y_1(t) - y_2(t)\|^2,$$

where we used $\alpha$-convexity of $\mathcal{E}$. Moreover, we have

$$\dot\gamma(t) = \langle \partial_t \mathrm{D}\mathcal{E}(t,y_1) - \partial_t \mathrm{D}\mathcal{E}(t,y_2), y_1 - y_2\rangle + \langle r_1^*, \dot y_1\rangle + \langle r_2^*, \dot z_2\rangle$$

where $r_j^* = 2(\mathrm{D}\mathcal{E}_j - \mathrm{D}\mathcal{E}_{3-j}) + b_j^*$ with $b_j^* = \mathrm{D}^2\mathcal{E}(t,y_j)[y_j - y_{3-j}] - \mathrm{D}\mathcal{E}_j + \mathrm{D}\mathcal{E}_{3-j}$. Using (C1) we find $\|b_j^*\|_* \leq C_{yyy}\|y_1 - y_2\|^2$ and $\|\partial_t \mathrm{D}\mathcal{E}(t,y_1) - \partial_t \mathrm{D}\mathcal{E}(t,y_2)\| \leq C_{tyy}\|y_1 - y_2\|$ which leads to

$$\begin{aligned} \dot\gamma &\leq C_{tyy}\|y_1 - y_2\|^2 + C_{yyy}\|y_1 - y_2\|^2(\mathrm{Lip}(y_1) + \mathrm{Lip}(y_2)) + 2\langle \mathrm{D}\mathcal{E}_1 - \mathrm{D}\mathcal{E}_2, \dot y_1 - \dot y_2\rangle \\ &\leq (C_{tyy} + C_{yyy}2K)\|y_1 - y_2\|^2 + 0 \ \leq\ (C_{tyy} + C_{yyy}2K)\gamma/\alpha, \end{aligned}$$

where we have used (4.2) in the second estimate. Thus, Gronwall's estimate implies the desired result if $\gamma(0) = 0$. ∎

Another way to establish uniqueness without the above strong smoothness condition (C1) and the $\alpha$-convexity in (C2) is possible, if the stable sets $\mathcal{S}(t)$ are convex, see [MT04, Sect. 5] for sufficient conditions.

**Theorem 4.2** *If $\mathcal{E}$ has the form $\mathcal{E}(t,y) = \mathcal{U}(y) - \langle \ell(t), y\rangle$ where $\mathcal{U}$ is strictly convex and if the stable sets $\mathcal{S}(t)$ are convex for all $t \in [0,T]$, then for each initial condition $y_0 \in \mathcal{S}(0)$ there is at most one solution to (S) & (E).*

**Proof:** Let $y_j : [0,T] \to Y$ be two solutions with $y_j(0) = y_0$ and $\widetilde{y}(t) = \frac{1}{2}(y_1(t)+y_2(t))$. By the convexity of the stable sets we know that $\widetilde{y}(t) \in \mathcal{S}(t)$ and thus $\widetilde{y}$ satisfies (S).

Now assume $y_2(t) \neq y_1(t)$ for some $t > 0$. Using strict convexity of $\mathcal{U}$, the energy balance (E) of (2.9) and the linearity of $\partial_t\mathcal{E}$, we obtain

$$\begin{aligned}&\mathcal{E}(t,\widetilde{y}(t))+\mathrm{Diss}_\Psi(\widetilde{y};[0,t]) < \tfrac{1}{2}[\mathcal{E}(t,y_2(t))+\mathcal{E}(t,y_1(t)) + \mathrm{Diss}_\Psi(y_2;[0,t])+\mathrm{Diss}_\Psi(y_1;[0,t])]\\ &= \tfrac{1}{2}[\mathcal{E}(0,y_2(0))+\mathcal{E}(0,y_1(0))] - \int_0^t \tfrac{1}{2}[\langle\dot{\ell},y_2\rangle + \langle\dot{\ell},y_1\rangle]\,\mathrm{d}s \;=\; \mathcal{E}(0,y_0) + \int_0^t \partial_t\mathcal{E}(s,\widetilde{y}(s))\,\mathrm{d}s.\end{aligned}$$

However, as in Proposition 5.7 it can be shown that (S) implies the opposite energy inequality, i.e., $\mathcal{E}(t,\widetilde{y}(t)) + \mathrm{Diss}_\Psi(\widetilde{y};[0,t]) \geq \mathcal{E}(0,y_0) + \int_0^t \partial_t\mathcal{E}(s,\widetilde{y}(s))\,\mathrm{d}s$. This produces a contradiction and we conclude $y_1 \equiv y_2$. ■

The following result was first established in [MT04, Sect. 7]. For the readers convenience we repeat the main steps, since the proof of Prop. 7.2 contains several wrong signs. The proof is an adaptation of the strong convergence result in [HR95, HR99] for the quadratic case.

**Theorem 4.3** *The assumption* (C1) *to* (C3) *hold. Then, the variational inequality* (4.1) *has for each $y_0 \in \mathcal{S}(0)$ a unique solution $y \in \mathrm{C}^{\mathrm{Lip}}([0,T],Y)$ with $y(0) = y_0$ depends Lipschitz continuously on the initial value $y_0 \in \mathcal{S}(0)$.*

*Moreover, the solutions $Y^\Pi : [0,T] \to Y$ (cf.* (3.9)*) of the incremental problem (IP) in* (3.4) *associated with a partition $\Pi$ are unique and converge strongly to the unique solution such that*

$$\|Y^\Pi - y\|_{\mathrm{L}^\infty([0,T],Y)} \leq C\sqrt{f(\Pi)},$$

*where $f(\Pi) = \min\{\, t_i - t_{i-1} \mid i{=}1,\ldots,N\,\}$ is the fineness of the partition and where $C$ is independent of the solution and of the partition.*

The uniqueness part and the Lipschitz continuity in the initial condition was established already in Proposition 4.1. For the existence and the strong convergence the following lemma is crucial.

**Proposition 4.4** *Let $\mathcal{E}^1$ and $\mathcal{E}^2$ satisfy the assumptions* (C1) *to* (C3) *and let $y_0 \in \mathcal{S}(0)$ be given. Set*

$$\rho = \sup\{\, \|\mathrm{D}\mathcal{E}^1(t,y) - \mathrm{D}\mathcal{E}^2(t,y)\|_* \mid \mathcal{E}(t,y) \leq (\mathcal{E}(0,y_0)+c_E^{(0)})\mathrm{e}^{c_E^{(1)}t} - c_E^{(0)}\,\}$$

*Then, there exists a constant $C_0 > 0$ such that for any partition $\Pi$ the associated solution $Y_1^\Pi$ and $Y_2^\Pi$ satisfy*

$$\|Y_1^\Pi - Y_2^\Pi\|_{\mathrm{L}^\infty([0,T],Y)} \leq C_0\sqrt{\rho}.$$

**Proof:** We introduce the notation $\sigma^j(t,y) = \mathrm{D}\mathcal{E}^j(t,y)$, $e_k = y_k^1 - y_k^2$ and the difference operator $\tau_k\zeta = \zeta_k - \zeta_{k-1}$ where $\zeta$ stands for $t, y^j, \sigma^j(t,y_k^l)$ or $e$.

The incremental solutions $(y_k^j)_{k=0,\ldots,N}$ are defined via the variational inequality

$$\forall\, v \in Y:\; \langle \sigma^j(t_k,y_k^j), v - \tau_k z^j\rangle + \Psi(v) - \Psi(\tau_k y^j) \geq 0.$$

Choosing $v = \tau_k y^{3-j}$ and adding the equations for $j = 1$ and $2$ gives the discrete counterpart to (4.2), namely

$$\langle \sigma^1(t_k, z_k^1) - \sigma^2(t_k, z_k^2), \tau_k e\rangle \le 0. \tag{4.3}$$

As in the uniqueness proof we introduce a energetic quantity $\gamma_k$ which controls the error $e_k = y_k^1 - y_k^2$ because of $\alpha$-convexity:

$$\gamma_k = \langle \sigma^1(t_k, y_k^1) - \sigma^1(t_k, y_k^2), e_k\rangle = \langle \mathrm{D}\mathcal{E}^1(t_k, y_k^1) - \mathrm{D}\mathcal{E}^1(t_k, y_k^2), y_k^1 - y_k^2\rangle \ge \alpha\|y_k^1 - y_k^2\|^2 = \alpha\|e_k\|^2.$$

The increment $\tau_k\gamma = \gamma_k - \gamma_{k-1}$ can be estimated via (4.3) as follows

$$\begin{aligned}\tau_k\gamma &= \langle \sigma^1(t_k, y_k^1) - \sigma^1(t_k, y_k^2), \tau_k e\rangle + \langle \tau_k(\sigma^1(t_k, y_k^1) - \sigma^1(t_k, y_k^2)), e_{k-1}\rangle \\ &= 2\langle \sigma^1(t_k, y_k^1) - \sigma^2(t_k, y_k^2), \tau_k e\rangle + \beta_k \ \le \ \beta_k\end{aligned}$$

where $\beta_k$ takes the form

$$\begin{aligned}\beta_k &= \langle A_k e_k - A_{k-1} e_{k-1}, e_{k-1}\rangle - \langle A_k e_k, \tau_k e\rangle - 2\langle \sigma^1(t_k, y_k^2) - \sigma^2(t_k, y_k^2), \tau_k e\rangle \\ &= -\langle A_k \tau_k e, \tau_k e\rangle + \langle (A_k - A_{k-1}) e_{k-1}, e_{k-1}\rangle - 2\langle \sigma^1(t_k, y_k^2) - \sigma^2(t_k, y_k^2), \tau_k e\rangle.\end{aligned}$$

The symmetric operators $A_k \in \mathrm{Lin}(Y, Y^*)$ are defined via $A_k = \int_0^1 \mathrm{D}^2\mathcal{E}^1(t_k, y_k^2 + \theta e_k)\,\mathrm{d}\theta$ and satisfy $A_k e_k = \sigma^1(t_k, y_k^1) - \sigma^1(t_k, y_k^2)$. By convexity and three–times differentiability we obtain

$$\langle A_k y, y\rangle \ge 0 \text{ and } \|A_k - A_{k-1}\| \le C_{tyy}\tau_k t + C_{yyy}\left(\|\tau_k y^1\| + \|\tau_k y^2\|\right) \text{ where } \tau_k = t_k - t_{k-1}.$$

Together with $\|\tau_k e\| \le \|\tau_k z^1\| + \|\tau_k z^2\| \le 2C_K\tau_k t$ (see Theorem 3.4) we find

$$\begin{aligned}\tau_k\gamma &\le 0 + (C_{tyy} + C_{yyy}2C_K)\tau_k t\,\|e_{k-1}\|^2 + \|\sigma^1(t_k, y_k^2) - \sigma^2(t_k, y_k^2)\|2C_K\tau_k t \\ &\le \Big(\tfrac{C_{tyy} + 2C_K C_{yyy}}{\alpha}\,\gamma_{k-1} + \rho 2C_K\Big)\tau_k t = (C_1\gamma_{k-1} + C_2\rho)(t_k - t_{k-1}).\end{aligned}$$

By induction over $k$ we find

$$\begin{aligned}\gamma_k &\le C_2\rho\textstyle\sum_{n=1}^k (t_n - t_{n-1})\prod_{j=n+1}^k [1 + C_1(t_j - t_{j-1})] \\ &\le C_2\rho\textstyle\sum_{n=1}^k (t_n - t_{n-1})\mathrm{e}^{C_1(t_k - t_n)} \ \le \ C_2\rho\int_0^{t_k} \mathrm{e}^{C_1(t_k - s)}\,\mathrm{d}s \ = \ \frac{C_2\rho}{C_1}\big(\mathrm{e}^{C_1 t_k} - 1\big).\end{aligned}$$

Together with $\|y_k^1 - y_k^2\|^2 \le \frac{1}{\alpha}\gamma_k$ this is the desired result $\|y_k^1 - y_k^2\|^2 \le \rho C_2 \mathrm{e}^{C_1 T}/(\alpha C_2)$. ■

**Proof of Theorem 4.3:** By convexity an the a priori assumptions we know that for all partitions the solutions of (IP) exist and lie in the sublevel $\mathcal{E}(t, y) \le (\mathcal{E}(0, y_0) + c_E^{(0)})\mathrm{e}^{c_E^{(1)} t} - c_E^{(0)}$.

We start with an arbitrary partition $\Pi$ of $[0, T]$ and define the a sequence of partitions $\Pi_m$ by by $\Pi_1 = \Pi$ and by successive dividing each subinterval into two equal intervals of half the length, in particular $f(\Pi_m) = 2^{1-m} f(\Pi)$. Denote by $Y^m : [0, T] \to Y$ the solution associated with the partition $\Pi_m$. For comparing $Y^M$ and $Y^{m+1}$ we want to apply Proposition 4.4. For this we define $\mathcal{E}^1$ and $\mathcal{E}^2$ as follows. For $t_k \in \Pi_{m+1}$ define $\widehat{t_k} \in \Pi_m$ via $\widehat{t_k} = \max\{\, s_j \in \Pi_m \mid s_j \le t_k \,\}$ and let

$$\mathcal{E}^1(t_k, y) = \mathcal{E}(\widehat{t_k}, y) \text{ and } \mathcal{E}^2(t_k, y) = \mathcal{E}(t_k, y) \text{ for } t_k \in \Pi_{m+1}.$$

For $t \notin \Pi_{m+1}$ we may define $\mathcal{E}^j$ by piecewise linear interpolation. The construction was done such that the incremental solution for $\mathcal{E}^2$ gives exactly $Y^{m+1}$, whereas $Y^m$ is exactly the incremental solution obtained with $\mathcal{E}^1$ on the partition $\Pi_{m+1}$, since $\widehat{t}_{2j} = \widehat{t}_{2j+1}$ leads to the fact that the unique incremental solution for $\mathcal{E}^1$ does only move on every second step.

Now $|t_k - \widehat{t}_k| \leq f(\Pi_{m+1}) = 2^{-m} f(\pi)$ implies $\|\mathrm{D}\mathcal{E}^1(t,y) - \mathrm{D}\mathcal{E}^2(t,y)\| \leq C_{ty} 2^{-m} f(\Pi)$ on the relevant sublevel. Thus, we conclude

$$\|Y^{m+1} - Y^m\|_{\mathrm{L}^\infty([0,T],Y)} \leq C_0 (C_{ty} 2^{-m} f(\Pi))^{1/2} = C_* \sqrt{f(\Pi)} 2^{-m/2},$$

and $(Y^m)_{m\in\mathbb{N}}$ form a Cauchy sequence in $\mathrm{L}^\infty([0,T],Y)$ with a limit $y : [0,T] \to Y$. Note that the total distance between $y$ and $Y^1 = Y^\Pi$ is less than or equal to $\sum_1^\infty C_* \sqrt{f(\Pi)} 2^{-m/2} \leq 3C_* \sqrt{f(\Pi)}$ as desired.

It remains to show that $y$ is a solution of (S) & (E) which is equivalent to (4.1). Using the a priori Lipschitz estimate of Theorem 3.4 shows that all $Y^m$ are uniformly Lipschitz when restricted to $\Pi_m$. Hence, $y$ satisfies $\|y(t) - y(s)\| \leq C_{ty}|t-s|/\alpha$. Moreover, using the stability of $Y^m$ at the points in $\Pi_m$, the strong convergence and the strong closedness of $\mathcal{S}_{[0,T]} = \{\, (t,y) | y \in \mathcal{S}(t) \,\}$ we easily conclude $y(t) \in \mathcal{S}(t)$ for all $t \in [0,T]$. Finally we are able to pass to the limit in the discrete energy estimates (ii) obtained in Theorem 3.2 and find that $y$ also satisfies (E). This concludes the proof of Theorem 4.3. ∎

## 4.2 Quasi-variational inequalities

In [BKS04] the evolution quasi-variational inequality

$$\begin{aligned}
&\text{(i)} && \ell(t) - Ay(t) \in C_*(g(t,\ell(t),y(t))) \quad \text{for all } t \in [0,T], \\
&\text{(ii)} && \ell(0) - Ay(0) = \sigma_0 \in C_*(g(0,\ell(0),y(0))) \\
&\text{(iii)} && \langle \ell(t) - Ay(t) - \widehat{\sigma}, \dot{y}(t) \rangle \geq 0 \text{ for } \widehat{\sigma} \in C_*(g(t,\ell(t),y(t))) \quad \text{for a.a. } t \in [0,T].
\end{aligned} \tag{4.4}$$

with $Y$ being a Hilbert space and $\langle Ay_1, y_2 \rangle = \langle y_1 | y_2 \rangle$ is considered. Here $g : [0,T] \times Y^* \times Y \to \mathcal{R}$ is a continuous map and $\mathcal{R}$ is a closed, convex subset of a Banach space $R$. Moreover, for each $g \in \mathcal{R}$ the set $C_*(g)$ is closed and convex and satisfies $B^*_{\rho_1}(0) \subset C_*(g) \subset B^*_{\rho_2}(0) \subset Y^*$. Under suitable conditions, which we explain below, an existence and uniqueness result is derived which corresponds to Theorem 4.3.

To compare the results in [BKS04] with the results presented so far, we translate it into our subdifferential framework. With $\mathcal{E}(t,y) = \frac{1}{2}\langle Ay, y \rangle - \langle \ell(t), y \rangle$ and $\Psi(g,\cdot) = \mathcal{L}I_{C_*(g)}$ eqn. (4.4) takes the form

$$0 \in \partial\Psi(g(t,\ell(t),y(t)), \dot{y}(t)) + \mathrm{D}\mathcal{E}(t,y(t)) \text{ a.e. on } [0,T], \quad y(0) = A^{-1}(\ell(0) - \sigma_0). \tag{4.5}$$

The above assumptions imply that the dissipation metric $\Psi(g,\cdot)$ satisfies the estimates

$$\forall\, r \in \mathcal{R}\ \forall\, v \in Y : \quad \rho_1 \|v\| \leq \Psi(r,v) \leq \rho_2 \|v\|. \tag{4.6}$$

The results in [BKS04] are formulated in terms of the Minkowski functional $\mathcal{M}_{C_*(r)} : Y^* \to [0,\infty)$ of the sets $C_*(r)$, namely

$$M(r,\sigma) = \mathcal{M}_{C_*(r)}(\sigma) = \inf\{\, s > 0 \mid \tfrac{1}{s}\sigma \in C_*(r) \,\}.$$

By the Legendre-Fenchel transform $\mathcal{L}$ we have, for $\mathbb{B}(r,\sigma)=\frac{1}{2}M(r,\sigma)^2$, the identity

$$\tfrac{1}{2}\Psi(r,v)^2=\mathcal{L}[\mathbb{B}(r,\cdot)](v)=\mathcal{L}[\tfrac{1}{2}M(r,\cdot)](v).$$

The main assumptions on $C_*(r)$ or $\Psi(r,\cdot)$ are now phrased in terms of $\mathbb{B}$:

$$\begin{array}{ll}
\text{(a)} & \mathbb{B}\in \mathrm{C}^1(\mathcal{R}{\times}Y^*;\mathbb{R}) \\
 & \text{with } J(r,\sigma)=\mathrm{D}_\sigma\mathbb{B}(r,\sigma)\in Y \text{ and } K(r,\sigma)=\mathrm{D}_r\mathbb{B}(r,\sigma)\in R^*; \\
\text{(b)} & \forall\, r\in\mathbb{R}\;\forall\,\sigma\in C_*(r):\;\|K(r,\sigma)\|_{R^*}\leq K_0; \\
\text{(c)} & \forall\, r_1,r_2\in\mathcal{R}\;\forall\,\sigma_1\in C_*(r_1)\;\forall\,\sigma_2\in C_*(r_2): \\
 & \|J(r_1,\sigma_1){-}J(r_2,\sigma_2)\|_Y\leq C_J\|(r_1,\sigma_1){-}(r_2,\sigma_2)\|_{R\times Y^*} \\
 & \|K(r_1,\sigma_1){-}K(r_2,\sigma_2)\|_{R^*}\leq C_K\|(r_1,\sigma_1){-}(r_2,\sigma_2)\|_{R\times Y^*}.
\end{array} \tag{4.7}$$

With these assumptions for the linear, time-dependent problem

$$0\in\partial\Psi(r(t),\dot y(t))+Ay(t)-\ell(t)\text{ a.a. on }[0,T],\quad y(0)=A^{-1}(\ell(0)-\sigma_0), \tag{4.8}$$

the following existence result and Lipschitz estimate are derived.

**Proposition 4.5** *If the assumptions* (4.6) *and* (4.7) *hold, then, for each* $\ell\in \mathrm{W}^{1,1}([0,T],Y^*)$, $r\in \mathrm{W}^{1,1}([0,T],\mathcal{R})$ *and* $\sigma_0\in\mathcal{C}_*(r(0))$ *eqn.* (4.8) *has a unique solution. Moreover, if* $y_1,y_2$ *are solutions of* (4.8) *associated with* $(r_1,\ell_1)$ *and* $(r_2,\ell_2)$*, respectively, then for a.a.* $t\in[0,T]$ *we have the estimate*

$$\begin{aligned}\tfrac{1}{\rho_2}\|\dot y_1(t){-}\dot y_2(t)\|+\tfrac{\mathrm{d}}{\mathrm{d}t}|\mathbb{B}_1(t){-}\mathbb{B}_2(t)|\;&\leq\;\tfrac{1}{\rho_1}\|\dot\ell_1(t){-}\dot\ell_2(t)\|_*+K_0\|\dot r_1(t){-}\dot r_2(t)\|_R\\ +\big(2C_J\|\dot\ell_1(t)\|_*{+}(C_K{+}\rho_2C_JK_0)\|\dot r_1(t)\|_R\big)&\big(\|r_1(t){-}r_2(t)\|_R+\|\ell_1(t){-}\ell_2(t)\|_*\big),\end{aligned} \tag{4.9}$$

*where* $\mathbb{B}_j(t)=\mathbb{B}(r_j(t),\ell_j(t){-}Ay_j(t))$.

The crucial new idea here is the introduction of the new quantity $\mathbb{B}$ which takes values in $[0,\frac{1}{2}]$, since $M(r,\sigma)\leq 1$ on $C_*(r)$. In fact, $\mathbb{B}$ measures the distance to the yield surface $\partial C_*(r(t))$, namely $\sigma\in\partial C_*(r)\;\Leftrightarrow\;\mathbb{B}(r,\sigma)=\frac{1}{2}$. Moreover, the identity

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\mathbb{B}(r(t),\ell(t){-}Ay(t))=\langle K(r(t),\Sigma(t)),\dot r(t)\rangle+\langle J(r(t),\Sigma(t)),\dot\ell(t){-}A\dot y(t)\rangle,$$

where $\Sigma(t)=\ell(t){-}Ay(t)$, together with the flow law $\dot y(t)=\lambda J(r(t),\Sigma(t))$ (where $\lambda\geq 0$ for $\mathbb{B}=\frac{1}{2}$ and $\lambda=0$ else) allows us to derive (4.9) in a direct manner.

Finally the original nonlinear problem is solved by a contraction argument. For this, the mapping $g:[0,T]{\times}Y^*{\times}Y\to\mathcal{R}$ needs to be suitable:

$$\begin{array}{ll}
\text{(a)} & g\in \mathrm{C}^1([0,T]{\times}Y^*{\times}Y,\mathcal{R}); \\
\text{(b)} & \forall\,(t,\ell,y):\;\|\mathrm{D}_\ell g(t,\ell,y)\|_{Y^*\to R}\leq C_\ell,\;\;\|\mathrm{D}_y g(t,\ell,y)\|_{Y\to R}\leq C_y; \\
\text{(c)} & \partial_t g(t,\cdot),\;\mathrm{D}_\ell g(t,\cdot),\;\mathrm{D}_y g(t,\cdot)\text{ are uniformly Lipschitz continuous.}
\end{array} \tag{4.10}$$

**Theorem 4.6** *If assumptions* (4.6)*,* (4.7) *and* (4.10) *hold and*

$$\rho_2 K_0 C_y < 1, \tag{4.11}$$

*then for each* $\ell \in \mathrm{W}^{1,1}([0,T], Y^*)$ *and each* $\sigma_0 \in Y^*$ *with* $\sigma_0 \in C_*(g(0,\ell(0), A^{-1}(\ell(0){-}\sigma_0)))$ *eqn.* (4.5) *(or, equivalently,* (4.4)*) has a unique solution* $y \in \mathrm{W}^{1,1}([0,T], Y)$*. Moreover, the solutions depend Lipschitz continuously on the data in the following way. For each* $\rho_3 > 0$ *there exists a constant* $C_3 > 0$ *such that for every* $\ell_1$ *and* $\ell_2$ *with* $\|\ell_j\|_{\mathrm{W}^{1,1}} \leq \rho_3$ *and every* $\sigma_0^j$ *with* $\sigma_0^j \in C_*(g(0,\ell_j(0), A^{-1}(\ell_0{-}\sigma_0^j)))$ *the unique solutions* $y_1$ *and* $y_2$ *satisfy*

$$\|y_1{-}y_2\|_{\mathrm{W}^{1,1}([0,T],Y)} \leq C_3\big(\|\sigma_0^1{-}\sigma_0^2\|_* + \|\ell_1{-}\ell_2\|_{\mathrm{W}^{1,1}([0,T],Y^*)}\big).$$

In [BKS04, Sect. 8] one finds a counterexample with $Y = \mathbb{R}^2$ showing that dropping the Lipschitz continuity (4.7)(c) for $J$ leads to nonuniqueness. Similar smallness conditions for the Lipschitz constant were obtained in [KM98].

In [RS04], the quasi-variational sweeping process

$$-\dot y \in \partial I_{C_*(t,y)}(y) \subset H \quad \text{for } y \in \mathrm{W}^{1,1}([0,T], H) \tag{4.12}$$

is considered in an ordered Hilbert space $(H, \leq)$. The assumptions of the time and state dependent set $C_*$ are the following:

(i) There exist functions $\mathbb{Y}_*(t,y)$ and $\mathbb{Y}^*(t,y)$ such that

$$C_*(t,y) = [\mathbb{Y}_*(t,y), \mathbb{Y}^*(t,y)] = \{\, \widetilde y \in H \mid \mathbb{Y}_*(t,y) \leq \widetilde y \leq \mathbb{Y}^*(t,y)\,\}.$$

(ii) For each $t \in [0,T]$ the functions $-\mathbb{Y}_*(t,\cdot) : H \to H$ and $-\mathbb{Y}^*(t,\cdot) : H \to H$ are
 (ii.1) maximal (for graph inclusion within monotone operators),
 (ii.2) T-monotone, i.e., $(-\mathbb{Y}_*^*(t,y_1){-}\mathbb{Y}_*^*(t,y_2) | (y_1{-}y_2)^+) \geq 0$ for all $y_1, y_2$,
 (ii.3) non-decreasing, i.e., $y_1 \leq y_2 \implies -\mathbb{Y}_*^*(t,y_1) \leq \mathbb{Y}_*^*(t,y_2)$.

Here $\mathbb{Y}_*^*$ means either $\mathbb{Y}_*$ or $\mathbb{Y}^*$.

In this general setting uniqueness of solutions can be established. An existence result is obtained for $H = \mathrm{L}^2(\Omega)$ with the usual ordering of functions and under additional assumptions on the functions $Y_*$ and $Y^*$.

## 4.3 General dissipation metrics

This subsection is speculative, in the sense that we propose a certain philosophy which is under investigation in [MR04b].

The general subdifferential formulation has the form

$$0 \in \partial_v \Psi(y, \dot y) + \mathrm{D}\mathcal{E}(t,y) \ \subset Y^*, \tag{4.13}$$

where $\partial_v \Psi(y,\dot y)$ denotes the subdifferential of $\Psi$ with respect to the second variable $v = \dot y$ at fixed $y$. This formulation includes the theory in [MT04] where $\mathrm{D}_z\Psi \equiv 0$ as well as the theory in [BKS04] with $\mathrm{D}\mathcal{E}(t,y) = Ay{-}\ell(t)$ if we let $\mathcal{R} = R = Y$ and $g(t,\ell,y) = y$. In the above two subsections the results heavily rely on convexity assumptions of $\mathcal{E}$. However, existence and uniqueness results for such problems should not depend on the coordinate system we use to describe the problem. However, under coordinate transformations convexity properties are usually destroyed. Denote by $y = \Phi(\widehat{y})$ a smooth coordinate transformation from $\widehat{Y}$ into $Y$ ($\Phi \in \mathrm{C}^2(\widehat{Y}, Y)$). Then, the transformed energy $\widehat{\mathcal{E}}$ and the transformed dissipation metric $\widehat{\Psi}$ are

$$\widehat{\mathcal{E}}(t, \widehat{y}) = \mathcal{E}(t, \Phi(\widehat{y})) \quad \text{and} \quad \widehat{\Psi}(\widehat{y}, \widehat{v}) = \Psi(\Phi(\widehat{y}), \mathrm{D}\Phi(\widehat{y})\widehat{v}).$$

Using the adjoint operator $\mathrm{D}\Phi(\widehat{y})^* : Y^* \to \widehat{Y}^*$, the transformed derivatives read $\partial_{\widehat{v}}\widehat{\Psi}(\widehat{y}, \widehat{v}) = \mathrm{D}\Phi(\widehat{y})^*\partial\Psi(\Phi(\widehat{y}), \mathrm{D}\Phi(\widehat{y})\widehat{v})$ and $\mathrm{D}\widehat{\mathcal{E}}(t, \widehat{y}) = \mathrm{D}\Phi(\widehat{y})^*\mathrm{D}\mathcal{E}(t, \Phi(\widehat{y}))$, such that (4.13) is equivalent to the transformed equation $0 \in \partial\widehat{\Psi}(\widehat{y}, \dot{\widehat{y}}) + \mathrm{D}\widehat{\mathcal{E}}(t, \widehat{y}) \subset \widehat{Y}^*$.

For proving existence and uniqueness for (4.13) with a method like in [BKS04] we would need the smallness condition $\|\mathrm{D}_z\Psi(y, v)\| \leq \delta \ll 1$ for all $y \in Y$ and $v$ with $\|v\| \leq 1$ which implies (4.11) since by Legendre transform one finds $K(y, \sigma) = -\Psi(y, J(y, \sigma))\mathrm{D}_y\Psi(y, J(y, \sigma))$. However, this condition in not invariant under coordinate changes, since

$$\mathrm{D}_y\widehat{\Psi}(\widehat{y}, \widehat{v})[\widehat{w}] = \mathrm{D}_y\Psi(\Phi(\widehat{y}), \mathrm{D}\Phi(\widehat{y})\widehat{v})[\mathrm{D}\phi(\widehat{y})\widehat{w}] + \mathrm{D}_v\Psi(\Phi(\widehat{y}), \mathrm{D}\Phi(\widehat{y})\widehat{v})[\mathrm{D}^2\phi(\widehat{y})[\widehat{v}, \widehat{w}]]$$

needs the second derivative of $\Phi$ as well as the derivative of $\Psi$ with respect to $v$ (whose existence we have to assume henceforth). In [MR04b] we argue that the smallness of $\|\mathrm{D}_y\Psi(y, v)\|$ can be replaced by a one-sided convexity condition which takes the form

$$\exists\, \delta > 0\ \forall\, (t, y) \in \mathcal{S}_{[0,T]}\ \forall v \in Y : \langle \mathrm{D}^2\mathcal{E}(t, y)v, v\rangle + \mathrm{D}_y\Psi(y, v)[v] \geq \alpha\Psi(y, v)^2. \tag{4.14}$$

Note that both sides in the estimate are 2-homogeneous, but the left-hand side in the estimate is not quadratic, since $\mathrm{D}_y\Psi(y, \alpha v)[\alpha v] = \alpha|\alpha|\mathrm{D}_y\Psi(y, v)[v]$.

Nevertheless the new condition is nothing else than the local version of the geodesic convexity if we use $d = \mathcal{D}$. In fact, using the expansion $\mathcal{D}(y, y{+}v) = \Psi(y, v) + \frac{1}{2}\mathrm{D}_y\Psi(y, v)[v] + o(\|v\|^2)$, our smoothness assumptions on $\mathcal{E}$ and (4.14) we find

$$\mathcal{E}(t, y{+}sv) + \mathcal{D}(y, y{+}sv) \geq \mathcal{E}(t, y) + (s\mathrm{D}\mathcal{E}(t, y)[v] + \Psi(y, sv)) + \alpha s^2 + o(s^2)_{s\to 0}.$$

The left-hand side in the estimate of (4.14) is not invariant under coordinate changes. However, taking it together with the stability condition the additional term involving the second derivative of $\Phi$ has a positive sign, namely

$$\begin{aligned}
&\langle \mathrm{D}^2\widehat{\mathcal{E}}(t, \widehat{y})\widehat{v}, \widehat{v}\rangle + \mathrm{D}_y\widehat{\Psi}(\widehat{y}, \widehat{v})[\widehat{v}]\\
&= \langle \mathrm{D}^2\mathcal{E}(t, \Phi(\widehat{y}))\mathrm{D}\Phi(\widehat{y})\widehat{v}, \mathrm{D}\Phi(\widehat{y})\widehat{v}\rangle + \mathrm{D}_y\Psi(\Phi(\widehat{y}), \mathrm{D}\Phi(\widehat{y})\widehat{v})[\mathrm{D}\Phi(\widehat{y})\widehat{v}]\\
&\qquad + \tfrac{1}{2}\mathrm{D}\mathcal{E}(t, \Phi(\widehat{y}))[\mathrm{D}^2\Phi(\widehat{y})[\widehat{v}, \widehat{v}]] + \tfrac{1}{2}\mathrm{D}_v\Psi(\Phi(\widehat{y}))[\mathrm{D}^2\Phi(\widehat{y})[\widehat{v}, \widehat{v}]]\\
&\geq_{(\mathrm{S})} \alpha\, \Psi(\Phi(\widehat{y}), \mathrm{D}\Phi(\widehat{y})\widehat{v})^2 + 0 \ = \ \alpha\, \widehat{\Psi}^2(\widehat{y}, \widehat{v})^2,
\end{aligned}$$

where for (S) we have used $y = \Phi(\widehat{y}) \in \mathcal{S}(t)$ which implies $0 \in \partial_v\Psi(y, 0) + \mathrm{D}\mathcal{E}(t, y)$.

A second condition appearing already in [MT04, App. C] is the so-called ***structure condition*** on the dissipation metric $\Psi : Y{\times}Y \to [0, \infty]$:

$$\begin{aligned}
&\forall\, R > 0\ \exists\, C_R \geq 0\ \forall\, y_1, y_2 \text{ with } \|y_j\| \leq R :\\
&\qquad 0 \in \partial_v\Psi(y_j, v_j){+}\sigma_j \text{ for } j = 1, 2\\
&\qquad \Longrightarrow \quad \langle \sigma_1{-}\sigma_2, v_1{-}v_2\rangle \leq C_R\|y_1{-}y_2\|^2\big(\|v_1\|{+}\|v_2\|\big).
\end{aligned} \tag{4.15}$$

In contrast to the above condition, (4.15) is not invariant under coordinate changes.

**Example 4.7** *We take* $Y = \mathbb{R}^2$, $\Psi(y,v) = |v_1|+|v_2|$ *and* $y = \Phi(\widehat{y}) = (y_1+y_2^2, y_2)$. *Then,* $\Psi$ *satisfies the structure condition with* $C_R = 0$, *since* $0 \in \partial\Psi(v_j)+\sigma_j$ *is equivalent to* $v_j \in \partial(\mathcal{L}\Psi)(-\sigma_j)$ *and* $\mathcal{L}\Psi = I_{C_*}$ *is convex. However,* $\widehat{\Psi}$ *does not satisfy* (4.15). *To see this, use the explicit form*

$$\partial_{\widehat{v}}\widehat{\Psi}(\widehat{y},\widehat{v}) = \mathrm{Sign}(\langle\binom{1}{2\widehat{y}_2},\widehat{v}\rangle)(1, 2\widehat{y}_2) + \mathrm{Sign}(\widehat{v}_2)(0,1).$$

*For* $\widehat{y} = (0,1)$ *and* $\widehat{v} = (1,-1)$ *we find* $\partial\widehat{\Psi}(\widehat{y},\widehat{v}) = \{(-1,-3)\}$. *For* $\widehat{y}_* = (0,1-\varepsilon)$ *with* $0<\varepsilon<1/2$ *and* $\widehat{v}_* = (1,-2)$ *we find* $\partial\widehat{\Psi}(\widehat{y}_*,\widehat{v}_*) = \{(-1,-3+2\varepsilon)\}$. *Thus, we find* $\widehat{\sigma} = (1,3)$ *and* $\sigma_* = (1,3-2\varepsilon)$ *and arrive at*

$$\langle\widehat{\sigma}-\widehat{\sigma}_*, \widehat{v}-\widehat{v}_*\rangle = \langle(0,2\varepsilon),(0,1)\rangle = 2\varepsilon.$$

*Since* $\|\widehat{y}-\widehat{y}_*\| = |\varepsilon|$ *we see that* (4.15) *does not hold. The problem here is not the missing differentiability of* $\Psi$ *but rather the fact, that* $\Psi^2$ *is not uniformly convex.*

Based on assumption (4.14) and the natural smoothness assumptions on $\mathcal{E}$ and $\Psi$ (e.g., $\mathcal{E} \in \mathrm{C}^3$, $\Psi^2 \in \mathrm{C}^2$ and $\Psi^2$ uniformly convex) analogues of Theorem 4.3 and 4.6 are derived in [MR04b].

A related notion of ***convex composite systems*** was introduced in [Che03]. There the notion of monotone operators is generalized to systems which are monotone after a suitable diffeomorphism is applied. See Section 6.2 for some details of this theory.

## 4.4 Higher temporal regularity and improved convergence

In the general convex case we know from the a priori estimates in Section 3.5 that the solutions are Lipschitz continuous. Under suitable conditions this regularity can be improved somewhat, but the best we can hope for, is that the time derivative $\dot{y}$ lies in $\mathrm{BV}([0,T],Y)$.

**Example 4.8** *Let* $Y = \mathbb{R}$ *and* $\mathcal{E}(t,y) = \frac{1}{2}(y-\ell(t))^2$, $\Psi(y,\dot{y}) = |\dot{y}|$ *and* $y(0) = 0$. *For* $\ell(t) = a\sin(t)$ *with* $a > 1$ *define* $t_1 \in (0,\pi/2)$ *via* $\sin t_1 = 1/a$ *and* $t_2 \in (\pi/2, 3\pi/2)$ *via* $\sin t_2 = 1-2/a$. *Then the unique solution reads*

$$y(t) = \begin{cases} 0 & \text{for } t \in [0,t_1], \\ a\sin(t)-1 & \text{for } t \in [t_1,\pi/2] \text{ and } t-2\pi k \in [\pi+t_2, 5\pi/2], \\ a-1 & \text{for } t-2\pi k \in [\pi/2, t_2], \\ a\sin(t)+1 & \text{for } t-2\pi k \in [t_2, 3\pi/2], \\ -a+1 & \text{for } t-2\pi k \in [3\pi/2, \pi+t_2], \end{cases} \qquad \text{for } k \in \mathbb{N}_0.$$

*Hence, the derivative* $\dot{y}$ *has jumps whenever* $\Sigma(t) = \ell(t)-y(t)$ *hits the yields surface (i.e., the boundary of* $C_* = [-1,1]$*), namely at* $t_1$ *and* $t_2+\pi\mathbb{N}$. *Note that the derivative* $\dot{y}$ *is continuous when* $y$ *leaves the yield surface.*

Using this observation the following result was derived in [MT04, Thm. 7.8].

**Theorem 4.9** *If* $\mathcal{E} \in \mathrm{C}^3([0,T]\times Y, \mathbb{R})$ *and* $\Psi = \mathcal{L}I_{C_*}$ *where* $C_*$ *satisfies* $B^*_{\rho_1}(0) \subset C_* \subset B^*_{\rho_2}(0)$ *for* $\rho_2 \geq \rho_1 > 0$ *and the boundary* $\partial C_*$ *is of class* $\mathrm{C}^2$*, then any solution of the variational inequality* (4.1) *satisfies* $\dot y \in \mathrm{BV}([0,T],Y)$.

In [Kre99, Thm. 7.2] a more abstract approach is used to prove a similar regularity result. It is based on the local Lipschitz continuity results as given in Theorems 2.7 and 2.8.

**Theorem 4.10** *If the hysteresis operator* $\mathcal{H} : C_*\times \mathrm{W}^{1,1}([0,T],Y^*) \to \mathrm{W}^{1,1}([0,T],Y)$ *is Lipschitz continuous on every bounded subset on* $C_*\times \mathrm{W}^{1,1}([0,T],Y^*)$*, then for every* $\sigma_0 \in \mathcal{C}_*$ *and every* $\ell \in \mathrm{W}^{1,1}([0,T],Y^*)$ *with* $\dot\ell \in \mathrm{BV}([0,T],Y^*)$ *the solution* $y = \mathcal{H}(\sigma_0,\ell)$ *satisfies* $\dot y \in \mathrm{BV}([0,T],Y)$.

**Proof:** The idea of the proof is simple. For $h > 0$ consider the inputs $\ell_h$ with $\ell_h(t) = \ell(0)$ for $t \in [0,h]$ and $\ell(t) = \ell(t-h)$ for $t \in [h,T]$. Because of the rate independence the unique solution $\mathcal{H}(\sigma_0,\ell_h)$ is $y_h$ which is obtained from $y$ in the same way as $\ell_h$ from $\ell$. Since the functions $(\ell_h)_{h\in[0,T]}$ are bounded in $\mathrm{W}^{1,1}([0,T],Y^*)$, we obtain a Lipschitz constant $L$ such that

$$\begin{aligned}&\textstyle\int_h^T \|\dot y(t)-\dot y(t-h)\|\,\mathrm{d}t \leq \|y-y_h\|_{\mathrm{W}^{1,1}} \leq L\|\ell-\ell_h\|_{\mathrm{W}^{1,1}}\\ &\leq C\textstyle\int_0^T \|\ell(t)-\ell_h(t)\|_*\,\mathrm{d}t \leq Ch\|\dot\ell\|_{\mathrm{BV}([0,T],Y^*)}.\end{aligned}$$

This implies $\|\dot y\|_{\mathrm{BV}([0,T],Y)} \leq C\|\dot\ell\|_{\mathrm{BV}([0,T],Y^*)}$, see [Kre99] for details. ■

In numerical approaches to elastoplasticity or other hysteresis problems such higher temporal regularity can be used to improve the convergence rates of the incremental problem. We do this for the linear variational inequality (2.2), namely

$$\langle Ay(t)-\ell(t), v-\dot y(t)\rangle + \Psi(v) - \Psi(\dot y(t)) \geq 0 \text{ for a.a. } t \in [0,T], \tag{4.16}$$

under the same assumptions as in Section 2. Instead of the fully implicit Euler scheme (IP), which was used above, one considers the more general semi-implicit scheme as follows:

$$(\mathrm{IP})_\vartheta \qquad \begin{array}{c}\text{Find } y_k \in Y \text{ such that } \forall\, \widehat v \in Y:\\ \langle Ay_k^\vartheta-\ell(t_k^\vartheta), \widehat v-(y_k-y_{k-1})\rangle - \Psi(y_k-y_{k-1}) + \Psi(\widehat v) \geq 0,\end{array}$$

where $t_k^\vartheta = (1-\vartheta)t_{k-1}+\vartheta t_k$ and $y_k^\vartheta = (1-\vartheta)y_{k-1}+\vartheta y_k$. For $\vartheta = 1$ this is the old fully implicit scheme and for $\vartheta = 1/2$ it is the midpoint rule (also called Crank-Nicholson scheme).

Under the proved assumption of Lipschitz continuity (i.e., $\|\dot y\|_{\mathrm{L}^\infty} < \infty$), it is shown in [HR95], that the convergence of the discrete solution to the exact solutions behaves like

$$\|y^\Pi - y\|_{\mathrm{L}^\infty} \leq C(f(\Pi))^s$$

with $s = 1/2$ if $\vartheta \in [1/2, 1]$, see Theorem 4.3 for the case $\vartheta = 1$. Under the unproved assumption $y \in \mathrm{W}^{2,2}([0,T],Y)$ this convergence was improved to the order $s = 1$ in [AC00, Rem. 4.1]. For the Crank-Nicholson scheme, i.e., $\vartheta = 1/2$, [HR95] obtained $s = 1$ if $y \in \mathrm{W}^{3,1}([0,T],Y)$ and this was improved to $s = 2$ in [AC00, Rem. 4.3].

Following the method in [AC00] we provide a short proof of the convergence rate $s = 1$ under the assumption $\dot y \in \mathrm{BV}([0,T],Y)$, which is the best we can expect for any true hysteretic behavior. We also add the convergence estimate for the derivative as derived there.

**Theorem 4.11** *Let* $y \in \mathrm{C}^{\mathrm{Lip}}([0,T],Y)$ *be a solution of the variation inequality* (4.16) *with* $\dot y \in \mathrm{BV}([0,T],Y)$. *Then there exists a constant* $C>0$ *such that for each partition* $\Pi$ *the piecewise linear interpolant* $\widehat{y}^\Pi$ *of the solution of (IP)*$_{\vartheta=1}$ *satisfies*

$$\|y-\widehat{y}^\Pi\|_{\mathrm{L}^\infty([0,T],Y)} \le f(\Pi)\big[1+\tfrac{f(\Pi)}{2}\big]\mathrm{Var}_A(\dot y;[0,T]),$$
$$\|\dot y-\dot{\widehat{y}}^\Pi\|_{\mathrm{L}^2([0,T],Y)} \le 2\tfrac{f(\Pi)}{\sqrt{f_*(\Pi)}}\mathrm{Var}_A(\dot y;[0,T])$$

*where* $f_*(\Pi)=\min\{\, t_k - t_{k-1} \mid k=1,\dots,N\,\} \le f(\Pi)$.

**Proof:** Like in Section 2 above we use the energetic scalar product $\langle v|w\rangle = \langle Av,w\rangle$ and the associated norm $\|v\|_A$.

Define the error function $e$ via $e(t)=y(t)-\widehat{y}^\Pi(t)$ and set $I_k=[t_{k-1},t_k]$, $\tau_k=t_k-t_{k-1}$, $\ell_k=\ell(t_k)$, $e_k=e(t_k)$ and $\dot e_k=\dot y(t_k)-\frac{1}{\tau_k}(y_k-y_{k-1})$. Since $\widehat{y}^\Pi$ is piecewise linear we have

$$\begin{aligned}\|e_k-e_{k-1}-\tau_k\dot e_k\|_A &= \|\textstyle\int_{I_k}\dot y(s)-\dot y(t_k)\,\mathrm{d}s\|_A \;\le\; \int_{I_k}\|\dot y(s)-\dot y(t_k)\|_A\,\mathrm{d}s\\ &\le \textstyle\int_{I_k}\mathrm{Var}_A(\dot y;[s,t_k])\,\mathrm{d}s \;\le\; \tau_k\mathrm{Var}_A(\dot y;I_k).\end{aligned} \tag{4.17}$$

Since $y$ satisfies (4.16) and $y_k$ satisfies (IP)$_{\vartheta=1}$ we obtain by using the test functions $v=\frac{1}{\tau_k}(y_k-y_{k-1})$ and $\widehat{v}=\tau_k\dot y(t_k)$, respectively:

$$0 \le \langle Ay(t_k)-\ell_k, \tfrac{1}{\tau_k}(y_k-y_{k-1})-\dot y(t_k)\rangle + \Psi(\tfrac{1}{\tau_k}(y_k-y_{k-1}))-\Psi(\dot y(t_k)),$$
$$0 \le \langle Ay_k-\ell_k, \tau_k\dot y(t_k)-(y_k-y_{k-1}) + \Psi(\tau_k\dot y(t_k))-\Psi(y_k-y_{k-1}).$$

Dividing the second equation by $\tau_k$ and adding it to the first one, we find

$$0 \le \langle A(y(t_k)-y_k), \tfrac{1}{\tau_k}(y_k-y_{k-1})-\dot y(t_k)\rangle = \langle e_k|-\dot e_k\rangle = -\langle e_k|\dot e_k\rangle\,.$$

Now the discrete error can be estimated via

$$\begin{aligned}\|e_k\|_A^2 &= \|e_{k-1}\|_A^2 + 2\langle e_k-e_{k-1}|e_k\rangle - \|e_k-e_{k-1}\|_A^2\\ &\le \|e_{k-1}\|_A^2 + 2\langle e_k-e_{k-1}-\tau_k\dot e_k|e_k\rangle - \|e_k-e_{k-1}\|_A^2\\ &\le \|e_{k-1}\|_A^2 - \|e_k-e_{k-1}\|_A^2 + \tau_k\mathrm{Var}_A(\dot y;I_k)\|e_k\|_A.\end{aligned} \tag{4.18}$$

Let $m$ be such that $\max\{\,\|e_k\|_A \mid k=1,\dots,N\,\}$ is attained at $k=m$. Then, adding the above estimate from $k=1$ to $m$, using $e_0=0$ and neglecting the terms $\|e_k-e_{k-1}\|_A^2$ gives

$$\|e_m\|_A^2 \le \textstyle\sum_{k=1}^m \tau_k\mathrm{Var}_A(\dot y;I_k)\|e_m\|_A \le f(\Pi)\mathrm{Var}_A(\dot y;[0,T])\|e_m\|_A,$$

which implies $\|e_k\|_A \le f(\Pi)\mathrm{Var}_A(\dot y;[0,T])$ for all $k$. Moreover, again using the fact that $\widehat{y}^\Pi$ is piecewise linear gives for $t\in(t_{k-1},t_k)$

$$e(t)=\tfrac{t-t_{k-1}}{\tau_k}e_k+\tfrac{t_k-t}{\tau_k}e_{k-1}+\textstyle\int_{I_k}\alpha_k(t,s)(\dot y(s)-\dot y(t_k))\,\mathrm{d}s \text{ with } \alpha_k(t,s)=\begin{cases}\frac{t_k-t}{\tau_k} & \text{if } s<t,\\ -\frac{t-t_{k-1}}{\tau_k} & \text{if } s>t;\end{cases}$$

since $\int_{I_k}\alpha_k(t,s)\,\mathrm{d}s=0$. By $\int_{I_k}|\alpha_k(t,s)|\,\mathrm{d}s \le \tau_k/2$ this implies $\|e(t)-(\frac{t-t_{k-1}}{\tau_k}\,e_k+\frac{t_k-t}{\tau_k}\,e_{k-1})\|_A \le \frac{\tau_k}{2}\mathrm{Var}_A(\dot y;I_k)$ and the first estimate is established.

Summing (4.18) once again we find, with $e_0 = 0$ and $f_*(\Pi) = \min \tau_k$,

$$\begin{aligned} &f_*(\Pi) \sum_1^N \tfrac{1}{\tau_k} \|e_k - e_{k-1}\|_A^2 \leq \sum_1^N \|e_k - e_{k-1}\|_A^2 \\ &\leq \sum_1^N \tau_k \mathrm{Var}_A(\dot y; I_k) \|e_k\|_A \leq \Big( f(\Pi) \mathrm{Var}_A(\dot y; [0,T]) \Big)^2 . \end{aligned}$$

We define $Y_\Pi$ to be the piecewise linear interpolant of the exact solution $y$. Then we have $\dot Y_\Pi - \dot{\widehat y}^\Pi = \frac{1}{\tau_k}(e_k - e_{k-1})$ on $(t_{k-1}, t_k)$ and the left-hand side in the above estimate is $\|\dot Y_\Pi - \dot{\widehat y}^\Pi\|^2_{\mathrm{L}^2([0,T],Y)}$. Let $v = y - Y_\Pi$, then $v(t_k) = 0$ for all $k$ and $\mathrm{Var}_A(\dot v; I_k) \leq \mathrm{Var}_A(\dot y; I_k)$. Hence,

$$\int_{I_k} \|\dot v(s)\|_A^2 \,\mathrm{d}s = \big[ \langle v(s) | \dot v(s) \rangle \big]_{t_{k-1}}^{t_k} - \int_{I_k} \langle v | \mathrm{d}\dot v \rangle \leq \|v\|_{\mathrm{L}^\infty(I_k,Y)} \mathrm{Var}_A(\dot v; I_k).$$

As above we have $\|v\|_{\mathrm{L}^\infty(I_k,Y)} \leq \frac{\tau_k}{2} \mathrm{Var}_A(\dot v; I_k)$ and conclude

$$\|\dot y - \dot Y_\Pi\|^2_{\mathrm{L}^2([0,T],Y)} = \|\dot v\|^2_{\mathrm{L}^2([0,T],Y)} \leq \frac{f(\Pi)}{2} \mathrm{Var}(\dot v; [0,T])^2 \leq f(\Pi) \mathrm{Var}(\dot y; [0,T])^2.$$

Together with the triangle inequality and $f_*(\Pi) \leq f(\Pi)$ we obtain the desired result. ■

The strong convergence of the derivative like $\sqrt{f(\Pi)}$ can also be shown using interpolation between the linear convergence in $\mathrm{L}^\infty$ and a uniform bound on $\mathrm{Var}_A(\widehat y^\Pi; [0,T])$.

# 5 Nonconvex and nonsmooth problems

We recall the three major conditions (A1) to (A3) from Sections 3.1 and 3.2:

$$\begin{aligned} &\text{(i)} \quad \forall\, z_1, z_2 \in \mathcal{Z} : \ \mathcal{D}(z_1, z_2) = 0 \quad \Longleftrightarrow \quad z_1 = z_2, \\ &\text{(ii)} \quad \forall\, z_1, z_2, z_3 \in \mathcal{Z} : \ \mathcal{D}(z_1, z_3) \leq \mathcal{D}(z_1, z_2) + \mathcal{D}(z_2, z_3). \end{aligned} \tag{A1}$$

$$\begin{aligned} &\text{There exist } c_E^{(1)}, c_E^{(0)} > 0 \text{ such that for all } y_* \in \mathcal{Y}: \\ &\mathcal{E}(t, y_*) < \infty \implies \partial_t \mathcal{E}(\cdot, y_*) : [0,T] \to \mathbb{R} \text{ is measurable} \\ &\qquad\qquad\qquad\qquad \text{and } |\partial_t \mathcal{E}(t, y_*)| \leq c_E^{(1)} (\mathcal{E}(t, y_*) + c_E^{(0)}). \end{aligned} \tag{A2}$$

$$\begin{aligned} &\forall\, t \in [0,T] : \mathcal{E}(t, \cdot) : \mathcal{Y} \to \mathbb{R}_\infty \text{ has compact sublevels}, \\ &\mathcal{D} : \mathcal{Z} \times \mathcal{Z} \to [0, \infty] \text{ is lower semi-continuous}. \end{aligned} \tag{A3}$$

The existence theory developed below will build on the incremental problem (IP) and the a priori estimates derived in Section 3.

## 5.1 Existence results

The general strategy for constructing solutions to (S) & (E) is to choose a sequence of partitions $\Pi^m$ with fineness $f_m$ tending to 0, to extract a convergent subsequence of $(Y^l)_l$ of $(Y^{\Pi^m})_{m \in \mathbb{N}}$ and then to show that the limit $Y : [0,T] \to \mathcal{Y}$ solves (S) & (E). A major problem arises from the fact that the temporal behavior of the elastic component $\varphi$ of $y = (\varphi, z)$ cannot be controlled, which is in contrast to the inelastic component $z$ whose variation is controlled via the dissipation.

For the dissipative part it is possible to extract a suitable limit function if the dissipation is strong enough. We need the following assumption for any sequence $(z_k)_k$ and any $z$ in $\mathcal{Z}$:

$$\min\{\mathcal{D}(z_k, z), \mathcal{D}(z, z_k)\} \to 0 \text{ for } k \to \infty \quad \Longrightarrow \quad z_k \xrightarrow{\mathcal{Z}} z \text{ for } k \to \infty. \tag{A4}$$

The following version of Helly's selection principle is a special case of [MM04a, Thm. 3.2]. The classical result of Helly relates to real-valued monotone functions. Versions for functions with values in Hilbert and Banach spaces can be found in [Mon93, BP86].

**Theorem 5.1** *Let $\mathcal{D} : \mathcal{Z}\times\mathcal{Z} \to [0, \infty]$ satisfy* (A1) *and* (A4). *Moreover, let $\mathcal{K}$ be a compact subset of $\mathcal{Z}$. Then, for every sequence $(Z^l)_{l\in\mathbb{N}}$ with $Z^l : [0, T] \to \mathcal{K}$ and bounded dissipation, i.e.,*

$$\sup_{l\in\mathbb{N}} \mathrm{Diss}_{\mathcal{D}}(Z^l; [0, T]) \le C < \infty,$$

*there exist a subsequence $(Z^{l_n})_{n\in\mathbb{N}}$, a function $z^\infty : [0, T] \to \mathcal{K}$ and a function $\delta^\infty : [0, T] \to [0, C]$ such that the following holds:*

*(a)* $\delta_{l_n}(t) := \mathrm{Diss}_{\mathcal{D}}(Z^{l_n}, [0, t]) \to \delta^\infty(t)$ *for all* $t \in [0, T]$*,*
*(b)* $Z_{l_n}(t) \xrightarrow{\mathcal{Z}} z^\infty(t)$ *for all* $t \in [0, T]$*,*
*(c)* $\mathrm{Diss}_{\mathcal{D}}(z^\infty, [t_0, t_1]) \le \lim_{t\searrow t_1} \delta_\infty(t) - \lim_{s\nearrow t_0} \delta_\infty(s)$ *for all* $0 \le t_0 < t_1 \le T$.

Like in the theory of BV functions in Banach spaces, all functions $z \in \mathrm{BV}_{\mathcal{D}}([0, T], \mathcal{Z})$ are continuous except at the discontinuity points of $t \mapsto \mathrm{Diss}_{\mathcal{D}}(z; [0, t])$. Moreover, for all $t \in [0, T]$ the right-hand and left-hand limits $z_+(t)$ and $z_-(t)$ exist, see Section 3 in [MM04a, Sect. 3].

For the main existence result we need two more conditions. One condition relates to the power of the external forces $\partial_t\mathcal{E}(t, y)$ which we assume to satisfy not only (A2) but also a uniform continuity property:

$$\begin{aligned}&\text{Condition (A2) holds and}\\ &\forall\, E^* > 0\ \forall\, \varepsilon > 0\ \exists\, \delta > 0 : \quad \mathcal{E}(t, y) \le E^* \text{ and } |t - s| \le \delta\\ &\qquad\qquad\qquad\qquad\qquad\quad \Longrightarrow \quad |\partial_t\mathcal{E}(t, y) - \partial_t\mathcal{E}(s, y)| < \varepsilon.\end{aligned} \tag{A5}$$

The above conditions for the topology $\mathcal{T}$ on $\mathcal{Y}$ and the associated lower semi-continuities of $\mathcal{E}$ and $\mathcal{D}$ appear very natural and are standard from the point of view of the calculus of variations. Condition (A5) concerns only the the power of the external forces, which is determined by the prescribed loading data, and thus is uncritical.

**Theorem 5.2** *Assume that $\mathcal{E}$ and $\mathcal{D}$ satisfy the assumptions* (A1)*,* (A3)*,* (A4)*, and* (A5). *Moreover, let one of the following two conditions* (A6) *or* (A7) *be satisfied:*

$$\mathcal{D} : \mathcal{Z}\times\mathcal{Z} \to [0, \infty] \quad \textit{is continuous.} \tag{A6}$$

$$\begin{aligned}&\textit{The set } \mathcal{S}_{[0,T]} \textit{ of stable states is closed in } [0, T]\times\mathcal{Y} \textit{ and}\\ &\forall\, E_0 > 0 :\ \partial_t\mathcal{E} : \{\, (t, y) \mid \mathcal{E}(t, y) \le E_0 \,\} \to \mathbb{R} \textit{ is continuous.}\end{aligned} \tag{A7}$$

*Then, for each $y_0 \in \mathcal{S}(0)$ there exists a solution $y = (\varphi, z) : [0,T] \to \mathcal{Y}$ of (S) & (E).*

*Moreover, if $\Pi^l \in \mathrm{Part}_{f_l}^{N_l}([0,T])$ is a sequence of partitions with fineness $f_l$ tending to 0 and $Y^{\Pi_l}$ is the interpolant of any solution of the associated incremental problem (IP), then there exist a subsequence $y_k = Y^{\Pi_{l_k}}$ and a solution $y = (\varphi, z) : [0,T] \to \mathcal{Y}$ of (S) & (E) such that the following holds:*

*(i)* $\forall\, t \in [0,T] :\; z_k(t) \xrightarrow{\mathcal{Z}} z(t)$;

*(ii)* $\forall\, t \in [0,T] :\; \mathrm{Diss}_{\mathcal{D}}(z_k; [0,t]) \to \mathrm{Diss}_{\mathcal{D}}(z; [0,t])$;

*(iii)* $\forall\, t \in [0,T] :\; \mathcal{E}(t, y_k(t)) \to \mathcal{E}(t, y(t))$;

*(iv)* $\partial_t \mathcal{E}(\cdot, y_k(\cdot)) \to \partial_t \mathcal{E}(\cdot, y(\cdot))$ *in* $\mathrm{L}^1((0,T))$

**Remark 5.3** *It is easy to see that conditions* (A4) *and* (A6) *can be weakened by assuming that the conditions are valid on each sublevel of $\mathcal{E}$.*

**Remark 5.4** *In Step 3 below, we will show that condition* (A6) *implies that $\mathcal{S}_{[0,T]}$ is closed. This central condition will be discussed in more detail in Section 5.2.*

**Remark 5.5** *The above theorem does not claim the convergence of the elastic component $\varphi^n(t)$ to $\varphi(t)$. In fact, since we do not have any control on the temporal oscillations of $\varphi$ we have no selection criterion. We will construct the limit $\varphi(t)$ by choosing a suitable $t$-dependent subsequence of $\varphi^k(t)$. As a consequence we do not obtain information on the continuity or on the measurability of the limit $\varphi : [0,T] \to \mathcal{F}$.*

*However, sometimes the $\varphi$-component is uniquely determined via $z$ in the following sense (cf. [MR03, Eqn.(3.18)]):*

$$y_1{=}(\varphi_1, z_1),\; y_2{=}(\varphi_2, z_2) \in \mathcal{S}(t) \text{ and } z_1 = z_2 \quad \Longrightarrow \quad \varphi_1 = \varphi_2. \tag{5.1}$$

*Then, the $\varphi$-component can be controlled more precisely. In [MM04a] the slightly stronger assumption*

$$y_k = (\varphi_k, z_k) \in \mathcal{S}(t) \text{ and } z_k \xrightarrow{\mathcal{Z}} z \quad \Longrightarrow \quad \varphi_k \xrightarrow{\mathcal{F}} \varphi \tag{5.2}$$

*is used to conclude the stronger result $y^n(t) \xrightarrow{\mathcal{Y}} y(t)$ for all $t \in [0,T]$ in Theorem 5.2 (i) as well as continuity of $t \mapsto y(t) \in \mathcal{Y}$ for all $t$ except at the (at most countable) jump points of* $\mathrm{Diss}_{\mathcal{D}}(z; [0,T])$.

The proof consists of several steps and uses the two following auxiliary results. The first result concerns the continuity of the power of the external forces as a function on $\mathcal{Y}$, i.e., of $y \mapsto \partial_t \mathcal{E}(t, y)$. Very often it is assumed that the loading acts linearly on $\varphi$. This gives the term $\partial_t \mathcal{E}(t, y) = -\langle \dot{\ell}(t), \varphi \rangle$ which is automatically weakly continuous. However, in the case of time-dependent Dirichlet conditions this is more difficult, since we need to control the stresses due to the boundary condition. This problem was first solved in [DFT04] by showing that the stresses in fact converge weakly if we know that the functions $\varphi^n$ as well as the energy converge. The following result is an abstract and much simpler version of this fact.

**Proposition 5.6** *If $\mathcal{E}$ satisfies* (A3) *and* (A5)*, then for all $t \in (0,T)$ the following implication holds.*

$$\left.\begin{array}{l} y_m \xrightarrow{\mathcal{Y}} y \text{ and} \\ \mathcal{E}(t, y_m) \to \mathcal{E}(t, y) < \infty \end{array}\right\} \quad \Longrightarrow \quad \partial_t \mathcal{E}(t, y_m) \to \partial_t \mathcal{E}(t, y). \tag{5.3}$$

**Proof:** Let $E_0, h_0 > 0$ be such that $t \pm h_0 \in [0,T]$ and $\mathcal{E}(t, y_m), \mathcal{E}(t, y) \le E_0$ for sufficiently large $m$. Then, condition (A5) implies the existence of a modulus of continuity $\omega_0 : [0, h_0] \to [0, \infty)$ (i.e., $\omega_0$ is monotone increasing and $\omega_0(h) \to 0$ for $h \searrow 0$) such that for $h \in (0, h_0)$ we have

$$|\tfrac{1}{h}\Big(\mathcal{E}(t{\pm}h, y_m){-}\mathcal{E}(t, y_m)\Big) \mp \partial_t\mathcal{E}(t, y_m)| \le \omega_0(h), \tag{5.4}$$

since the difference quotient can be replaced by a derivative at an intermediate value. The same estimate also holds for $y$. By $h > 0$, the lower semi-continuity of $\mathcal{E}(t, \cdot)$ from (A3) and the assumed convergence of the energy we find

$$\liminf_{m\to\infty} \tfrac{1}{h}\Big(\mathcal{E}(t{\pm}h, y_m){-}\mathcal{E}(t, y_m)\Big) \ge \tfrac{1}{h}\Big(\mathcal{E}(t{\pm}h, y){-}\mathcal{E}(t, y)\Big).$$

Combining the case "+" with (5.4) we find

$$\begin{aligned}&\liminf_{m\to\infty} \partial_t\mathcal{E}(t, y_m) \;\ge\; \liminf_{m\to\infty} \tfrac{1}{h}(\mathcal{E}(t{+}h, y_m){-}\mathcal{E}(t, y_m)) - \omega_0(h)\\ &\ge\; \tfrac{1}{h}(\mathcal{E}(t{+}h, y){-}\mathcal{E}(t, y)) - \omega_0(h) \;\ge\; \partial_t\mathcal{E}(t, y) - 2\omega_0(h).\end{aligned}$$

Similarly, the case "$-$" gives $\limsup_{m\to\infty} \partial_t\mathcal{E}(t, y_m) \le \partial_t\mathcal{E}(t, y) + 2\omega_0(h)$. Since $h$ can be made arbitrarily small, the result is proved. ■

The second result shows that the stability property (S) already implies a lower energy estimate, as can be seen in the proof of Theorem 3.2(ii). Thus, it will be sufficient to keep track of the upper energy estimate only, see $(\mathrm{E})_{\mathrm{discr}}$ in Cor. 3.3. This was observed first in [MTL02].

**Proposition 5.7** *Assume that* (A1) *and* (A5) *hold. Let* $y = (\varphi, z) : [0,T] \to \mathcal{Y}$ *be given such that* $\mathrm{Diss}_{\mathcal{D}}(z; [0,T]) < \infty$*,* $t \to \mathcal{E}(t, y(t))$ *is bounded,* $\partial_t\mathcal{E}(\cdot, y(\cdot)) \in \mathrm{L}^\infty((0,T))$*, and* $y(t) \in \mathcal{S}(t)$ *for all* $t \in [0,T]$*. Then, for all* $0 \le r < s \le T$ *we have the lower energy inequality*

$$\mathcal{E}(s, y(s)) + \mathrm{Diss}_{\mathcal{D}}(y; [r,s]) \ge \mathcal{E}(r, y(r)) + \textstyle\int_r^s \partial_t\mathcal{E}(t, y(t))\,\mathrm{d}t. \tag{5.5}$$

**Proof:** Since $\theta : t \mapsto \partial_t\mathcal{E}(t, y(t))$ is integrable there exists a sequence of partitions $\Pi^m \in \mathrm{Part}^{N_m}_{\delta_m}([r,s])$ with $\delta_m \to 0$ such that the Lebesgue integral can be approximated by the corresponding Riemann sums, namely $\int_r^s \theta(t)\,\mathrm{d}t = \lim_{m\to\infty} \sum_{j=1}^{N_m} \theta(t_j^m)(t_j^m{-}t_{j-1}^m)$. We refer to [Mai05] or [FM04] for this result, or to [DFT04] for a more general version.

In each subinterval $[t_{j-1}^m, t_j^m]$ we use the stability (S), see the proof of Thm. 3.2(ii), and (A5) to obtain

$$\begin{aligned}\mathcal{E}(t_j^m, y(t_j^m)) + \mathcal{D}(y(t_{j-1}^m), y(t_j^m)) \quad &\ge_{(\mathrm{S})} \quad \mathcal{E}(t_{j-1}^m, y(t_{j-1}^m)) + \textstyle\int_{t_{j-1}^m}^{t_j^m} \partial_s\mathcal{E}(s, y(t_j^m))\,\mathrm{d}s\\ &\ge_{(\mathrm{A5})} \quad [\theta(t_j^m){-}\varepsilon](t_j^m{-}t_{j-1}^m),\end{aligned}$$

where $\varepsilon > 0$ can be made as small as we like by choosing $m$ sufficiently large and hence $\delta_m$ sufficiently small. Adding over $j = 1, \ldots, N_m$ and taking the limit $m \to \infty$ gives the desired result, since $\varepsilon > 0$ is arbitrary. ■

**Proof of Theorem 5.2.** For a simplified version of this proof we refer to the proof of Theorem 2.1, where the same steps are followed but for the much simpler case of a quadratic energy on a Banach space $Y$.

We first prove the result under the assumption that (A6) is satisfied. The differences in the proof for the case when (A7) holds are given afterwards.

*Step 1: A priori estimates.* We choose an arbitrary sequence of partitions $\Pi^m$ whose fineness $f_m$ tends to 0. According to Section 3.2 the time-incremental minimization problems (IP) are solvable and the piecewise constant interpolants $Y^m : [0,T] \to \mathcal{Y}$ satisfy the a priori estimates

$$\mathrm{Diss}_{\mathcal{D}}(Z^m;[0,T]) \leq C \quad \text{and} \quad \forall\, t\in[0,T]:\ \mathcal{E}(t,Y^m(t)) \leq C,$$

where $C$ is given explicitly in Cor. 3.3(3).

*Step 2: Selection of subsequences.* Our version of Helly's selection principle in Thm. 5.1 allows us to select a subsequence of $(Z^m)_{m\in\mathbb{N}}$ which converges pointwise and which makes the dissipation converge as well. Moreover, the functions $\Theta^m : t \mapsto \partial_t\mathcal{E}(t,Y^m(t))$ form a bounded sequence in $\mathrm{L}^\infty((0,T))$. Thus, by choosing a further subsequence $(Y^{m_k})_{k\in\mathbb{N}}$ we may assume the following convergence properties for $k\to\infty$, where we write $y^k=(\varphi^k,z^k)$ as shorthand for $Y^{m_k}$ and $\theta_k$ for $\Theta^{m_k}$ :

$$\forall\, t\in[0,T]:\ \delta_k(t) := \mathrm{Diss}_{\mathcal{D}}(z_k;[0,t]) \to \delta(t) \quad \text{and} \quad z_k(t) \xrightarrow{\mathcal{Z}} z(t);$$
$$\theta_k \overset{*}{\rightharpoonup} \theta \quad \text{in } \mathrm{L}^\infty((0,T)).$$

Note that the limit functions $\delta$, $z$, and $\theta$ exist. We further define the function $\theta_{\sup} : t \mapsto \limsup_{k\to\infty}\theta_k(t)$ such that $\theta_{\sup}\in\mathrm{L}^\infty((0,T))$ and $\theta\leq\theta_{\sup}$ by Fatou's lemma.

To define $\varphi(t)$, fix $t\in[0,T]$ and we choose a $t$-dependent subsequence $k=K_n^t$ such that

$$\theta_{K_n^t}(t) \to \theta_{\sup}(t) \quad \text{and} \quad \varphi_{K_n^t}(t) \xrightarrow{\mathcal{F}} \varphi(t).$$

Here we use the a priori bound $\mathcal{E}(t,y_k(t))\leq C$ and the compactness of the sublevels assumed in (A3). Hence, $y(t)=(\varphi(t),z(t))$ is defined.

*Step 3: Stability of the limit function.* We first show that (A6) implies the closedness of $\mathcal{S}_{[0,T]}$. For a sequence $(t_l,y_l)_{l\in\mathbb{N}}$ in $\mathcal{S}_{[0,T]}$ with limit $(t,y)$ consider any test state $\widehat{y}$. Since $\mathcal{D}$ is continuous and $\mathcal{E}$ lower semi-continuous, we have

$$\mathcal{E}(t,y) \leq \liminf_{l\to\infty}\mathcal{E}(t_l,y_l) \leq_{\text{(S)}} \liminf_{l\to\infty}\mathcal{E}(t_l,\widehat{y})+\mathcal{D}(y_l,\widehat{y}) = \mathcal{E}(t,\widehat{y})+\mathcal{D}(y,\widehat{y}),$$

which is the desired stability of $y$.

Using the closedness of $\mathcal{S}_{[0,T]}$ it is easy to show that the limit function $y : t \mapsto (\Pi(t),z(t))$ is stable. For $t\in[0,T]$ fixed define $\tau_k^t=\min\{\,\tau\in\Pi^{m_k} \mid \tau\geq t\,\}$, then $y_k(t)=y_k(\tau_k^t)\in\mathcal{S}(\tau_k^t)$ by the definition of the interpolant $y_k=Y^{m_n}$. Thus,

$$(\tau^t_{K_n^t}, y_{K_n^t}(t)) \in \mathcal{S}_{[0,T]}, \quad \tau_k^t \to t, \quad \text{and} \quad y_{K_n^t}(t) \xrightarrow{\mathcal{Y}} y(t).$$

Hence, the closedness gives $(t,y(t))\in\mathcal{S}_{[0,T]}$, i.e., $y(t)\in\mathcal{S}(t)$.

*Step 4: Upper energy estimate.* We define the functions

$$e_k(t) := \mathcal{E}(t, y_k(t)), \quad \delta_k(t) := \mathrm{Diss}_{\mathcal{D}}(z_k; [0,t]), \quad w_k(t) := \int_0^t \partial_t \mathcal{E}(s, y_k(s))\,\mathrm{d}s = \int_0^t \theta_k(s)\,\mathrm{d}s.$$

Cor. 3.3 and the boundedness of $\partial_t \mathcal{E}$ by a constant $C_1$ (use (A2) and Step 1) give

$$e_k(t) + \delta_k(t) \leq w_k(t) + C_1 f_{m_k}. \tag{5.6}$$

Since $\mathcal{E}$ is lower semi-continuous and $\delta_k$ and $\theta_k$ converge according to Step 2, the limit $k = K_n^t \to \infty$ for $n \to \infty$ leads to

$$\mathcal{E}(t, y(t)) + \mathrm{Diss}_{\mathcal{D}}(z; [0,t]) \leq e(t) + \delta(t) \leq e(0) + \int_0^t \theta(s)\,\mathrm{d}s \leq e(0) + \int_0^t \theta_{\sup}(s)\,\mathrm{d}s,$$

where $\mathcal{E}(t, y(t)) \leq e(t) = \liminf_{k\to\infty} e_k(t)$. In fact, we have $e(t) = \lim_{n\to\infty} \mathcal{E}(t, y_{K_n^t}(t))$, since

$$\mathcal{E}(t, y(t)) =_{\text{(A6)}} \lim_{n\to\infty} \mathcal{E}(t, y(t)) + \mathcal{D}(z_{K_n^t}(t), z(t)) \geq_{\text{(S)}} \limsup_{n\to\infty} \mathcal{E}(t, y_{K_n^t}(t)) \geq e(t).$$

Thus, together with $y_{K_n^t}(t) \xrightarrow{\mathcal{Y}} y(t)$ the assumptions of Prop. 5.6 are satisfied and we conclude

$$\theta_{\sup}(t) = \lim_{n\to\infty} \theta_{N_n^t}(t) = \lim_{n\to\infty} \partial_t \mathcal{E}(t, y_{K_n^t}(t)) = \partial_t \mathcal{E}(t, y(t)).$$

Together with the above, this is the desired upper energy estimate.

*Step 5: Lower energy estimate.* As we have established that $\theta_{\sup} = \partial_t \mathcal{E}(\cdot, y(\cdot))$ lies in $\mathrm{L}^\infty((0,T))$ we are able to apply Prop. 5.7 and obtain the lower energy estimate

$$\mathcal{E}(t, y(t)) + \mathrm{Diss}_{\mathcal{D}}(z; [0,t]) \geq \mathcal{E}(0, y(0)) + \int_0^t \partial_t \mathcal{E}(s, y(s))\,\mathrm{d}s.$$

*Step 6: Improved convergence.* Steps 1 to 5 show that the constructed limit $y : [0,T] \to \mathcal{Y}$ is a solution. In the last step we show that the convergences (i)–(iv) stated at the end of the theorem hold. Part (i) is already shown. The lower and upper energy estimate imply

$$e(0) + \int_0^t \theta_{\sup}\,\mathrm{d}s \leq e(t) + \mathrm{Diss}_{\mathcal{D}}(z; [0,t]) \leq e(t) + \delta(t) \leq e(0) + \int_0^t \theta\,\mathrm{d}s \leq e(0) + \int_0^t \theta_{\sup}\,\mathrm{d}s.$$

Hence, all inequalities are in fact equalities and we conclude $\mathrm{Diss}_{\mathcal{D}}(z, [0,t]) = \delta(t)$ and $\theta = \theta_{\sup}$ a.e. in $[0,T]$. The first identity is (ii) and the second identity implies (iv), cf. [FM04, Prop. A2]. Finally note that the energy $\mathcal{E}(t, y_k(t))$ convergences not only on the $t$-dependent subsequence $k = N_n^t$, but along the whole sequence. This follows since we have shown that $e_k(t) + \delta_k(t)$ always has a limit and $\delta_k$ is convergent.

This completes the proof of Theorem 5.2 in the case that (A6) holds. Now assume that (A7) holds instead.

Step 1 to 3 work identical. In Step 4 the identity $\theta_{sup}(t) = \partial_t \mathcal{E}(t, y(t))$ follows directly from the continuity of $\partial_t \mathcal{E}$ assumed in (A7). Thus, the upper and lower energy estimates follow as above and Step 4 and 5 are done.

In Step 6 the convergence of the energy is not yet established. However, with (5.6) and $\delta_k(t) \to \delta(t) = \mathrm{Diss}_{\mathcal{D}}(z;[0,t])$ we again find, by the lower semi-continuity of $\mathcal{E}$,

$$\begin{array}{l}\mathcal{E}(t,y(t)) + \mathrm{Diss}_{\mathcal{D}}(z;[0,t]) \ \le\ \liminf_{k\to\infty} e_k(t) + \delta(t) \\ \le\ \limsup_{k\to\infty} e_k(t) + \delta(t) \ \le\ \mathcal{E}(0,y(0)) + \int_0^t \theta_{\sup}(s)\,\mathrm{d}s.\end{array}$$

Together with the lower energy estimate this proves $e_k(t) \to \mathcal{E}(t,y(t))$ as desired. The remaining parts of Step 6 are the same.

Thus, Theorem 5.2 is proved. ■

We formulate now a special version of Theorem 5.2, which is based on Banach spaces and which can be easily applied to several models in continuum mechanics.

**Theorem 5.8** *Let $Y_1$ and $Y$ be Banach spaces. Suppose that $Y_1$ is compactly embedded in $Y$ and that $\{\, y \in Y_1 \mid \|y\|_{Y_1} \le 1 \,\}$ is closed in $Y$. The dissipation distance $\mathcal{D} : Y \times Y \to \mathbb{R}$ is the $Y$-norm, i.e., $\mathcal{D}(y_1,y_2) = \|y_1-y_2\|_Y$. Furthermore the functional $\mathcal{E} : [0,T]\times Y \to [\mathcal{E}_{\min},\infty]$ has the following properties:*

*(a) $\mathcal{E}$ is lower semi-continuous on $[0,T]\times Y$ (with respect to the norm topology of $Y$).*

*(b) For some real numbers $c_1 > 0$, $C_2$ and $\alpha > 0$ we have*

$$\mathcal{E}(t,y) \ge c_1\|y\|_{Y_1}^{\alpha} - C_2 \quad (\textit{i.e., } \mathcal{E}(t,y) = \infty \textit{ for } y \in Y\setminus Y_1). \tag{5.7}$$

*(c) Condition* (A5) *is satisfied.*

*Then, for each $y_0 \in \mathcal{S}(0)$ there exists at least one solution $y \in \mathrm{BV}_{\mathcal{D}}([0,T],Y) \cap B([0,T],Y_1)$ of (S) & (E) with $y(0) = y_0$ and all the conclusions of Theorem 5.2 also hold.*

*Here $B([0,T],Y_1)$ denotes the set of mappings $y$ such that $t \mapsto \|y(t)\|_{Y_1}$ is bounded.*

The result is an easy consequence if we choose $\mathcal{Y} = Y$ equipped with its norm topology. Obviously, $\mathcal{D}$ is continuous and satisfies (A2) and (A4). Moreover, the lower semi-continuity of $\mathcal{E}$ and its coerciveness in the compactly embedded space $Y_1$ show that $\mathcal{E}$ has compact sublevels. Thus, (A1) to (A6) hold and Thm. 5.2 is applicable.

A possible application of this result is the partial differential inclusion

$$\begin{array}{r}0 \in \kappa(x)\,\mathrm{Sign}(\dot y(t,x)) - \mathrm{div}\,\big[a(x)\mathrm{D}_x y(t,x)\big] + \mathrm{D}_y F(x,y(t,x)) - \ell(t,x) \text{ in } \Omega, \\ z(t,x) = 0 \text{ on } \partial\Omega.\end{array}$$

To this end, take $Y = \mathrm{L}^1(\Omega)$, $Y_1 = \mathrm{H}^1_0(\Omega)$ and define $\mathcal{D}$ and $\mathcal{E}$ via

$$\begin{array}{rcl}\mathcal{D}(y_0,y_1) &=& \int_\Omega \kappa(x)|y_1(x){-}y_0(x)|\,\mathrm{d}x \quad \text{and} \\ \mathcal{E}(t,y) &=& \int_\Omega \frac{a(x)}{2}|\mathrm{D}_x y(x)|^2 + F(x,y(x)) - \ell(t,x)y(x)\,\mathrm{d}x.\end{array}$$

If we assume that $F : \overline{\Omega}\times\mathbb{R} \to [0,\infty]$ is continuous, then (a) and (b) hold with $\alpha = 2$. Moreover, with $\ell \in \mathrm{C}^{\mathrm{Lip}}([0,T],\mathrm{H}^{-1}(\Omega))$ we obtain $|\partial_t\mathcal{E}(t,y)| = |\langle \partial_t\ell(t),y\rangle| \le C\|y\|_{\mathrm{H}^1} \le c_E^{(1)}(\mathcal{E}(t,y){+}c_E^{(0)})$ for suitable $c_E^{(1)}, c_E^{(0)} > 0$.

## 5.2 Closedness of the stable set

The major assumptions of our existence result in Theorem 5.2 are the compactness of the sublevels of $\mathcal{E}$ and the closedness of $\mathcal{S}_{[0,T]}$. Before deriving abstract results in this direction we give two simple nontrivial applications of the theorem and thus highlight that the choice of the topology $\mathcal{T}$ is crucial. For both examples let $Y = \mathrm{L}^1(\Omega)$ with $\Omega \subset \mathbb{R}^d$ be open and bounded, and choose the dissipation distance $\mathcal{D}(y_0, y_1) = \|y_1 - y_0\|_{\mathcal{Y}} = \int_\Omega |y_1(x) - y_0(x)| \,\mathrm{d}x$.

For the first example consider

$$\mathcal{E}_1(t, y) = \int_\Omega a(x)|y(x)|^\alpha - g(t, x)y(x) \,\mathrm{d}x,$$

where $a(x) \geq a_0 > 0$, $\alpha > 1$, and $g \in \mathrm{C}^1([0, T], \mathrm{L}^\infty(\Omega))$. Since $\mathcal{E}_1(t, \cdot)$ is convex and lower semi-continuous, the sublevels of $\mathcal{E}$ are closed, convex set which are contained an $\mathrm{L}^\alpha$-ball. Hence, taking $\mathcal{T}$ to be the weak topology on $Y = \mathrm{L}^1(\Omega)$, the compactness condition (A4) holds. Note that using the norm topology of $\mathrm{L}^1(\Omega)$ would not supply the desired compactness. The stable sets for $\mathcal{E}_1$ are given by

$$\mathcal{S}_1(t) = \{\, y \in \mathrm{L}^1(\Omega) \mid |y(x)|^{\alpha-2}y(x) \in [\tfrac{g(t,x)-1}{a(x)\alpha}, \tfrac{g(t,x)+1}{a(x)\alpha}] \text{ for a.a. } x \in \Omega \,\},$$

which are closed with respect to $\mathcal{T}$, since they are convex and closed in the norm topology. Hence, with $\mathcal{T}$ as weak topology in $\mathcal{Y} = \mathrm{L}^1(\Omega)$ all conditions of Theorem 5.2 can be satisfied.

For the second example consider the nonconvex energy functional

$$\mathcal{E}_2(t, y) = \int_\Omega \tfrac{1}{2}|\mathrm{D}y(x)|^2 + f(t, x, y(x)) \,\mathrm{d}x \text{ for } y \in \mathrm{H}^1(\Omega) \quad \text{and } +\infty \text{ else},$$

where $f : [0, T] \times \Omega \times \mathbb{R} \to \mathbb{R}$ and $\partial_t f$ are continuous and bounded. Because of the gradient term the sublevels of $\mathcal{E}$ are already compact in the norm topology of $Y = \mathrm{L}^1(\Omega)$, since they are closed and contained in a $Y_1$-ball, where $Y_1 = H^1(\Omega)$ is compactly embedded in $Y$. With these properties, it can be shown that all conditions of Theorem 5.8 are satisfied.

The proof of the first abstract result is contained in Step 3 of the proof of Thm. 5.2. We repeat the result here for convenience.

**Proposition 5.9** *Let* (A2) *hold. Assume that $\mathcal{E}$ is lower semi-continuous on $[0, T] \times \mathcal{Y}$ and that $\mathcal{D}$ is continuous on $\mathcal{Z} \times \mathcal{Z}$. Then, $\mathcal{E} : \mathcal{S}_{[0,T]} \to \mathbb{R}_\infty$ is continuous as well and the set $\mathcal{S}_{[0,T]}$ is closed.*

**Proof:** For $(s, y_s), (t, y_t) \in \mathcal{S}_{[0,T]}$ we have by stability

$$-C_{\mathcal{E}}|t-s| - \mathcal{D}(y_s, y_t) \leq \mathcal{E}(t, y_t) - \mathcal{E}(s, y_s) \leq C_{\mathcal{E}}|t-s| + \mathcal{D}(y_t, y_s).$$

This estimate together with the continuity of $\mathcal{D}$ implies the continuity of $\mathcal{E}$.

Now, consider a sequence $(t_k, y_k)_{k\in\mathbb{N}}$ in $\mathcal{S}_{[0,T]}$ with $t_k \to t^*$ and $y_k \xrightarrow{\mathcal{Y}} y^*$. It remains to show that $y^* \in \mathcal{S}(t^*)$. For an arbitrary $y \in \mathcal{Y}$ we have $\mathcal{E}(t_k, y_k) \leq \mathcal{E}(t_k, y) + \mathcal{D}(y_k, y)$ for all $k \in \mathbb{N}$. Taking the limit $k \to \infty$ the continuities yield $\mathcal{E}(t^*, y^*) \leq \mathcal{E}(t^*, y) + \mathcal{D}(y^*, y)$. Since $y \in \mathcal{Y}$ is arbitrary, it follows that $y^* \in \mathcal{S}(t^*)$. ■

The next result is a strengthened version of the previous one.

**Proposition 5.10** *Let* (A2) *hold. Assume that for each sequence* $(t_k, y_k)_{k\in\mathbb{N}}$ *with* $(t_k, y_k) \in \mathcal{S}_{[0,T]}$, $t_k \to t^*$ *and* $y_k \xrightarrow{\mathcal{Y}} y^*$ *in* $\mathcal{Y}$ *the following condition holds:*

$$\forall\, y \in \mathcal{Y} : \liminf_{k\to\infty} \big[\mathcal{E}(t_k, y_k) - \mathcal{D}(y_k, y)\big] \geq \mathcal{E}(t^*, y^*) - \mathcal{D}(y^*, y). \tag{5.8}$$

*Then, the set* $\mathcal{S}_{[0,T]}$ *is closed.*

**Proof:** Let $y \in \mathcal{Y}$ be arbitrary. We have to show that $\mathcal{E}(t^*, y^*) \leq \mathcal{E}(t^*, y) + \mathcal{D}(y^*, y)$. Since $(t_k, y_k) \in \mathcal{S}_{[0,T]}$ we have the following estimates

$$\begin{aligned}\mathcal{E}(t^*, y^*) &= \mathcal{E}(t^*, y^*) - \mathcal{E}(t_k, y_k) + \mathcal{E}(t_k, y_k) \leq \mathcal{E}(t^*, y^*) - \mathcal{E}(t_k, y_k) + \mathcal{E}(t_k, y) + \mathcal{D}(y_k, y)\\ &= \mathcal{E}(t^*, y) + \mathcal{D}(y^*, y) + (\mathcal{E}(t_k, y) - \mathcal{E}(t^*, y)) - [\mathcal{E}(t_k, y_k) - \mathcal{D}(y_k, y) - \mathcal{E}(t^*, y^*) + \mathcal{D}(y^*, y)].\end{aligned}$$

Taking the limit $k \to \infty$, using (A2) (i.e., $|\partial_t \mathcal{E}| \leq C_{\mathcal{E}}$) and condition (5.8) we obtain the desired result. ∎

For an application to the delamination problem we use the following result, which uses continuity of $\mathcal{E}$ and some approximation property for $\mathcal{D}$. This approximation property is weaker than the continuity assumed in Prop. 5.9. A similar idea, but not in such an abstract setting, is used in [FL03, Thm. 2.1] and [DFT04], where the corresponding result is named ***jump transfer lemma***.

**Proposition 5.11** *Let* (A1), (A2), (A3), *and* (A4) *hold and assume that* $\mathcal{E}$ *and* $\mathcal{D}$ *satisfy the following condition:*

$$\begin{aligned}&\textit{For all } (t, \widehat{y}), (t_k, y_k) \in \mathcal{S}_{[0,T]} \textit{ with } (t_k, y_k) \xrightarrow{\mathcal{Y}} (t, y) \textit{ there exists } \widehat{y}_k \in \mathcal{Y}\\ &\textit{such that} \quad \widehat{y}_k \xrightarrow{\mathcal{Y}} \widehat{y} \textit{ and } \liminf_{k\to\infty} \mathcal{E}(t_k, \widehat{y}_k) + \mathcal{D}(z_k, \widehat{z}_k) \leq \mathcal{E}(t, \widehat{y}) + \mathcal{D}(z, \widehat{z}).\end{aligned} \tag{5.9}$$

*Then, the set* $\mathcal{S}_{[0,T]}$ *is closed.*

Remark: For the case that $\mathcal{Z}$ is a Banach space and $\mathcal{D}(z, \widehat{z}) = \Delta(\widehat{z} - z)$ with $c_1\|z\| \leq \Delta(z) \leq c_2\|z\|$, we simply choose $\widehat{y}_k = (\varphi_k, \widehat{z} - z + z_k)$. Then $\mathcal{D}(z_k, \widehat{z}_k) = \Delta(\widehat{z} - z) = \mathcal{D}(z, \widehat{z})$, and the assumption holds trivially.

**Proof:** Take any sequence $(t_k, y_k) \in \mathcal{S}_{[0,T]}$ with $(t_k, y_k) \xrightarrow{\mathcal{Y}} (t, y)$. We have to show that $y \in \mathcal{S}(t)$. For arbitrary $\widehat{y} \in \mathcal{S}(t)$ we choose $\widehat{y}_k \in \mathcal{M}(t_k) \supset \mathcal{S}(t_k)$ according to condition (5.9). Using the lower semi-continuity of $\mathcal{E}$ and $y_k \in \mathcal{S}(t_k)$ we obtain

$$\mathcal{E}(t, y) = \liminf_{k\to\infty} \mathcal{E}(t_k, y_k) \leq \liminf_{k\to\infty} \mathcal{E}(t_k, \widehat{y}_k) + \mathcal{D}(y_k, \widehat{y}_k) \leq \mathcal{E}(t, \widehat{y}) + \mathcal{D}(y, \widehat{y}),$$

which is the desired stability result. ∎

If $\mathcal{Y}$ is a Banach space $Y$, then it is often easy to show that $\mathcal{D}$ is continuous with respect to the strong topology. However, compactness is often only obtained in the weak topology. Hence, it is desirable to know, under which conditions we can show convexity of the stable sets $\mathcal{S}(t)$. The most important case involves a quadratic energy $\mathcal{E}(t, y) = \langle Ay, y\rangle - \langle \ell(t), y(t)\rangle$ and a translationally invariant dissipation metric $\Psi = \mathcal{L}I_{C_*}$. As we have seen in Section 2 we have $\mathcal{S}(t) = A^{-1}(\ell(t){-}C_*)$. Under suitable conditions on a general dissipation distance $\Psi : Y{\times}Y \to [0,\infty]$ (like (4.11)) it is still possible to show the characterization

$$\mathcal{S}(t) = \{\, y \in Y \mid 0 \in \partial\Psi(y,0) + Ay{-}\ell(t) \,\},$$

and in some cases the convexity may be established from this. However, in general the stable sets are not convex and fortunately this condition is not needed in Section 4 where we always prove strong convergence.

**Example 5.12** *Let $Y = \mathbb{R} \times H$ where $H$ is a Hilbert space. Let $y = (a,h) \in X$ and*

$$\mathcal{E}(t,y) = \tfrac{1}{4}(a^2{+}\|h\|^2)^2 - \gamma(t)a, \quad \Delta(y) = \sqrt{a^2 + \|h\|^2}.$$

*Then for $\gamma(t_1) = 2$ it can be shown that $\mathcal{S}(t_1)$ is not convex and not weakly closed. In fact, for any $h_*$ with $\|h_*\| = (3{\cdot}5^3/2^{16})^{1/6}$ we have $((3^5/2^8)^{1/3}, h_*) \in \mathcal{S}(t_1)$ but $((3^5/2^8)^{1/3}, 0) \notin \mathcal{S}(t_1)$, see [MT04, Ex. 5.5] and on the left in Figure 1.*

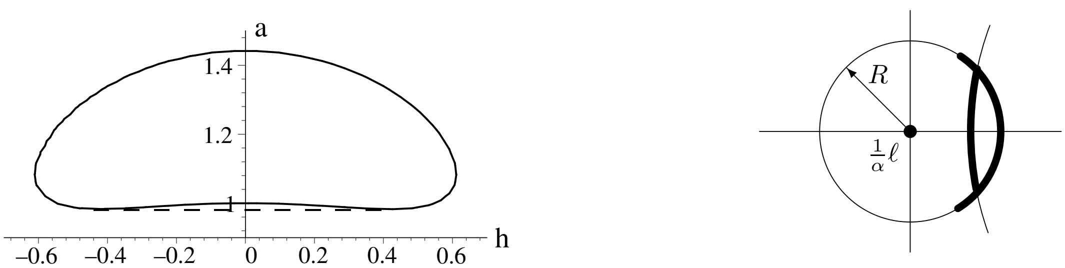


Figure 1: Visualizations of nonconvex stable sets.
Left: Example 5.12 with $H = \mathbb{R}$. Right: Example 5.13 with $Y = \mathbb{R}^2$.

**Example 5.13** *In this example $\mathcal{E}$ is quadratic plus a characteristic function. Let $Y$ be a Hilbert space with $\dim Y \geq 2$ and*

$$\mathcal{E}(t,y) = \tfrac{\alpha}{2}\|y\|^2 + I_{B_R(0)}(y) - \langle \ell(t), y\rangle, \quad \Psi(v) = \|v\|.$$

*Then $y$ with $\|y\| < R$ is stable if and only if $\|\alpha y{-}\ell(t)\| \leq 1$. For $y$ with $\|y\| = R$ the boundary of $B_R(0)$ enlarges the stable set. Stability holds if there exists $\gamma \in [\alpha,\infty)$ such that $\|\gamma y{-}\ell(t)\| \leq 1$. Thus, in the case $\|\ell(t)\| \leq \sqrt{1{+}\alpha^2R^2}$ we have the convex stable set $\mathcal{S}(t) = \{z \in E : \|\alpha z{-}\ell(t)\| \leq 1\} \cap B_R(0)$, which is the intersection of two balls. In the case $\|\ell(t)\| > \sqrt{1{+}\alpha^2R^2}$ we have*

$$\mathcal{S}(t) = \{\, y \mid \|\alpha y{-}\ell(t)\|{\leq}1 \,\} \cup \{\, y \mid \|y\|{=}R,\ \|\big(\|\ell(t)\|^2{-}1\big)^{1/2} y{-}R\ell(t)\|{\leq}R \,\}$$

*which contains a nonconvex part of the boundary of the sphere, see on the right of Figure 1.*

## 5.3 An example of nonconvergence

Here we provide an example where the incremental problem (IP) is solvable and the associated interpolants converge to a limit $z^\infty : [0,\tau] \to \mathcal{Z}$. However, the limit is not a solution despite the fact that the energetic problem (S) & (E) has many solutions.

Let $(\ell_1, \|\cdot\|_1)$ and $(c_0, \|\cdot\|_\infty)$ be the Banach spaces of absolutely summable sequences and sequences converging to 0, respectively. Consider $\mathcal{Z} = \{\, z = (z^{(j)})_{j\in\mathbb{N}} \in \ell_1 \mid \|z\|_1 \le 1 \,\}$,

$$\mathcal{E}(t,z) = -\textstyle\sum_{j=1}^\infty z^{(j)} - \langle \ell(t), z\rangle \quad \text{and} \quad \mathcal{D}(z_0,z_1) = \|z_1 - z_0\|_1\,,$$

where $\ell \in \mathrm{C}^1([0,3],c_0)$ is given via

$$\ell(t) = \textstyle\sum_{k=1}^\infty (\frac14)^k \varphi(2^k t) e_k \text{ where } e_k = (0,\dots,0,1,0,\dots) \in c_0$$

and $\varphi \in \mathrm{C}^1(\mathbb{R})$ with $\mathrm{supp}(\varphi) = [1/2, 1]$ and $\varphi(t) \in [0,1]$. Hence $\|\ell(t)\|_\infty \le ct^2$ and for each $t \in [0,3]$ there exist $k \in \mathbb{N}$ and $\lambda \in [0,1]$ with $\ell(t) = \lambda e_k$.

The stable sets can be easily computed, since $\mathcal{E}(t,\cdot)$ is linear:

$$\begin{aligned} \ell(t) = 0 \quad &\Longrightarrow \quad \mathcal{S}(t) = \mathcal{Z} \\ \ell(t) = \lambda e_k \text{ with } \lambda \in (0,1) \quad &\Longrightarrow \quad \mathcal{S}(t) = \{\, z \in \mathcal{Z} \mid z^{(k)} = 1 - \|z - z^{(k)} e_k\|_1 \,\}. \end{aligned}$$

For the incremental problem (IP) we prescribe the initial condition $z_0 = 0$ and $t_0 = 0$. In the first step we have to minimize

$$z \mapsto \mathcal{E}(t_1, z) + \mathcal{D}(0, z_0) = -\langle \widehat{e} + \ell(t_1), z\rangle + \|z\|_1,$$

where $\widehat{e} = (1,1,1,\dots) \in \ell_\infty$. If $\ell(t_1) = 0$, then any $z_1 \in \mathcal{Z}$ with $z_1^{(j)} \ge 0$ for all $j \in \mathbb{N}$ is a minimizer. If $\ell(t_1) = \lambda_1 e_{n(1)}$ with $\lambda_1 \in (0,1)$, then the unique minimizer is $z_1 = e_{n(1)}$. Generically, for small time increments $t_1 - t_0$ the second case occurs and $n(1) \to \infty$ for $t_1 \searrow 0$.

In the second step, $\ell(t_2) = \lambda_2 e_{n(2)}$ with $n(2) \le n(1)$ and $\lambda_2 \in [0,1]$, and we have to minimize

$$z \mapsto \mathcal{E}(t_2, z) + \mathcal{D}(z_1, z) = -\langle \widehat{e} + \lambda_2 e_{n(2)}, z\rangle + \|z - e_{n(1)}\|_1.$$

It is easy to see that $z_2 = z_1 = e_{n(1)}$ remains the unique global minimizer, since for $n(2) < n(1)$ we have

$$\begin{aligned} &\mathcal{E}(t_2, e_{n(2)}) + \mathcal{D}(e_{n(1)}, e_{n(2)}) \;=\; -(1+\lambda_2) + 2 \\ &> \;\mathcal{E}(t_2, e_{n(1)}) + \mathcal{D}(e_{n(1)}, e_{n(1)}) \;=\; -1 + 0. \end{aligned}$$

Finally, for all further steps we find $z_k = e_{n(1)}$. Thus, for all partitions $\Pi$ the piecewise constant interpolant $z^\Pi : [0,T] \to \mathcal{Z}$ has the form

$$z^\Pi(t) = 0 \text{ for } t \in [0, t_1) \quad \text{and} \quad z^\Pi(t) = e_{n(1)} \text{ for } t \in [t_1, 3],$$

where $n(1)$ is determined via $\ell(t_1) = \lambda_1 e_{n(1)}$ and hence for $f(\Pi) \to 0$ we find $n(1) \to \infty$.

To study convergence, we fix the topology on $\mathcal{Z}$ as the weak$^*$ topology on $\ell_1 = c_0^*$. Then, $\mathcal{Z}$ is a compact space, but $\mathcal{E}(t,\cdot) : \mathcal{Z} \to \mathbb{R}$ is not weakly$^*$ lower semi-continuous. Even worse, the stable sets $\mathcal{S}(t)$ are not weakly$^*$ closed for $\ell(t) = \lambda e_k$ with $\lambda \in (0,1)$. However, we find

$$z^{\Pi_m}(t) \overset{*}{\rightharpoonup} 0 \text{ for all } t \in [0,3].$$

Thus, the limit function $z^\infty : [0,3] \to \mathcal{Z}$ with $z^\infty(t) = 0$ is well-defined. Obviously, $z^\infty$ solves (E) but the stability (S) fails for all $t$ with $\ell(t) \neq 0$.

Nevertheless, (S) & (E) has many solutions. Choose any $z_* \in \mathcal{Z}$ with $\|z_*\|_1 = 1$ and $z_*^{(j)} \geq 0$ for all $j \in \mathbb{N}$. Define $z : [0,3] \to \mathcal{Z}$ with $z(0) = 0$ and $z(t) = z_*$ for $t > 0$. Then, (S) holds, since $z(t) \in \mathcal{S}(t)$ for each $t \in [0,3]$. Moreover, (E) holds since $\mathcal{E}(0, z(0)) = 0$ and for $t > 0$ we have

$$\begin{aligned}\mathcal{E}(t, z(t)) = -1 - \langle \ell(t), z_* \rangle, \quad \mathrm{Diss}_{\mathcal{D}}(z; [0,t]) = 1\\ \textstyle\int_0^t \partial_s \mathcal{E}(s, z(s))\,\mathrm{d}s = -\int_0^t \langle \dot{\ell}(s), z_* \rangle\,\mathrm{d}s = -\langle \ell(t), z_* \rangle.\end{aligned}$$

## 5.4 Formulations which resolve jumps

A major disadvantage of the global energetic formulation using (S) & (E) is that the stability condition (S) is a ***global*** stability condition. Thus, jumps from $y_-$ to $y_+$ can occur despite the fact that any continuous path $\widetilde{y} : [0,1] \to \mathcal{Y}$ from $y_0$ to $y_1$ would have to pass a potential barrier higher than $\mathcal{E}(t, y_0)$, i.e., there is always an $s \in (0,1)$ with $\mathcal{E}(t, \widetilde{y}(s)) + \mathcal{D}(y_0, \widetilde{y}(s)) > \mathcal{E}(t, y_0)$. However, considering ***continuous paths*** we need to specify a topology with respect to which we ask for continuity. This topology may be different from $\mathcal{T}$, which was used for the existence theory, it should rather be modelled on physical grounds, or it should be chosen for mathematical convenience. In particular, it is desirable to use semi-distances $d : \mathcal{Y} \times \mathcal{Y} \to [0, \infty]$ such that the choice $d = \mathcal{D}$ is possible.

It was first proposed in [Mie03a] to study a version of the incremental problem, where global minimization is replaced by a local one, namely inside a ball in the $d$-distance of small radius $\delta > 0$:

$$(\mathrm{IP})_\delta \qquad y_k \in \operatorname{Arg\,min}\{\, \mathcal{E}(t_k, y) + \mathcal{D}(z_{k-1}, z) \mid y \in \mathcal{Y},\ d(y_{k-1}, y) \leq \delta \,\}. \tag{5.10}$$

Of course, it will be essential that the additional parameter $\delta$ tends to 0 slower than the fineness $f(\Pi)$ of the partition, e.g., $\delta = f(\Pi)^{1/2}$. Then, the solutions of $(\mathrm{IP})_\delta$ will display standard rate-independent behavior in many regions but will have inbetween phases where the solution performs a fast jump.

Following [Vis01] we define a rate-independent version of ***$\Phi$-minimal paths*** as follows. The set of arc-length parametrized paths is defined via

$$\begin{aligned}\mathcal{A}(y_0) = \{\, (t, y) \in \mathrm{C}^0([0, \mathcal{T}], \mathbb{R} \times \mathcal{Y}) \mid t(0) = 0,\ y(0) = y_0,\ t'(\tau) \geq 0 \text{ a.e.},\\ t(\tau) + \mathrm{Diss}_d(y; [0, \tau]) = \tau \text{ for all } \tau \,\},\end{aligned}$$

where $\mathrm{Diss}_d$ is the dissipation associated with the new metric $d$. Thus, the curves in $\mathcal{A}(z_0)$ are parametrized by the arc-length variable $\tau$ instead of the usual process time $t$. In particular we find $t'(\tau) \in [0,1]$ a.e. and $d(y(\tau_1), y(\tau_2)) \leq |\tau_1 - \tau_2|$. On $\mathcal{A}(y_0)$ we define the mapping $\Phi : \mathcal{A}(y_0) \to \mathrm{L}^\infty([0, \mathcal{T}])$ via

$$\Phi[t, y](\tau) = \mathcal{E}(t(\tau), y(\tau)) + \mathrm{Diss}_{\mathcal{D}}(y; [0, \tau]) - \int_0^\tau \tfrac{\partial}{\partial t} \mathcal{E}(t(\tau), y(\tau)) t'(\tau)\,\mathrm{d}\tau,$$

then the path $(t, y) \in \mathcal{A}(y_0)$ is called $\Phi$-minimal, if $(t, y) \leq_\Phi (\widehat{t}, \widehat{y})$ for all $(\widehat{t}, \widehat{y}) \in \mathcal{A}(y_0)$, where the relation $\leq_\Phi$ is defined as follows. For two paths $(t, y), (\widehat{t}, \widehat{y}) \in \mathcal{A}(y_0)$ define the time of "equality" via $\widetilde{\tau}^0_{y,\widehat{y}} = \inf\{\,\tau \in [0,\mathcal{T}\,] \mid (t(\tau),y(\tau)) \neq (\widehat{t}(\tau),\widehat{y}(\tau))\,\}$, then

$$(t,y) \leq_\Phi (\widehat{t},\widehat{y}) \quad\Longleftrightarrow\quad \forall\,\tau_2 > \widetilde{\tau}^0_{y,\widehat{y}}\;\exists\,\tau_1 \in (\widetilde{\tau}_0,\tau_2):\; \Phi[t,y](\tau_1) \leq \Phi[\widehat{t},\widehat{y}](\tau_1). \tag{5.11}$$

This formulation can be weakened and localized as follows. Define

$$M(t,y) = \liminf_{\varepsilon\to 0} \tfrac{1}{\varepsilon} \inf\{\, \mathcal{E}(t,y) + \mathcal{D}(y,\widehat{y}) \mid d(y,\widehat{y}) \leq \varepsilon \,\}.$$

Then $(t,y) \in \mathcal{A}(y_0)$ is called ***locally*** $\Phi$***-minimal***, if

$$\tfrac{\mathrm{d}}{\mathrm{d}\tau}\Phi[t,y](\tau) \leq M(t(\tau),y(\tau)) \quad \text{for a.a. } t \in [0,\mathcal{T}]. \tag{5.12}$$

The two above formulations are still derivative free in the sense that the underlying space $\mathcal{Y}$ does not need to have a differentiable structure, such that derivatives of $y:[0,\mathcal{T}] \to \mathcal{Y}$ need not be defined. Only the energetic, real-valued quantities $\mathcal{E}$, $\mathrm{Diss}_\mathcal{D}$ and $\mathrm{Diss}_d$ need to be absolutely continuous.

If the state space $\mathcal{Y}$ has a differentiable structure, then we may assume that the dissipation distance $\mathcal{D}$ and the semi-distance $d$ are generated by local metrics $\Psi : \mathrm{T}\mathcal{Y} \to [0,\infty]$ and $\eta : \mathrm{T}\mathcal{Y} \to [0,\infty]$, respectively. Moreover, we consider now solutions which are absolutely continuous. Then, the condition $(t,y) \in \mathcal{A}(y_0)$ implies $t'(\tau) + \eta(y(\tau),y'(\tau)) = 1$ for a.a. $\tau \in [0,\mathcal{T}]$. If additionally $\mathcal{E}$ is differentiable in $y$, then

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}\tau}\Phi[t,y](\tau) &= \langle \mathrm{D}\mathcal{E}(t,y), y'\rangle + \Psi(y,y') \\ M(t,y) &= \inf\{\, \langle \mathrm{D}\mathcal{E}(t,y), v\rangle + \Psi(y,v) \mid \eta(y,v) \leq 1 \,\}. \end{aligned}$$

Thus, condition (5.12) can be reformulated via the combined functional $\Psi_\eta : \mathrm{T}\mathcal{Y} \to [0,\infty]$

$$\Psi_\eta(y,v) = \begin{cases} \Psi(y,v) & \text{for } \eta(y,v) \leq 1, \\ \infty & \text{else.} \end{cases}$$

We obtain the following differentiable version of (5.12):

$$\left.\begin{aligned} 0 \in \partial_v \Psi_\eta(y(\tau),y'(\tau)) + \mathrm{D}_y\mathcal{E}(t(\tau),y(\tau)) \in \mathrm{T}^*_{y(\tau)}\mathcal{Y} \\ 0 \leq t'(\tau) = 1 - \eta(y(\tau),y'(\tau)) \end{aligned}\right\} \quad \text{for a.a. } \tau \in [0,\mathcal{T}]. \tag{5.13}$$

In [EM04b] this local formulation is investigated for the case that $\mathcal{Y}$ is a finite-dimensional Banach space $Y$ and that both dissipation metrics are translation invariant and nondegenerate, i.e., there exists a $c > 0$ such that $\Psi(v), \eta(v) \geq c\|v\|$ for all $v \in Y$. It is shown that the piecewise linear interpolants of the solutions of the localized incremental problem $(\mathrm{IP})_\delta$ converge, after arc-length parametrization, to a solution of the first equation in (5.13). In general, the limit function will not have arc-length parametrization, but it can be reparametrized to provide a full solution of (5.13). Using a Young measure argument it can be shown that the limit is always in arc-length parametrization if the two metrics $\Psi$ and $\eta$ satisfy a certain compatibility condition (which holds for instance for $\eta = \Psi$).

Moreover, it is shown in [EM04b], that (5.13) appears as a limit problem if the following viscously regularized problem is considered:

$$0 \in \partial\Psi_\varepsilon(\dot{y}(t)) + \mathrm{D}\mathcal{E}(t, y(t)) \in Y^* \quad \text{for a.a. } t \in [0, T], \tag{5.14}$$

where $\Psi_\varepsilon(v) = \Psi(v)+\frac{\varepsilon}{2}\eta(v)^2$. Existence of solutions $z_\varepsilon \in \mathrm{H}^1([0,T], Y)$ follows under mild assumptions on $\mathcal{E}$, since now $\Psi_\varepsilon$ grows quadratically, see [CV90]. Reparametrizing these solutions as above, one can show that the limits for $\varepsilon \to 0$ exist and satisfy (5.13).

A similar arc-length reparametrization was used in [And95] for the surface friction problem studied in Section 6.3. There, the differential inclusion $0 \in \mathcal{R}(y(t), \dot{y}(t)) + Ay(t) - \ell(t)$ is solved for by using a delay in the form $0 \in \mathcal{R}(y(t-\varepsilon), \dot{y}(t)) + Ay(t) - \ell(t)$ which produces a unique solution $y^\varepsilon$. It is then shown that the reparametrized solutions contain a subsequence which converges to a generalized solutions which, in the original time $t$ (not reparametrized) may have jumps.

## 5.5 Time-dependent state spaces $\mathcal{Y}(t)$

In some situations it is necessary to introduce time-dependent state spaces which arise from time-dependent boundary conditions. In the most general situation we have a big state $\widetilde{\mathcal{Y}}$ on which the energy functional $\mathcal{E} : [0,T]\times\mathcal{Y} \to \mathbb{R}_\infty$ and the dissipation distance $\mathcal{D} : \mathcal{Y}\times\mathcal{Y} \to \mathbb{R}_\infty$ are defined. Then, the functional $\mathcal{E}(t, \cdot)$ may be $+\infty$ outside a set $\mathcal{Y}(t) \subset \mathcal{Y}$, which may be defined via time-dependent Dirichlet conditions. The problem is that in such situations it is not possible to satisfy the condition (A2) concerning the time derivative $\partial_t\mathcal{E}$.

In continuum mechanics we often have $y = (\varphi, z) \in \widetilde{\mathcal{F}}\times\mathcal{Z} = \mathcal{Y}$ and the time-dependence comes into play only through a set $\mathcal{F}(t) \subset \mathcal{F}$. Then, one may introduce a transformation $\varphi(t) = \Phi_t(\widetilde{\varphi}(t))$ such that $\Phi_t$ maps $\widetilde{\mathcal{F}}$ homeomorphically into $\mathcal{F}(t)$ (e.g., by subtracting the time-dependent boundary conditions). Then, one defines the transformed energy

$$\widetilde{\mathcal{E}}(t, \widetilde{\varphi}, z) = \mathcal{E}(t, \Phi_t(\widetilde{\varphi}), z) \quad \text{for } \widetilde{\varphi} \in \widetilde{\mathcal{F}},\ t \in [0, T] \text{ and } z \in \mathcal{Z},$$

and the problem is reduced to the time-independent case. We refer to [FM04] for a careful treatment of time-dependent Dirichlet boundary data in the case of small strains as well as in the case of finite-strain elasticity.

However, in some situations this decoupling does not work and we now present a way how this situation can be modelled via the energetic formulation. The stability condition is easily transfered to the time-dependent case, as it is a static condition involving only one time instant. However, for the energy balance we need a replacement of the power of the external forces, previously written as $\partial_t\mathcal{E}(t, y)$.

For this purpose, we assume that $\mathcal{Y}$ is a Banach space and there exist a fixed subset $\widetilde{\mathcal{Y}} \subset \mathcal{Y}$ and invertible transformations

$$\Phi_t : \mathcal{Y} \to \mathcal{Y} \quad \text{with } \Phi_t(\widetilde{\mathcal{Y}}) = \mathcal{Y}(t) \text{ for all } t \in [0, T].$$

We define the functionals $\widetilde{\mathcal{E}} : [0,T]\times\mathcal{Y} \to \mathbb{R}_\infty$ and $\widetilde{\mathcal{D}}_{s,t} : \widetilde{\mathcal{Y}}\times\widetilde{\mathcal{Y}} \to [0, \infty]$ via

$$\widetilde{\mathcal{E}}(t, \widetilde{y}) = \mathcal{E}(t, \Phi_t(\widetilde{y})) \quad \text{and} \quad \widetilde{\mathcal{D}}_{s,t}(\widetilde{y}, \widehat{y}) = \mathcal{D}(\Phi_s(\widetilde{y}), \Phi_t(\widehat{y})).$$

Hence, we introduce a time-dependent dissipation on the time-independent state space $\widetilde{\mathcal{Y}}$. Note that the solutions to be constructed have to lie in $\widetilde{\mathcal{Y}}$, but the functional $\widetilde{\mathcal{E}}$ is defined on all of $\mathcal{Y}$.

For all $s,t \in [0,T]$ we also define the transfer operators

$$\widetilde{\Phi}_{s,t} : \mathcal{Y} \to \mathcal{Y},\ \widetilde{y} \mapsto \Phi_t^{-1}(\Phi_s(\widetilde{y}));$$

which satisfy the evolution property $\widetilde{\Phi}_{r,s} \circ \widetilde{\Phi}_{s,t} = \widetilde{\Phi}_{r,t}$ and, by the definitions, we find

$$\widetilde{\mathcal{D}}_{s,t}(\widetilde{y}_0, \widetilde{\Phi}_{r,t}(\widetilde{y}_1)) = \mathcal{D}(\Phi_s(\widetilde{y}_0), \Phi_r(\widetilde{y}_1)) = \widetilde{\mathcal{D}}_{s,r}(\widetilde{y}_0, \widetilde{y}_1) \quad \text{and } \widetilde{\mathcal{D}}_{s,t}(\widetilde{y}, \widetilde{\Phi}_{s,t}(\widetilde{y})) = 0. \tag{5.15}$$

If $\mathcal{D}$ is generated from a dissipation metric $\Psi : \mathrm{T}\mathcal{Y} \to [0,\infty]$, then $\widetilde{\mathcal{D}}_{s,t}$ associates with the time-dependent dissipation metric $\widetilde{\Psi}$ given by

$$\widetilde{\Psi}(t, \widetilde{y}, \widetilde{v}) = \Psi(\Phi_t(\widetilde{y}), \mathrm{D}\Phi_t(\widetilde{y})v - \partial_t \Phi_t(\widetilde{y})).$$

The main assumption on the model, replacing the former condition (A2), is now that for each $(t,\widetilde{y}) \in [0,T]\times\widetilde{\mathcal{Y}}$ with $\widetilde{\mathcal{E}}(t,\widetilde{y}) < \infty$ the function $s \mapsto \widetilde{\mathcal{E}}(s, \widetilde{\Phi}_{t,s}(\widetilde{y}))$ is continuously differentiable and

$$\exists\, c_E^{(1)}, c_E^{(0)} > 0\ \forall\, s \in [0,T] :\ |\tfrac{\partial}{\partial s}\widetilde{\mathcal{E}}(s, \widetilde{\Phi}_{t,s}(\widetilde{y}))| \le c_E^{(1)}(\widetilde{\mathcal{E}}(t,\widetilde{y}) + c_E^{(0)}). \tag{5.16}$$

We now define the power of external forces via

$$\widetilde{p}(t,\widetilde{y}) = \tfrac{\mathrm{d}}{\mathrm{d}t}\widetilde{\mathcal{E}}(t, \Phi_t^{-1}(w))\big|_{w=\Phi_t(\widetilde{y})} = \tfrac{\mathrm{d}}{\mathrm{d}s}\widetilde{\mathcal{E}}(s, \widetilde{\Phi}_{t,s}(\widetilde{y}))\big|_{s=t}.$$

**Example 5.14** *Consider a smooth situation with $\mathcal{Y} = \mathbb{R}^N$, $\mathcal{E}(t,y) = \frac{1}{2}\langle Ay, y\rangle - \langle \ell(t), y\rangle$, $\Psi(y,v) = \widehat{\Psi}(v)$ and $\mathcal{Y}(t) = b(t) + V$ where $V$ is an arbitrary, fixed subspace. Hence, the variational inequality reads*

$$0 \in \partial\widehat{\Psi}(\dot{y}(t)) + Ay(t) - \ell(t) + \partial I_{\mathcal{Y}(t)}(y(t)) \subset \mathbb{R}^N.$$

*With $y = \Phi_t(\widetilde{y}) = b(t) + Q(t)\widetilde{y}$, where $Q(t) \in \mathrm{Lin}(V,V)$, we obtain $\widetilde{\mathcal{E}}(t,\widetilde{y}) = \mathcal{E}(t, \Phi_t(\widetilde{y})) = \frac{1}{2}\langle \widetilde{A}\widetilde{y}, \widetilde{y}\rangle - \langle \widetilde{\ell}(t), \widetilde{y}\rangle + \widetilde{e}(t)$ with $\widetilde{A}(t) = Q^{\mathsf{T}}AQ$ and $\widetilde{\ell} = \ell - Ab$. Using $\dot{\widetilde{A}} = \dot{Q}^{\mathsf{T}}AQ + Q^{\mathsf{T}}A\dot{Q}$ and $\dot{\widetilde{\ell}} = \dot{\ell} - A\dot{b}$ we find*

$$\begin{aligned}
&\partial_t\widetilde{\mathcal{E}}(t,\widetilde{y}) = \langle AQ(\widetilde{y}+b) - \ell, \dot{Q}y\rangle - \langle \dot{\ell} - A\dot{b}, Q\widetilde{y} + b\rangle - \langle \ell, \dot{b}\rangle,\\
&\mathrm{D}\widetilde{\mathcal{E}}(t,\widetilde{y})[Q^{-1}(\dot{Q}\widetilde{y} + \dot{b})] = \langle AQ(\widetilde{y}+b) - \ell, \dot{Q}y + \dot{b}\rangle,\ \textit{and hence}\\
&\widetilde{p}(t,\widetilde{y}) = \partial_t\widetilde{\mathcal{E}}(t,\widetilde{y}) - \mathrm{D}\widetilde{\mathcal{E}}(t,\widetilde{y})[Q^{-1}(\dot{Q}\widetilde{y} + \dot{b})] = -\langle \dot{\ell}, Q\widetilde{y} + b\rangle = \langle A\dot{b} - \dot{\widetilde{\ell}}, Q\widetilde{y} + b\rangle.
\end{aligned}$$

*Thus, we see the two contributions of the power of the changing boundary conditions via $\dot{b}$ and the power of the external forces via $\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\ell}$. Moreover, the rate $\dot{Q}$ of the (unnecessary) transformation $Q(t)$ does not contribute to the power. With $a(t,\widetilde{y}) = \mathrm{D}\Phi_t(\widetilde{y})\partial_t\Phi_t(\widetilde{y}) = Q^{-1}(\dot{Q}\widetilde{y} + \dot{b})$ the transformed system in $V$ takes the form*

$$0 \in \Big(\widehat{\Psi}(Q(t)[\dot{\widetilde{y}}(t) - a(t, \widetilde{y}(t))]) + \widetilde{A}(t)\widetilde{y}(t) - \widetilde{\ell}(t)\Big) \cap V^* \subset V^*.$$

For fixed times we define the set $\mathcal{S}(t) \subset \mathcal{Y}(t)$ of stable states via $\mathcal{S}(t) = \{\, y \in \mathcal{Y}(t) | \mathcal{E}(t,y) \leq \mathcal{E}(t,\widehat{y}) + \mathcal{D}(y,\widehat{y})$ for all $\widehat{y} \in \mathcal{Y}(t)\,\}$ as well as the transformed set

$$\widetilde{\mathcal{S}}(t) = \{\, \widetilde{y} \in \widetilde{\mathcal{Y}} \mid \widetilde{\mathcal{E}}(t,\widetilde{y}) \leq \widetilde{\mathcal{E}}(t,\widehat{y}) + \widetilde{\mathcal{D}}_{t,t}(\widetilde{y},\widehat{y}) \text{ for all } \widehat{y} \in \widetilde{\mathcal{Y}}\,\} = \Phi_t^{-1}(\mathcal{S}(t)).$$

The dissipation of a curve $\widetilde{y} : [0,T] \to \widetilde{\mathcal{Y}}$ on the internal $[r,s] \subset [0,T]$ is defined via

$$\mathrm{Diss}_{\widetilde{\mathcal{D}}}(\widetilde{y},[r,s]) = \sup\{\, \sum_{j=1}^{N} \widetilde{\mathcal{D}}_{\tau_{j-1},\tau_j}(\widetilde{y}(\tau_{j-1}),\widetilde{y}(\tau_j)) \mid N \in \mathbb{N}, r \leq \tau_0 < \tau_1 < \ldots < \tau_N \leq s\,\},$$

such that $y : t \mapsto \Phi_t(\widetilde{y}(t)) \in \mathcal{Y}(t)$ satisfies $\mathrm{Diss}_{\mathcal{D}}(y;[r,s]) = \mathrm{Diss}_{\widetilde{\mathcal{D}}}(\widetilde{y},[r,s])$. We also define the power of the external forces in the original coordinates via

$$p(t,y) = \widetilde{p}(t,\Phi_t^{-1}(y)) \quad \text{ for } y \in \mathcal{Y}(t).$$

A simple application of the chain rule shows that, in the case that $\mathcal{Y}(t)$ is constant and $\mathcal{E}(t,y)$ is differentiable in $t$, we have $p(t,y) = \partial_t\mathcal{E}(t,y)$ as expected.

The following two energetic formulations (S) & (E) and $(\widetilde{\mathrm{S}})$ & $(\widetilde{\mathrm{E}})$ are equivalent via the transformation $y(t) = \Phi_t(\widetilde{y}(t))$.

**Definition 5.15** *A process $y : [0,T] \to \mathcal{Y}$ is called an **energetic solution** of the rate-independent problem for $(\mathcal{Y}(t))_{t\in[0,T]}$, $\mathcal{E}$ and $\mathcal{D}$, if (S) & (E) hold for all $t \in [0,T]$:*

**(S)** $y(t) \in \mathcal{S}(t) \subset \mathcal{Y}(t)$; **(E)** $\mathcal{E}(t,y(t)) + \mathrm{Diss}_{\mathcal{D}}(y,[0,t]) = \mathcal{E}(0,y(0)) + \int_0^t p(s,y(s))\,\mathrm{d}s.$

*A process $\widetilde{y} : [0,T] \to \widetilde{\mathcal{Y}}$ is called an **energetic solution** of the rate-independent problem for $\widetilde{\mathcal{Y}}$, $\widetilde{\mathcal{E}}$ and $(\widetilde{\mathcal{D}}_{s,t})_{0\leq s\leq t\leq T}$, if $\widetilde{(S)}$ & $\widetilde{(E)}$ hold for all $t \in [0,T]$:*

**$(\widetilde{\mathrm{S}})$** $\widetilde{y}(t) \in \widetilde{\mathcal{S}}(t) \subset \widetilde{\mathcal{Y}}$; **$(\widetilde{\mathrm{E}})$** $\widetilde{\mathcal{E}}(t,\widetilde{y}(t)) + \mathrm{Diss}_{\widetilde{\mathcal{D}}}(\widetilde{y},[0,t]) = \widetilde{\mathcal{E}}(0,\widetilde{y}(0)) + \int_0^t \widetilde{p}(s,\widetilde{y}(s))\,\mathrm{d}s.$

The important point is that both energetic formulations are strongly related to their associated time-incremental minimization problem (IP) and $(\widetilde{\mathrm{IP}})$, respectively. For a discretization $0 = t_0 < t_1 < \ldots < t_N = T$ and $y_0 \in \mathcal{Y}(0)$ we let $\widetilde{y}_0 = \Phi_0^{-1}(y_0)$ and consider the two incremental problems

$$\text{(IP)} \qquad y_k \in \operatorname{Arg\,min}\{\, \mathcal{E}(t_k,y)+\mathcal{D}(y_{k-1},y) \mid y \in \mathcal{Y}(t_k)\,\}. \tag{5.17}$$

$$(\widetilde{\mathrm{IP}}) \qquad \widetilde{y}_k \in \operatorname{Arg\,min}\{\, \widetilde{\mathcal{E}}(t_k,\widetilde{y})+\widetilde{\mathcal{D}}_{t_{k-1},t_k}(\widetilde{y}_{k-1},\widetilde{y}) \mid \widetilde{y} \in \widetilde{\mathcal{Y}}\,\}. \tag{5.18}$$

Of course, these two incremental problems are equivalent via $y_k = \Phi_{t_k}(\widetilde{y}_k)$. The following result shows that the basic a priori estimates for $(\widetilde{\mathrm{IP}})$ hold as in the case of a time-independent dissipation distance, cf. Theorem 3.2.

**Theorem 5.16** *If the above assumptions hold and $\widetilde{y}_0 \in \widetilde{\mathcal{S}}(0)$, then every solution $(\widetilde{y}_k)_{k=1,\ldots,N}$ of $(\widetilde{\mathrm{IP}})$ satisfies the following properties:*

*(i) For $k = 0,\ldots,N$ the state $\widetilde{y}_k$ is stable at time $t_k$, i.e., $\widetilde{y}_k \in \widetilde{\mathcal{S}}(t_k)$;*

*(ii) For $k = 1,\ldots,N$ we have*
$\int_{t_{k-1}}^{t_k} \widetilde{p}(s,\widetilde{\Phi}_{t_k,s}(\widetilde{y}_k))\,\mathrm{d}s \leq \widetilde{e}_k - \widetilde{e}_{k-1} + \widetilde{\delta}_k \leq \int_{t_{k-1}}^{t_k} \widetilde{p}(s,\widetilde{\Phi}_{t_{k-1},s}(\widetilde{y}_{k-1}))\,\mathrm{d}s,$

*where* $\widetilde{e}_j = \widetilde{\mathcal{E}}(t_j, \widetilde{y}_j)$ *and* $\widetilde{\delta}_k = \widetilde{\mathcal{D}}_{t_{k-1},t_k}(\widetilde{y}_{k-1}, \widetilde{y}_k)$.

*(iii) With* $E_0 = \widetilde{\mathcal{E}}(0, \widetilde{y}_0) + c_E^{(0)}$ *we have*

$$\sum_{k=1}^N \widetilde{\mathcal{D}}_{t_{k-1},t_k}(\widetilde{y}_{k-1}, \widetilde{y}_k) \leq E_0 \,\mathrm{e}^{c_E^{(1)} T} \quad \textit{and} \quad \widetilde{\mathcal{E}}(t_k, \widetilde{y}_k) \leq E_0 \,\mathrm{e}^{c_E^{(1)} t_k} - c_E^{(0)} \textit{ for } k = 1, \ldots, N.$$

**Proof:** For (i) use that $\widetilde{y}_k$ minimizes and the triangle inequality. For $\widetilde{y} \in \widetilde{\mathcal{Y}}$ we have

$$\widetilde{\mathcal{E}}(t_k, \widetilde{y}_k) \leq \widetilde{\mathcal{E}}(t_k, \widetilde{y}) + \widetilde{\mathcal{D}}_{t_{k-1},t_k}(\widetilde{y}_{k-1}, \widetilde{y}) - \widetilde{\mathcal{D}}_{t_{k-1},t_k}(\widetilde{y}_{k-1}, \widetilde{y}_k) \leq \widetilde{\mathcal{E}}(t_k, \widetilde{y}) + \widetilde{\mathcal{D}}_{t_k,t_k}(\widetilde{y}_k, \widetilde{y}).$$

To obtain the upper estimate of (ii) we use $\widehat{y}^* = \widetilde{\Phi}_{t_{k-1},t_k}(\widetilde{y}_{k-1}) \in \widetilde{\mathcal{Y}}$ as a test function in ($\widetilde{\mathrm{IP}}$) at the $k$-th step and employ (5.15).

$$\begin{aligned}
&\widetilde{\mathcal{E}}(t_k, \widetilde{y}_k) + \widetilde{\mathcal{D}}_{t_{k-1},t_k}(\widetilde{y}_{k-1}, \widetilde{y}_k) - \widetilde{\mathcal{E}}(t_{k-1}, \widetilde{y}_{k-1}) \;\leq\; \widetilde{\mathcal{E}}(t_k, \widehat{y}^*) + \widetilde{\mathcal{D}}_{t_{k-1},t_k}(\widetilde{y}_{k-1}, \widehat{y}^*) - \widetilde{\mathcal{E}}(t_{k-1}, \widetilde{y}_{k-1}) \\
&= \widetilde{\mathcal{E}}(t_k, \widetilde{\Phi}_{t_{k-1},t_k}(\widetilde{y}_{k-1})) - \widetilde{\mathcal{E}}(t_{k-1}, \widetilde{y}_{k-1}) + \widetilde{\mathcal{D}}_{t_{k-1},t_k}(\widetilde{y}_{k-1}, \widetilde{\Phi}_{t_{k-1},t_k}(\widetilde{y}_{k-1})) \\
&= \textstyle\int_{t_{k-1}}^{t_k} \frac{\mathrm{d}}{\mathrm{d}s} \widetilde{\mathcal{E}}(s, \widetilde{\Phi}_{t_{k-1},s}(y_{k-1}))\,\mathrm{d}s + 0 = \int_{t_{k-1}}^{t_k} \widetilde{p}(s, \widetilde{\Phi}_{t_{k-1},s}(y_{k-1}))\,\mathrm{d}s.
\end{aligned}$$

Similarly, we obtain the lower estimate in (ii) by using $\widehat{y}_* = \widetilde{\Phi}_{t_k,t_{k-1}}(\widetilde{y}_k)$ as a comparison function in the stability condition for $\widetilde{y}_{k-1}$:

$$\begin{aligned}
&\widetilde{\mathcal{E}}(t_k, \widetilde{y}_k) - \widetilde{\mathcal{E}}(t_{k-1}, \widetilde{y}_{k-1}) + \widetilde{\mathcal{D}}_{t_{k-1},t_k}(\widetilde{y}_{k-1}, \widetilde{y}_k) \\
&\geq \widetilde{\mathcal{E}}(t_k, \widetilde{y}_k) - \widetilde{\mathcal{E}}(t_{k-1}, \widehat{y}_*) - \widetilde{\mathcal{D}}_{t_{k-1},t_{k-1}}(\widetilde{y}_{k-1}, \widehat{y}_*) + \widetilde{\mathcal{D}}_{t_{k-1},t_k}(\widetilde{y}_{k-1}, \widetilde{y}_k) \\
&= \widetilde{\mathcal{E}}(t_k, \widetilde{y}_k) - \widetilde{\mathcal{E}}(t_{k-1}, \widetilde{\Phi}_{t_k,t_{k-1}}(\widetilde{y}_k)) - \mathcal{D}(\Phi_{t_{k-1}}(\widetilde{y}_{k-1}), \Phi_{t_k}(\widetilde{y}_k)) + \mathcal{D}(\Phi_{t_{k-1}}(\widetilde{y}_{k-1}), \widetilde{\Phi}_{t_k}(\widetilde{y}_k)) \\
&= \textstyle\int_{t_{k-1}}^{t_k} \frac{\mathrm{d}}{\mathrm{d}s} \widetilde{\mathcal{E}}(s, \widetilde{\Phi}_{t_k,s}(\widetilde{y}_k))\,\mathrm{d}s + 0 = \int_{t_{k-1}}^{t_k} \widetilde{p}(s, \widetilde{\Phi}_{t_k,s}(\widetilde{y}_k))\,\mathrm{d}s.
\end{aligned}$$

Estimate (iii) follows in the same way as shown in Section 3.3 by induction over $k$ and using (5.16) and the upper estimate in (ii). ∎

Following the lines of Section 5.1 it should be possible to develop a suitable existence theory.

## 5.6 Relaxation of rate-independent systems

Rate-independent systems can also be used to study systems which develop microstructure. In mathematics, we say that a system develops microstructure if energy minimization for a functional $\mathcal{I} : \mathcal{Y} \to \mathbb{R}_\infty$ leads to infimizing sequences $(y^{(j)})$, whose weak limit $y^\infty$ does not minimize $\mathcal{I}$. More precisely, we have

$$\mathcal{I}(y^{(j)}) \to \alpha = \inf\{\, \mathcal{I}(y) \mid y \in \mathcal{Y}\,\}, \quad y^{(j)} \xrightarrow{\mathcal{Y}} y^\infty \quad \text{and} \quad \mathcal{I}(y^\infty) > \alpha. \tag{5.19}$$

This means that the sublevels of $\mathcal{I}$ are not closed and the construction of minimizers via infimizing sequences does not work. In fact, the existence of minimizers may fail. In such a situation the functional $\mathcal{I}$ is usually relaxed to a new functional $\mathbf{I} : \mathbf{Y} \to \mathbb{R}_\infty$, which is lower semi-continuous and, hence, has a global minimizer $\mathbf{y}$ which is connected to the limit $y^\infty$ from above and may also retain some information on the infimizing sequence $(y^{(j)})$.

Since rate-independent problems are strongly connected to energy minimization via the energetic formulation (S) & (E), a related philosophy may be applied to the associated incremental problems. This was first observed in [OR99, ORS00] where the occurrence of certain microstructures in plasticity was explained, see also [MSL02, ML03]. Independently this idea was used for the derivation of evolution equations for shape-memory alloys in [MT99, MTL02, The02, MR03]. The abstract framework presented here was developed in [Mie03b, Mie04a].

We return to the energetic formulation (S) & (E) via the functionals $\mathcal{E} : [0,T]\times\mathcal{Y} \to \mathbb{R}_\infty$ and $\mathcal{D} : \mathcal{Y}\times\mathcal{Y} \to \mathbb{R}_\infty$, where now $\mathcal{E}$ and $\mathcal{D}$ need no longer be lower semi-continuous. The motivation for the suggested relaxation relies on the incremental problem (IP), see (3.4), which is in general no longer solvable due to formation of microstructure, see (5.19). In this situation we suggest the following approximate incremental problem.

(AIP)$_\varepsilon$ Given $\varepsilon > 0$ and $y_0 \in \mathcal{Y}$, find $y_k^\varepsilon \in \mathcal{Y}$ with
$$\mathcal{E}(t_k, y_k^\varepsilon)+\mathcal{D}(y_{k-1}^\varepsilon, y_k^\varepsilon) \leq \varepsilon + \mathcal{E}(t_k, y)+\mathcal{D}(y_{k-1}^\varepsilon, y) \text{ for all } y \in \mathcal{Y}. \tag{5.20}$$

Obviously, this problem has solutions for all $\varepsilon > 0$. The difficult, remaining question is how the solutions $y_k^\varepsilon$ behave for $\varepsilon \to 0$. As we have seen in the above example, we cannot expect pointwise convergence but certain macroscopic quantities should have limits for $\varepsilon \to 0$.

To define an abstract notion of relaxation we introduce a generalized convergence "$\xrightarrow{\mathbf{Y}}$" on an enlarged space $\mathbf{Y}$, whose elements are denoted by $\mathbf{y}$. This space is connected to $\mathcal{Y}$ via a continuous embedding $\mathcal{J} : \mathcal{Y} \mapsto \mathbf{Y}$. Moreover, generalized functionals $\mathbf{E} : [0,T]\times\mathbf{Y} \to \mathbb{R}$ and $\mathbf{D} : \mathbf{Y}\times\mathbf{Y} \to [0,\infty]$ replace the elastic functional $\mathcal{E}$ and the dissipation distance $\mathcal{D}$. The relaxation must be such that the associated ***relaxed incremental problem*** (RIP) for an initial datum $\mathbf{y}_0 \in \mathbf{Y}$ and the time discretization $0 = t_0 < t_1 < \ldots < t_N = T$ is solvable.

(RIP) For given $\mathbf{y}_0 \in \mathbf{Y}$ find, for $k = 1, \ldots, N$,
$$\mathbf{y}_k \in \operatorname{Arg\,min}\{\, \mathbf{E}(t_k, \mathbf{y})+\mathbf{D}(\mathbf{y}_{k-1}, \mathbf{y}) \mid \mathbf{y} \in \mathbf{Y} \,\}. \tag{5.21}$$

We do not ask for the conditions $\mathbf{D}(\mathcal{J}(0, z_0), \mathcal{J}(0, z_1)) = \mathcal{D}(z_0, z_1)$ and $\mathbf{E}(t, \mathcal{J}(\boldsymbol{\varphi}, z)) = \mathcal{E}(t, \boldsymbol{\varphi}, z)$. Hence, in general the relaxation will not be an extension.

**Definition 5.17** *A 4-tuple* $(\mathbf{Y}, \mathcal{J}, \mathbf{E}, \mathbf{D})$ *as defined above is called a* ***lower (or upper) incremental relaxation*** *of* $(\mathcal{Y}, \mathcal{E}, \mathcal{D})$ *if the following four conditions hold:*

**(R1)** ***Solvability:*** *For each* $\mathbf{y}_0 \in \mathbf{Y}$ *the relaxed incremental problem (RIP) has a solution.*

**(R2)** ***Approximation:*** $\mathcal{J}(\mathcal{Y})$ *is dense in* $\mathbf{Y}$.

**(R3)** ***Incremental consistency:*** *If* $(y_k)_{k=1,\ldots,N}$ *solves (IP), then* $\mathcal{J}(y_k)_{k=1,..,N}$ *solves (RIP); and if* $(\mathbf{y}_k)_{k=1,..,N}$ *satisfies* $\mathbf{y}_k = \mathcal{J}(y_k)$ *and solves (RIP), then* $(y_k)_{k=1,..,N}$ *solves (IP).*

**(R4)$_{\text{low}}$** ***Lower incremental relaxation:*** *For each solution* $(\mathbf{y}_k)_{k=1,\ldots,N}$ *of (RIP) there exist solutions* $(y_k)_{k=1,\ldots,N}$ *of (AIP)*$_\varepsilon$ *with* $\mathcal{J}(y_k^\varepsilon) \xrightarrow{\mathbf{Y}} \mathbf{y}_k$ *for* $\varepsilon \to 0$.

**(R4)$_{\text{upp}}$** ***Upper incremental relaxation:*** *If* $\mathcal{J}(y_k^\varepsilon) \xrightarrow{\mathbf{Y}} \mathbf{y}_k$ *and* $(y_k^\varepsilon)_{k=1,\ldots,N}$ *solves (AIP)*$_\varepsilon$*, then* $(\mathbf{y}_k)_{k=1,\ldots,N}$ *solves (RIP).*

Our definition implies that the relaxed problem has to be of the same energetic kind as the original one; we just give up the clear distinction between $\boldsymbol{\varphi} \in \mathcal{F}$ and $z \in \mathcal{Z}$. Condition **(R1)** forces us to consider only useful relaxations, namely those which have solutions. If the original problem is already solvable, then we can choose $\mathbf{Y} = \mathcal{F}\times\mathcal{Z}$, $\mathbf{E} = \mathcal{E}$ and $\mathbf{D} = \mathcal{D}$, since no relaxation is necessary. Condition **(R2)** says that the new state space $\mathbf{Y}$ should not be unnecessarily big in the sense that every $\mathbf{y} \in \mathbf{Y}$ can be approximated by a sequence $(\boldsymbol{\varphi}^\varepsilon, z^\varepsilon)_{\varepsilon>0}$ of classical elements in $\mathcal{F}\times\mathcal{Z}$, i.e., $\mathcal{J}(\boldsymbol{\varphi}^\varepsilon, z^\varepsilon) \xrightarrow{\mathbf{Y}} \mathbf{y}$ for $\varepsilon \to 0$. Condition **(R3)** is very important as it says that the relaxation must maintain classical solutions, if they exist for (IP) or if they are found by solving (RIP). Conditions **(R4)$_{\text{low}}$** and **(R4)$_{\text{upp}}$** link the rate-independent evolution of $(\mathcal{F}\times\mathcal{Z}, \mathcal{E}, \mathcal{D})$ to that of $(\mathbf{Y}, \mathbf{E}, \mathbf{D})$ via the approximate incremental problem **(AIP)**$_\varepsilon$.

Moreover the relaxed incremental problem (RIP) can be interpreted as the incremental problem associated to the following relaxed energetic formulation of a rate-independent time-continuous problem: A function $\mathbf{y} : [0,T] \mapsto \mathbf{Y}$ is a solution of the ***relaxed energetic problem*** associated with $(\mathbf{Y}, \mathbf{E}, \mathbf{D})$, if ($\mathbb{S}$) and ($\mathbb{E}$) hold for all $t \in [0,T]$:

$$
\begin{array}{ll}
(\mathbb{S}) & \forall\, \widetilde{\mathbf{y}} \in \mathbf{Y} :\ \mathbf{E}(t, \mathbf{y}(t)) \leq \mathbf{E}(t, \widetilde{\mathbf{y}}) + \mathbf{D}(\mathbf{y}(t), \widetilde{\mathbf{y}}); \\
(\mathbb{E}) & \mathbf{E}(t, \mathbf{y}(t)) + \mathbf{Diss}(\mathbf{y}; [0,t]) = \mathbf{E}(0, \mathbf{y}(0)) + \int_0^t \partial_s \mathbf{E}(s, \mathbf{y}(s))\, \mathrm{d}s,
\end{array}
$$

where the relaxed dissipation **Diss** is calculated via the relaxed dissipation distance $\mathbf{D}$.

A further desirable property for relaxations is the consistency for the time continuous problem:

**(R5)** ***Time-continuous consistency:*** *If $(\boldsymbol{\varphi}, z) : [0,T] \mapsto \mathcal{Y}$ solves (S) & (E), then $\mathcal{J} \circ y : [0,T] \mapsto \mathbf{Y}$ solves $(\mathbb{S})$ & $(\mathbb{E})$; and if $\mathbf{y} : [0,T] \mapsto \mathbf{Y}$ satisfies $\mathbf{y}(t) = \mathcal{J}(y(t))$ and solves $(\mathbb{S})$ & $(\mathbb{E})$, then $y : [0,T] \mapsto \mathcal{Y}$ solves (S) & (E).*

The major question is how suitable relaxations can be constructed. This problem is still unsolved. Following the ideas in [MTL02] the abstract setting in [Mie03b, Mie04a] suggest to do a separate relaxation for $\mathcal{E}$ and $\mathcal{D}$ independently and to use for $\mathbf{Y}$ the set of associated Young measures generated by the convergence "$\xrightarrow{\mathcal{Y}}$" in $\mathcal{Y}$. It is then easy to show that the conditions (R1)–(R3) hold. However, proving the validity of (R4)$_{\text{low}}$ or (R4)$_{\text{upp}}$ is very difficult.

Another way to define relaxations for rate independent problems of the type (S) & (E) is proposed in [The02]. This definition avoids totally the usage of incremental problems but needs instead a sequence of approximation operators $\mathcal{S}_n : \mathbf{Y} \mapsto \mathcal{Y}$ such that:

(R.i) For all $y \in \mathcal{Y}$ we have $\mathcal{S}_n(\mathcal{J}(y)) \xrightarrow{\mathbf{Y}} y$ for $n \to \infty$.
(R.ii) For all $t \in [0,T]$ and all $\mathbf{y} \in \mathbf{Y}$ we have $\mathcal{E}(t, \mathcal{S}_n(\mathbf{y})) \to \mathbf{E}(t, \mathbf{y})$ for $n \to \infty$.
(R.iii) For all $\mathbf{y}_0, \mathbf{y}_1 \in \mathbf{Y}$ we have $\mathcal{D}(\mathcal{S}_n(\mathbf{y}_0), \mathcal{S}_n(\mathbf{y}_1)) \to \mathbf{D}(\mathbf{y}_0, \mathbf{y}_1)$ for $n \to \infty$.

An application of this theory to phase transformations in elastic solids (see also Section 7.3) is given in [The02], where it is also shown that for the problem under consideration the relaxation axioms (R1)–(R3), (R5) and, most importantly, (R4)$_{\text{low}}$ are satisfied. See also [CT03] for a successful relaxation in a special situation in an elastoplastic problem at finite strains.

Formally, the same ideas were used in [MSL02, MTL02, HH03, ML03, KO03, RK04], however, the proofs of the important condition (R4) is missing.

# 6 Non-associated dissipation laws

The above energetic formulations have the major advantage that the dissipational forces are derived from the dissipation potential $\Psi$. In the more nonlinear setting the energetic formulation could be reduced to the stability condition (S) and the energy balance (E). In several applications the dissipational forces are more general and we replace the subdifferential $\partial_v\Psi(y,\dot{y})$ by a more general set $\mathcal{R}(y,\dot{y})$ of dissipation forces. However, we will stay in the framework of rate-independent systems, which means that $\mathcal{R}(y,\cdot)$ is homogeneous of degree 0.

Typical applications in mechanics occur in plastic behavior of materials, in particular for soils (cf. [vVd99, CDTV02]), and in Coulomb friction for elastic bodies, where the dissipation is proportional to the product of the modulus of the sliding velocity and the normal pressure. Several new phenomena occur in such problems and so far the theory is much less developed than for associated flow rules. New types of instabilities and bifurcations occur [MMG94, MK99, MP00, MR02] as well as ill-posedness [VP96]. Some positive results in non-associated plasticity are obtained in [Mró63, BKR98], but they are restricted to the finite-dimensional case of point mechanics.

## 6.1 General setup

We consider a reflexive Banach manifold $\mathcal{Y}$ and assume that for each $t\in[0,T]$ and each $(y,\dot{y})\in \mathrm{T}\mathcal{Y}$ a closed set $\mathcal{R}(t,y,\dot{y})\subset \mathrm{T}^*_y\mathcal{Y}$ for the dissipational forces is given. Rate independence is encoded into the problem by the assumption that $\mathcal{R}(t,y,\cdot)$ is homogeneous of degree 0, i.e.,

$$\forall\,\gamma>0\;\forall\,(t,y,v)\in[0,T]{\times}\mathrm{T}\mathcal{Y}:\;\mathcal{R}(t,y,\gamma v)=\mathcal{R}(t,y,v).$$

In the framework of multi-valued mappings $\mathcal{R}(t,y,\cdot):\mathrm{T}_y\mathcal{Y}\to\mathcal{P}(\mathrm{T}^*_y\mathcal{Y})$ we always find an inverse operator $\mathcal{V}(t,y,\cdot):\mathrm{T}^*_y\mathcal{Y}\to\mathcal{P}(\mathrm{T}_y\mathcal{Y})$ such that

$$\sigma\in\mathcal{R}(t,y,\gamma v)\quad\Longleftrightarrow\quad v\in\mathcal{V}(t,y,\sigma).$$

Rate independence now means that each $\mathcal{V}(t,y,\sigma)$ is a cone, i.e., $\gamma>0$ and $v\in\mathcal{V}(t,y,\sigma)$ imply $\gamma v\in\mathcal{V}(t,y,\sigma)$.

Moreover, the state $y\in\mathcal{Y}$ and the process time $t\in[0,T]$ determine the set of reaction forces $\Sigma(t,y)\subset\mathrm{T}^*_y\mathcal{Y}$, which may also be multi-valued. In the above energetic setting we obviously have $\mathcal{R}(t,y,v)=\partial_v\Psi(y,v)$, $\mathcal{V}(t,y,\sigma)=\partial\mathcal{L}(\Psi(y,\cdot))(\sigma)$ and $\Sigma(t,y)=-\mathrm{D}\mathcal{E}(t,y)$. The problem to be solved is now the following differential inclusion:

$$\begin{aligned}&\text{For given } y_0\in\mathcal{Y} \text{ find } y\in\mathrm{W}^{1,1}([0,T],\mathcal{Y}) \text{ with}\\ &0\in\mathcal{R}(t,y(t),\dot{y}(t))-\Sigma(t,y(t))\subset\mathrm{T}^*_{y(t)}\mathcal{Y} \text{ for a.a. } t\in[0,T],\quad y(0)=y_0.\end{aligned}\tag{6.1}$$

Very often the forces are assumed to have the form $\Sigma(t,y)=\Sigma_0(y)+\ell(t)$, then (6.1) takes the more familiar form

$$\ell(t)\in\mathcal{R}(t,y,\dot{y})-\Sigma_0(y)\subset\mathrm{T}^*_{y(t)}\mathcal{Y}.\tag{6.2}$$

Using the inverse $\mathcal{V}$ we can also write (6.1) as

$$\dot{y}(t)\in\mathcal{V}(t,y(t),\Sigma(t,y(t)))\subset\mathrm{T}_{y(t)}\mathcal{Y},\tag{6.3}$$

where the composition of the multi-valued maps $\mathcal{V}$ and $\Sigma$ is defined via $\mathcal{V}(t, y, \Sigma(t, y)) := \{ v \in \mathrm{T}_y\mathcal{Y} \mid \exists \sigma \in \Sigma(t, y) : \ v \in \mathcal{V}(t, y, \sigma) \}$.

A general theory for equations of the type (6.1) is not to be expected, since only additional structures will enable us to develop a suitable existence and uniqueness theory, see [AC84]. One such structure arises from thermodynamics. The forces in $\mathcal{R}$ are called ***dissipative*** (also called pre-monotone in [Alb98, Che03]), if for all $r \in \mathcal{R}(t, y, v)$ we have $\langle r, v\rangle \le 0$. By 0-homogeneity of $\mathcal{R}$ we may assume that there exists a function $\Psi_{\mathrm{low}} : [0, T]\times\mathrm{T}\mathcal{Y} \to [0, \infty]$ which is 1-homogeneous and satisfies

$$\forall\, (t, y, v) \in [0, T]\times\mathrm{T}\mathcal{Y}\ \forall\, r \in \mathcal{R}(t, y, v) : \ \langle r, v\rangle \ge \Psi_{\mathrm{low}}(v). \tag{6.4}$$

If additionally $\Sigma$ is obtained as a (sub-) differential of an energy functional $\mathcal{E}$, i.e., $\Sigma(t, y) = \mathrm{D}\mathcal{E}(t, y)$, then any solution of (6.1) satisfies the energy inequality

$$\mathcal{E}(t, y(t)) + \textstyle\int_0^t \Psi_{\mathrm{low}}(s, y(s), \dot y(s))\,\mathrm{d}s \le \mathcal{E}(0, y_0) + \int_0^t \partial_s\mathcal{E}(s, y(s))\,\mathrm{d}s.$$

## 6.2 Existence theory

So far, the main approach to an existence theory for such problems is via the theory of monotone operators in Hilbert spaces or accretive operators on general Banach spaces. In Theorem 2.3 we already gave one such result. If $\mathcal{Y}$ is a Hilbert space, $\mathcal{R}(y, v) = \partial\Psi(v)$ with $\Psi : Y \to [0, \infty]$ being 1-homogeneous and weakly continuous, and $\Sigma_0 : Y \to Y^*$ is Lipschitz continuous and strongly monotone, then (6.2) has a solution for suitable initial data.

Note that the theory of monotone operators applied to (6.3) does not give anything new. In fact, if $\mathcal{M} : Y \to \mathcal{P}(Y)$ is a maximal monotone operator such that all sets $\mathcal{M}(y)$ are closed convex cones, then maximality implies that the set $K = \{ y \in Y \mid 0 \in \mathcal{M}(y) \}$ is convex and closed and equals $D(\mathcal{M})$. Moreover, for each $y \in D(\mathcal{M})$ monotonicity implies $\mathcal{M}(y) \subset \mathrm{N}_K(y) = \{ v \in Y \mid \langle v|y{-}\widehat{y}\rangle \ge 0 \text{ for all } \widehat{y} \in K \}$. Hence, maximality implies $\mathcal{M} = \partial I_K$. See [Alb98, Ch. 7] for rate-independent material models, which can be transformed into this setting.

Here we want to discuss a more general result which is based on [Gui00, Che03]. If $y = (\varphi, z) \in F\times Z = Y$ with an elastic, dissipationless part $\varphi$, then usually $\mathcal{R}$ takes the form $\mathcal{R}(\varphi, z, \dot\varphi, \dot z) = \{0\}\times\mathcal{R}_z(z, \dot z) \subset F^*\times Z^*$ and $\Sigma(t, \varphi, z) = \binom{\Sigma_\varphi(t,\varphi,z)}{\Sigma_z(t,\varphi,z)}$. Using the inverse $\mathcal{V}_z$ of $\mathcal{R}_z$, (6.1) may be written in the explicit form

$$0 = \Sigma_\varphi(t, \varphi, z) \in F^*, \quad 0 \in \dot z + \mathcal{V}_z(z, \Sigma_z(t, \varphi, z)) \subset Z^*,$$

see [Che03, eqn. (CC)]. Assuming further that $\Sigma_\varphi(t, \varphi, z) = 0$ can be solved uniquely for $\varphi = \phi(t, z)$, we may insert this into the second equation and we are left with a general differential inclusion

$$0 \in \dot z(t) + B(t, z(t)) \subset Z^*, \quad z(0) = z_0. \tag{6.5}$$

This is a generalized form of (DI) (cf. (2.3)) which reads $0 \in \dot y + \partial I_{-C_*}(Ay{-}\ell(t))$. Moreover, the equation also includes equations of the type $0 \in \partial\Psi(\dot y){+}\Sigma_0(y){-}\ell(t)$, which were treated in Theorem 2.3. For this, just use the Legendre transform to obtain $0 \in \dot y + \partial I_{-C_*}(\Sigma_0(y){-}\ell(t))$. A closely related result was provided in [KM97].

The following result is the abstract version of [Che03, Thm. 2.6].

**Theorem 6.1** *Let $Z$ be a Hilbert space and $C_*$ a closed convex subset of $Z^*$. Moreover, assume that $\Phi : Z^* \to Z^*$ is a $\mathrm{C}^{1,\mathrm{Lip}}$ diffeomorphism (i.e., $\Phi$, $\Phi^{-1}$, $\mathrm{D}\Phi$ and $\mathrm{D}\Phi^{-1}$ exist and are globally Lipschitz continuous). Moreover, let $A : Z \to Z^*$ be bounded, symmetric and positive definite (as in Section 2). Finally assume $\ell \in \mathrm{C}^{1,\mathrm{Lip}}([0,T], Z^*)$. If $B$ in* (6.5) *has the form $B(t,z) = \mathcal{B}(Az-\ell(t))$ with the multi-valued map $\mathcal{B}(\sigma) = \mathrm{D}\Phi(\sigma)^*\partial I_{-C_*}(\Phi(\sigma))$, then* (6.5) *has for each $z_0$ with $0 \in \Phi(Az_0-\ell(0))+C_*$ a unique solution $z \in \mathrm{C}^{\mathrm{Lip}}([0,T], Z)$.*

Note that the equation has the form $0 \in \dot{z} + \mathrm{D}\Phi(\sigma)^*\partial I_{-C_*}(\Phi(\sigma))$ where $\sigma = Az-\ell(t)$. Using $\Psi = \mathcal{L}I_{C_*}$ this can be rewritten by the Legendre transform as

$$-\Phi(\sigma) \in \partial\Psi(\mathrm{D}\Phi(\sigma)^{-*}\dot{z}) \quad \Longleftrightarrow \quad 0 \in \mathcal{R}(\sigma, \dot{z}) + \sigma$$
$$\text{with } \mathcal{R}(\sigma, v) = \Phi^{-1}\big(\partial\Psi(\mathrm{D}\Phi(\sigma)^{-*}v)\big) \text{ and } \sigma = Az-\ell(t).$$

We see here, that $\mathcal{R}(\sigma, v)$ is obtained by applying $\Phi^{-1}$ to the convex set $\partial\Psi(\mathrm{D}\Phi(\sigma)^{-*}v)$, which means that $\mathcal{R}(\sigma, v)$ is not convex in general.

The ***Skorokhod problem*** forms another class of rate-independent systems with non-associated flow rules, see [KV01, KV03]. It is classically formulated in a Hilbert space using its scalar product, however, to stay consistent with the previous formulations we use our general notation where $Y$ is a Hilbert space with dual $Y^*$ and $A : Y \to Y^*$ is a positive definite isomorphism. As in (2.29) we start with a polyhedral convex set

$$C_* = \{\, \sigma \in Y^* \mid \forall\, j = 1, \ldots, K : \ \langle \sigma, n_j \rangle \leq \beta_j \,\} \subset Y^*,$$

with $\beta_j \geq 0$ and normal vectors $n_j \in Y \backslash \{0\}$.

In contrast to the classical subdifferential equation (SF) or the classical differential inclusion (DI) $\dot{y} \in \mathrm{N}_{C_*}(\ell(t)-Ay)$, which is treated in Theorem 2.8, we do not use the friction law $\partial(\mathcal{L}I_{C_*})$ but generalize it as follows. We define the multi-valued operator $\mathcal{J} : Y^* \to \mathcal{P}(\{1, ..., K\})$ of active indices via

$$\mathcal{J}(\sigma) = \{\, j \in \{1, ..., K\} \mid \langle \sigma, n_j \rangle = \beta_j \,\} \text{ for } \sigma \in C_* \quad \text{and} \quad \mathcal{J}(\sigma) = \emptyset \text{ for } \sigma \notin C_*.$$

Moreover, the reflection cone $\mathcal{V} : Y^* \to \mathcal{P}(Y)$ is given via vectors $m_1, \ldots, m_K \in Y \backslash \{0\}$ as

$$\mathcal{V}(\sigma) = \{\, \textstyle\sum_{j \in \mathcal{J}(\sigma)} \mu_j m_j \mid \mu_j \geq 0 \,\}.$$

Note that $m_j = n_j$ for all $j$ implies $\mathcal{V}(\sigma) = \partial I_{C_*}(\sigma)$. The inverse $\mathcal{R}$ of $\mathcal{V}$ reads

$$\mathcal{R}(v) = \{\, \sigma \in C_* \mid \exists\, \mu_j \geq 0 : \ v = \textstyle\sum_{j \in \mathcal{J}(\sigma)} \mu_j m_j \,\}.$$

The Skorokhod problem can now be written in the following two equivalent and dual forms:

$$0 \in \mathcal{R}(\dot{y}(t)) + Ay(t) - \ell(t) \subset Y^* \quad \text{or} \quad \dot{y}(t) \in \mathcal{V}(\ell(t)-Ay(t)) \subset Y. \tag{6.6}$$

We give an example of such a system in the next subsection.

Without loss of generality it is possible to assume further on that $Y$ is finite-dimensional and equal to $\mathrm{span}\{m_1, ..., m_K, n_1, ..., n_k\}$, since the $A$-orthogonal complement can be decoupled like at the end of Section 2.4. The crucial property which has to be satisfied by the vectors $\{m_1, ..., m_K, n_1, ..., n_k\}$ is

$$\langle Am_j, n_j\rangle > 0 \quad \text{for } j = 1, \dots, K, \tag{6.7}$$

and $\ell$-paracontractivity. The set $\{\, Q_j \mid j = 1, \dots, K \,\}$ containing the projections

$$Q_j : Y \to Y; y \mapsto \tfrac{\langle Ay, n_j\rangle}{\langle Am_j, n_j\rangle}\, m_j,$$

is called $\ell$-paracontracting, if there exist a norm $\|\cdot\|$ on $Y$ and a constant $\gamma > 0$ such that

$$\forall\, y \in Y\ \forall\, j = 1, \dots, K :\ \|Q_j y\| + \gamma\|Q_j y - y\| \le \|y\|. \tag{6.8}$$

The following results are established in [KV01, Thm. 3.1 & Thm. 5.8]. Further results can be found in [KV03], where the case of time-dependent $\beta_j$ is considered.

**Theorem 6.2** *Let $Y$ and $m_1, ..., m_K, n_1, ..., n_k$ be given as above and such that* (6.7) *and* (6.8) *hold. Then, for each $\ell \in \mathrm{W}^{1,1}([0,T], Y^*)$ and each $\sigma_0 \in C_*$ problem* (6.6) *has a solution $y$ with $y(0) = A^{-1}(\ell(0) - \sigma_0)$ and $y \in \mathrm{W}^{1,1}([0,T], Y)$.*

*Under the additional transversality condition*

$$\forall\, J' \subset \{1, \dots, K\} :\ \dim \operatorname{span}\{\, n_j \mid j \in J' \,\} = \dim \operatorname{span}\{\, m_j \mid j \in J' \,\}$$

*the solution is unique and $(\sigma_0, \ell) \mapsto y$ is Lipschitz continuous from $C_* \times \mathrm{W}^{1,1}([0,T], Y^*)$ into $\mathrm{W}^{1,1}([0,T], Y)$ as well as from $C_* \times \mathrm{C}^0([0,T], Y^*)$ into $\mathrm{C}^0([0,T], Y)$.*

**Example 6.3** *This simple example from queuing theory is taken from [KV01, Sect. 8], and it has the form of a Skorokhod problem.*

*With $w_{\mathrm{ord}}$ and $w_{\mathrm{priv}}$ we denote the number of ordinary and privileged customers waiting at a service point whose set of possible states is*

$$W = \{\, w = (w_{\mathrm{ord}}, w_{\mathrm{priv}}) \in [0,1]^2 \mid w_{\mathrm{ord}} - w_{\mathrm{priv}} \le c_{tot} \,\},$$

*where $c_{tot} > 0$ is the total capacity of the waiting room. The customers which arrived in the time interval $[0,t]$ is the input $\widetilde{\ell}(t) = (\widetilde{\ell}_{\mathrm{ord}}(t), \widetilde{\ell}_{\mathrm{priv}}(t))$ and the number of customers which left the service point during $[0,t]$ is $\widetilde{y}(t)$, served or refused because of missing capacity. Thus, we have $\widetilde{y} + w = \widetilde{\ell}$. There are the two counters: O for ordinary customers and P for privileged customers. The following service rules apply:*
*(i) All customers are served at their respective counters, which work with their maximal capacities $c_{\mathrm{ord}}$ and $c_{\mathrm{priv}}$, respectively, as long as there are customers.*
*(ii) If there is unused capacity at counter O, then it can be used by privileged customers.*
*(iii) If the waiting room is full, then for each refused privileged customer, there must be at least $\rho$ refused ordinary customers, where $\rho > 0$ is fixed.*

*Define $c = (c_{\mathrm{ord}}, c_{\mathrm{priv}})$ and the final variables $y(t) = \widetilde{y}(t) - tc$ and $\ell(t) = \widetilde{\ell}(t) - tc$, such that $w(t) = \ell(t) - y(t)$. The evolution of $y$ can be formulated as*

$$\dot{y}(t) \in B(\partial I_W[\ell(t) - y(t)]) \text{ for a.a. } t \in [0,T], \quad y(0) = y_0,$$

*where $B$ maps the cones $\{\, \alpha n_j \mid \alpha \geq 0 \,\}$ into the cones $\{\, \beta m_j \mid \beta \geq 0 \,\}$, where $n_1 = (-1, 0)$, $n_2 = (0, -1)$ and $n_3 = (1, 1)$ are the normal vectors at the edges of $W$ and $m_1 = (-1, 0)$, $m_2 = (1, -1)$ and $m_3 = (\rho, 1)$ are the reflection vectors.*

*The above theorem can be applied to show that this problem has a unique solution for each $\ell \in \mathrm{C}^{\mathrm{Lip}}([0, T], \mathbb{R}^2)$ and each $y_0 \in \ell(0) - W$.*

## 6.3 Dry friction on surfaces

The most important problem with non-associated flow law is that of dry friction of elastic bodies on surfaces. There are two mostly disjoint areas. The first case is the finite-dimensional one which involves a structure composed of rigid bodies which are connected with elastic interactions and may slide along given surfaces. The second case concerns an elastic body which touches a surface along a part of its boundary and which is assumed to have only small deformations such that linearized elasticity theory and linearized contact laws can be used. However, in both cases the difficulty arises that the tangential frictional force is proportional to the normal pressure. See [MK99, AK01, MR02] for surveys in this area.

In the first case the state of the structure is given by an element $y$ of a smooth, finite-dimensional manifold $\widetilde{\mathcal{Y}}$. The contact surfaces are modeled via smooth constraints $c_j : \widetilde{\mathcal{Y}} \to \mathbb{R}$, $j = 1, \ldots, p$, such that the state space is given by

$$\mathcal{Y} = \{\, y \in \widetilde{\mathcal{Y}} \mid c_j(y) \leq 0 \text{ for } j = 1, \ldots, p \,\}.$$

We assume that the derivatives $\mathrm{D}c_j(y)$ do not vanish on the boundary pieces $\Gamma_j = \mathcal{Y} \cap \{c_j(y) = 0\}$. Hence, the (outward) unit normal vectors $n_j(y) \in \mathrm{T}^*_y\mathcal{Y}$ exist on $\Gamma_j$. Several of the sets $\Gamma_j$ may intersect in a lower-dimensional manifold, which just means that several bodies of the structure are in contact.

The elastic interactions between the bodies are given through a smooth, time-dependent energy functional $\mathcal{E} : [0, T] \times \mathcal{Y} \to \mathbb{R}$. For simplicity, we assume that there are no frictional forces other than the one arising if $y(t)$ touches the boundary $\partial\mathcal{Y} = \Gamma = \cup_{j=1}^p \Gamma_j$. For $(y, v) \in \mathrm{T}_y\Gamma$ we denote by $\mathcal{R}(y, v)$ the set of possible reaction forces of the boundary at the given velocity $v$. For $y$ in the interior $\mathrm{int}(\mathcal{Y}) = \mathcal{Y} \setminus \Gamma$ of $\mathcal{Y}$ we simply set $\mathcal{R}(y, v) = \{0\}$. Then, the rate-independent friction problem takes the form

$$0 \in \mathcal{R}(y(t), \dot{y}(t)) + \mathrm{D}\mathcal{E}(t, y(t)) \subset \mathrm{T}^*_{y(t)}\widetilde{\mathcal{Y}}, \tag{6.9}$$

and the friction law is implemented through specifying $\mathcal{R}$.

For each contact point $y \in \Gamma_j$ we specify a static friction cone $\mathcal{R}_j(y) \subset \mathrm{T}^*_y\widetilde{\mathcal{Y}}$ which is closed, convex and contains $n_j(y)$. If a single body $y^j \in \mathbb{R}^d$ is in contact, this is usually done by decomposing the reaction forces $r^j \in \mathrm{T}^*_{y^j}\mathbb{R}^d$ into a tangential part $r^j_{\mathrm{t}}$ and a normal part $r^j_{\mathrm{n}} = \alpha n_j(y)$ and by setting

$$\mathcal{R}^j(y) = \{\, r^j = r^j_{\mathrm{t}} + r^j_{\mathrm{n}} \mid |r^j_{\mathrm{t}}| \leq \mu^j(y) r^j_{\mathrm{n}} \,\} \subset \mathrm{T}^*_{y^j}\mathbb{R}^d,$$

where $\mu^j(y) \geq 0$ is the coefficient of (isotropic) friction for the $j$-th body. To obtain now $\mathcal{R}_j(y) \subset \mathrm{T}^*_y\widetilde{\mathcal{Y}}$ we simply fill in $0$ for all reaction forces of the other bodies.

For situations in which $y \in \Gamma$ has several contacts, we make the assumptions that the different contacts do not influence each other. To describe this mathematically, we extend the vectors $n_j : \Gamma_j \to \mathrm{T}^*\mathcal{Y}$ and the cones $\mathcal{R}_j(y) \subset \mathrm{T}^*_y\mathcal{Y}$ to all of $\mathcal{Y}$ by $0$ and $\{0\}$, respectively. The tangential directions $\mathcal{T}(y)$ and the outward normal cones $\mathcal{N}(y)$ are

$$\mathcal{T}(y) = \{\, v \in \mathrm{T}_y\widetilde{\mathcal{Y}} \mid \langle n_j(y), v\rangle = 0 \text{ for } j = 1, \ldots, p \,\}, \quad \mathcal{N}^*(y) = \{\, \textstyle\sum_{j=1}^p \alpha_j n_j(y) \mid \alpha_j \geq 0 \,\}.$$

Additionally, we prescribe at each $y \in \Gamma$ a projection $P(y)$ which maps $\mathrm{T}_y\widetilde{\mathcal{Y}}$ onto $\mathcal{T}(y)$. The adjoint projector $P(y)^*$ has the kernel $\mathrm{span}(\mathcal{N}(y))$ and it decomposes reaction forces $r \in \mathrm{T}^*_y\widetilde{\mathcal{Y}}$ into its tangential part $r_\mathrm{t} = P(y)^* r$ and its normal part $r_\mathrm{n} = r - r_\mathrm{t} \in \mathcal{N}(y)$. With this, we define the total static friction cone as the sum

$$\mathcal{R}^*(y) = \textstyle\sum_{j=1}^p \mathcal{R}_j(y) = \{\, \sum_{j=1}^p r_j \mid r_j \in \mathcal{R}_j(y) \,\}$$

of the cones $\mathcal{R}_j(y)$, which gives again a closed, convex cone with $\mathcal{N}^*(y) \subset \mathcal{R}^*(y)$, and the velocity-dependent friction cone via

$$\mathcal{R}(y, v) = \begin{cases} \{\, r \in \mathcal{R}^*(y) \mid v \in P(y)\mathrm{N}_{\mathcal{R}^*(y)}(r) \,\} & \text{if } v \in \mathcal{T}(y), \\ \{0\} & \text{if } v \notin \mathcal{T}(y). \end{cases}$$

In particular, we have $\mathcal{R}(y, 0) = \mathcal{R}^*(y)$ for the sticking particle. However, sliding can only occur in that direction where the critical tangential force (relative to the normal force) is reached.

The easiest example is $\widetilde{\mathcal{Y}} = \mathbb{R}^3$, $c_1(y) = y_3$ and $\mathcal{R}_1((y_1, y_2, 0)) = \{\, r \mid (r_1^2 - r_2^2)^{1/2} \leq \mu r_3 \,\}$ and gives the time-dependent friction cone

$$\mathcal{R}((y_1, y_2, 0), v) = \begin{cases} \{0\} & \text{for } v_3 \neq 0, \\ \mathcal{R}_1(y_1, y_2, 0) & \text{for } v = 0, \\ \{\, \alpha(-\mu v_1, -\mu v_2, |v|) \mid \alpha \geq 0 \,\} & \text{for } v = (v_1, v_2, 0) \text{ with } |v| > 0. \end{cases} \tag{6.10}$$

Thus, it can be easily seen that there exists no $\Psi : \mathbb{R}^3 \to [0, \infty]$ such that $\mathcal{R}(0, v) = \partial\Psi(v)$.

There is a substantial body of work for this type of finite-dimensional friction problems, however, in most cases the inertia terms are used to regularize the problem, i.e., an equation like $0 \in M(t, y)\ddot{y} + \mathcal{R}(t, y, \dot{y}) + \mathrm{D}\mathcal{E}(t, y)$ is considered. We refer to [MK99, MR02] for surveys and to [AK97, MPS02, PM03, MR02, MMP04] for some relevant mathematical work. In [GMM98] it was shown that in quasistatic problems even in simple linear systems we have to expect jumps in the solution.

There is a way to reformulate the problem such that it almost looks like a rate-independent system with a dissipation potential. Using the decomposition $\mathrm{T}^*_y\widetilde{\mathcal{Y}} = \mathrm{span}(\mathcal{N}(y)) \oplus \mathcal{T}^*(y)$ with $\mathcal{T}^*(y) = P(y)^*\mathrm{T}^*_y\widetilde{\mathcal{Y}}$ we define, for $y \in \mathcal{Y}$ and $r_n \in \mathcal{N}^*(y)$, the set of possible tangential forces via

$$C_*(y, r_\mathrm{n}) = \{\, r_\mathrm{t} \in \mathcal{T}^*(y) \mid r_\mathrm{n} + r_\mathrm{t} \in \mathcal{R}^* y \,\} = P(y)^*\big(\mathcal{R}^*(y) \cap \{\, r \mid (\mathbf{1} - P(y))^* r = r_\mathrm{n} \,\}\big).$$

Using this set we use the Legendre transform on $\mathcal{T}(y)$ to define the dissipation functional $\Psi(y, r_\mathrm{n}, \cdot) : \mathrm{T}_y\widetilde{\mathcal{Y}} \to [0, \infty]$ via

$$\Psi(y, r_\mathrm{n}, v) = [\mathcal{L} I_{C_*(y, r_\mathrm{n})}](P(y)v) = \sup\{\, \langle r_\mathrm{t}, P(y)v\rangle \mid r_\mathrm{t} \in C_*(y, r_\mathrm{n}) \,\}.$$

For $r_{\rm n} \notin \mathcal{N}(y)$ we have $C_*(y,r_{\rm n}) = \emptyset$ and hence $\Psi(y,r_{\rm n},\cdot) \equiv \infty$. Some elementary calculations show that if $v = P(y)v$, then $r \in \mathcal{R}(y,v)$ is equivalent to $r_{\rm t} \in \partial_v \Psi(y,r_{\rm n},y)$. Thus, the friction laws are reduced to an associated flow law (a principle of maximal dissipation) in the tangential direction, if the normal forces are considered to be given.

Note that $\Psi$ is defined for all velocities, but only the tangential part $v_{\rm t} = P(y)v$ contributes. Thus, $\partial\Psi$ always includes the whole space $\mathrm{span}(\mathcal{N}(y))$, which is useful in the following equivalent rewriting of (6.9):

$$0 \in \partial\Psi(y(t),\sigma_{\rm n}(t),\dot y(t)) - \sigma(t) \subset \mathrm{T}^*_y\widetilde{\mathcal{Y}}, \quad \text{where } \sigma(t) = -\mathrm{D}\mathcal{E}(t,y(t)) \text{ and } \sigma_{\rm n} = (\mathbf{1}-P(y))^*\sigma.$$

Thus, the structure is somewhat similar to the case of general dissipation metrics. However, the main difficulty coming into play here is that the function $\Psi$ which is built using the normal vectors $n$ and the projections $P$ is not continuous. Whenever a new contact arises or a contact disappears, then there are jump discontinuities.

**Example 6.4** *In [Mon93, Sect. 5.3] the following friction problem is solved. Let* $\widetilde{\mathcal{Y}} = \mathbb{R}^3$ *and* $c_1(y) = y_3$ *which gives* $\mathcal{Y} = \{\, y \in \mathbb{R}^3 \mid y_3 \leq 0 \,\}$. *As an energy functional we choose* $\mathcal{E}(t,y) = \frac{\alpha}{2}|y-\ell(t)|^2$, *where* $|\cdot|$ *denotes the Euclidean norm. The friction law on* $\Gamma = \partial\mathcal{Y} = \{y_3 = 0\}$ *is defined via* (6.10) *and the friction coefficient function* $\mu : \Gamma \to (0,\mu_{max}]$ *with* $|\mu(y)-\mu(\widetilde y)| \leq \beta|y-\widetilde y|$ *for* $y,\widetilde y \in \Gamma$.

*It is proved that the corresponding friction problem has for each loading* $\ell \in \mathrm{W}^{1,1}([0,T];\mathbb{R}^3)$ *and each equilibrated (i.e., stable) state* $y^0$ *a solution* $y$ *with* $y(0) = y^0$ *and* $y \in \mathrm{W}^{1,1}([0,T];\mathbb{R}^3)$, *if additionally the smallness condition* $\beta\|\ell_3(\cdot)\|_\infty < 1$ *holds. Here a state* $y^0$ *is called equilibrated with* $\ell(0)$, *if for* $\ell_3(0) \leq 0$ *we have* $y^0 = \ell(0)$ *and for* $\ell_3(0) > 0$ *we have* $y_3^0 = 0$ *and* $|y^0-(\ell_1(0),\ell_2(0),0)| \leq \ell_3(0)\mu(y^0)$.

Counterexamples to uniqueness and existence of solutions are given in [Kla90, Bal99], and [GMM98, And95] provides examples in which the solutions are in general not continuous. However, general systems of the type described above need much further study.

The second class of friction problems involves a linearly elastic body, which may touch a surface with parts of its boundary. Throughout we assume small displacements, since the general case seems out of reach at the present stage of research. The first major steps in this field were done in [DL76], where the static problem was solved and simplified evolution variational inequalities were considered. The time-dependent problem was first studied including inertia terms and sometimes viscoelastic damping, which keep the solution from making undesirable jumps, see [MO87, Kut97, Eck02, EJ03]. Here we restrict ourselves to the rate-independent case, which is usually called the quasistatic case in contrast to the dynamic case.

The system consists of the elastic bulk energy $\frac12\langle Au,u\rangle - \langle \ell(t),u\rangle$, where $A : Y \to Y^*$ is the usual symmetric elastic operator with $Y = \mathrm{H}^1_{\Gamma_{\rm D}}(\Omega) = \{\, u \in \mathrm{H}^1(\Omega;\mathbb{R}^d) \,|\, u|_{\Gamma_D} = 0 \,\}$, where $\Omega \subset \mathbb{R}^d$ is a domain with Lipschitz boundary and $\Gamma_{\rm D} \subset \partial\Omega$, and $\langle Au,u\rangle = \int_\Omega C_{ijkl}\partial_i u_j \partial_k u_l \,\mathrm{d}x \geq c\|u\|^2_{\mathrm{H}^1}$ for some $c > 0$. At a contact part $\Gamma_{\rm c} \subset \partial\Omega$, which has positive distance from $\Gamma_{\rm D}$, the body may touch a given obstacle which is prescribed by the function $g : \Gamma_{\rm c} \to \mathbb{R}$. Let $\nu$ be the normal vector on $\partial\Omega$, then there is no contact, if the normal component $u_{\rm n} = u\cdot\nu$ satisfies $u_{\rm n} < g$. Contact means that the penetration depth $u_{\rm n} - g$ is nonnegative. Note that the tangential displacement $u_\mathrm{t} = u - u_\mathrm{n}\nu$ is not involved in the contact condition, since we are in a situation of small displacements.

The normal stress vector $\sigma \in \mathbb{R}^d$ and its normal and tangential components at a point $x \in \Gamma_\mathrm{c}$ are defined via

$$\sigma[u] = (\textstyle\sum_{ijk} C_{ijkl}\partial_i u_j \nu_k)_{l=1,\ldots,d}, \quad \sigma[u]_\mathrm{n} = \sigma[u]\cdot\nu \quad \text{and} \quad \sigma[u]_\mathrm{t} = \sigma[u] - \sigma[u]_\mathrm{n}\nu.$$

In the case of normal compliance, one assumes that the penetration depth can become positive due to some elastic behavior of the obstacle. This induces a normal stress according to a compliance law

$$-\sigma[u]_\mathrm{n} = H(x, u_\mathrm{n} - g), \quad \text{where } H(x,\delta) = 0 \text{ for } \delta \le 0.$$

Usually one chooses $H(x,\delta) = \lambda(x)\max\{0,\delta\}^m$ for suitable parameters $\lambda, m > 0$. Hard Coulomb friction is modeled via $H(\delta) = +\infty$ for $\delta > 0$. Associated with this elasticity law is the functional

$$\mathcal{H}(u) = \textstyle\int_{\Gamma_\mathrm{c}} h(x, u(x) - g(x))\,\mathrm{d}a(x), \quad \text{where } h(x,u) = \int_0^u H(x,\delta)\,\mathrm{d}\delta.$$

The total stored energy now defines the energy functional

$$\mathcal{E}_\mathcal{H}(t,u) = \tfrac{1}{2}\langle Au, u\rangle - \langle \ell(t), u\rangle + \mathcal{H}(u).$$

As in the rigid-body case, the friction law is now specified best by a local dissipation function $\psi$ in the form

$$\psi(x, \sigma_\mathrm{n}, v) = \begin{cases} \infty & \text{for } \sigma_\mathrm{n} > 0, \\ -\sigma_\mathrm{n}\,\mu(x)|v_\mathrm{t}| & \text{for } \sigma_\mathrm{n} \le 0. \end{cases}$$

The friction law now asks the stress vector $\sigma$ and the velocity $v$ to satisfy $\sigma_\mathrm{t} \in \partial\Psi(x, \sigma_\mathrm{n}, v)$.

Thus, the whole problem can be written as a variational inequality using the stress-dependent dissipation functional

$$\Psi(\sigma_\mathrm{n}; v) = \textstyle\int_{\Gamma_\mathrm{c}} \psi(x, \sigma_\mathrm{n}(x), v(x))\,\mathrm{d}a(x)$$

in the following way:

$$\langle \mathrm{D}\mathcal{E}_\mathcal{H}(t,u), v - \dot{u}\rangle + \Psi(\sigma[u]_\mathrm{n}; v) - \Psi(\sigma[u]_\mathrm{n}; \dot{u}) \ge 0 \quad \text{for all } v \in Y = \mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega). \tag{6.11}$$

This model with $H(\delta) = \lambda\max\{0,\delta\}^m$ and $m \in [1, d/(d-2))$ was treated in a series of papers [And91, And95, And99]. Under the assumption of small friction ($\|\mu\|_{\mathrm{L}^\infty} \ll 1$) existence of solutions is shown. The approach follows exactly the one explained in Section 3.6 for general state-dependent dissipation metrics. The smallness of the friction coefficient corresponds to the smallness of $\psi_*$ in (3.16), which controls the deviation from the convex part obtained from the energy. Roughly spoken, the result is the one which one expects, namely that for each loading $\ell = (f_\mathrm{vol}, f_\mathrm{surf}) \in \mathrm{W}^{1,1}([0,T]; \mathrm{L}^2(\Omega;\mathbb{R}^d) \times \mathrm{H}^{-1/2}(\partial\Omega;\mathbb{R}^d))$ with $\ell(0) = 0$ there exists a function $u \in \mathrm{W}^{1,1}([0,T]; Y)$ with $u(0) = 0$ such that (6.11) holds a.e. on $[0,T]$.

The case of a real hard unilateral constraint with $h(\delta) = \infty$ for $\delta > 0$ is handled in [And00], again using the smallness of the friction coefficient $\mu$. Studying the solutions of the compliance problem (6.11) for the compliance parameter $\lambda$ tending to $\infty$, it is shown that the Coulomb friction problem has also a solution. Defining

$$K_g = \{\, u \in Y \mid u_\mathrm{n}|_{\Gamma_\mathrm{c}} \leq g \,\} \quad \text{and} \quad \mathcal{E}_0(t,u) = \tfrac{1}{2}\langle Au, u\rangle - \langle \ell(t), u\rangle,$$

the variational inequality now reads with $Y = \mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)$:

$$\begin{array}{l} \forall\, v \in Y:\ \langle \mathrm{D}\mathcal{E}_0(t,u), v - \dot u\rangle - \langle \sigma_\mathrm{n}[u], v_\mathrm{n} - \dot u_\mathrm{n}\rangle - \Psi(\sigma[u]_\mathrm{n}; v) - \Psi(\sigma[u]_\mathrm{n}; \dot u) \geq 0, \\ \forall\, w \in K_g:\ \langle \sigma_\mathrm{n}[u], w_\mathrm{n} - u_\mathrm{n}\rangle \geq 0. \end{array} \tag{6.12}$$

In fact, both inequalities can be put into one equation by introducing the potential $\mathcal{E}_\infty(t,y) = \mathcal{E}_0(t,u) + I_{K_g}(u)$:

$$0 \in \partial\Psi(\sigma[u]_\mathrm{n}; \dot u) + \partial\mathcal{E}_\infty(t,u).$$

After doing some slight modifications and specifying the assumptions fully, it is shown in [And00] that (6.12) has for each $\ell = (f_\text{vol}, f_\text{surf})$ and for each suitable initial data a solution $u \in \mathrm{W}^{1,1}([0,T], Y)$ with $u(t) \in K_g$ for all $t \in [0,T]$.

The case of large friction coefficient $\mu$ and normal compliance is handled in [And95]. For this the solution concept needs to be modified, since solutions will no longer be continuous and the variational inequality (6.11) has to be replaced by a more energetic formulation.

# 7 Applications to continuum mechanics

To unify the presentation of the applications in continuum models we refrain from the full generality and restrict ourselves to standard situations like, for instance, simple (dead) loadings and time-independent Dirichlet boundary conditions. For time-dependent boundary conditions we refer to [FM04], where they are treated with similar ideas as explained in Section 5.5.

Throughout we will consider a body $\Omega \subset \mathbb{R}^d$ with $d \in \{1,2,3\}$, which is open, bounded and has a Lipschitz boundary such that integration by parts and Sobolev embeddings are available. The deformation is $\varphi : \Omega \to \mathbb{R}^d$ and we will use $u : \Omega \to \mathbb{R}^d; x \mapsto \varphi(x) - x$ to denote the displacement in the case of linearized elasticity. For the general situation we use $y : \Omega \to \mathbb{R}^d$ to denote $\varphi$ or $u$. In addition, there will be an internal variable $z : \Omega \to Z \subset \mathbb{R}^m$. The two constitutive functions are the stored-energy density (stress potential) $W : \Omega \times \mathbb{R}^{d\times d} \times Z \to \mathbb{R}_\infty$ and the dissipation potential $\psi : \Omega \times \mathrm{T}Z \to [0,\infty]$. The latter generates the dissipation distance $D : \Omega \times Z \times Z \to [0,\infty]$ such that the functionals have the form

$$\begin{array}{rcl} \mathcal{E}(t,y,z) & = & \int_\Omega W(x, \mathrm{D}y(x), z(x)) + \frac{\kappa}{r}|\mathrm{D}z(x)|^r \,\mathrm{d}x - \langle \ell(t), y\rangle, \\ \mathcal{D}(z_0,z_1) & = & \int_\Omega D(x, z_0(x), z_1(x))\,\mathrm{d}x, \end{array} \tag{7.1}$$

where $t \mapsto \ell(t)$ denotes the loading which is considered as input data. We add the regularizing term $\frac{\kappa}{r}|\mathrm{D}z|^r$ with suitable $r > 1$ which is also called “nonlocal” in mechanics terminology. For $\kappa > 0$ it provides helpful compactness properties.

Throughout the following subsections we will assume that the deformations or displacements are taken from a space $F = \mathrm{W}^{1,p}_{\Gamma_\mathrm{Dir}}(\Omega; \mathbb{R}^d)$ (denoted $\mathrm{H}^1_{\Gamma_\mathrm{Dir}}(\Omega;\mathbb{R}^d)$ for $p = 2$), where

$\Gamma_{\rm Dir} \subset \partial\Omega$ is such that in the case of linearized elasticity Korn's inequality holds in $F$, i.e., there exists a constant $c > 0$ such that

$$\forall\, u \in F: \ \|\varepsilon(u)\|_{\mathrm{L}^p} \geq c\|u\|_{\mathrm{W}^{1,p}} \quad \text{where } \varepsilon(u) = \tfrac{1}{2}(\mathrm{D}u + (\mathrm{D}u)^{\mathsf{T}}).$$

Here, $\varepsilon(u) \in \mathbb{R}^{d\times d}_{\rm sym}$ is called the linearized strain tensor. In the case of finite elasticity, we only impose Poincaré's inequality:

$$\forall\, \varphi \in F: \ \|\mathrm{D}\varphi\|_{\mathrm{L}^p} \geq c\|\varphi\|_{\mathrm{W}^{1,p}}.$$

The different applications below differ in the form of the variable $z$ and in the nonlinearities or nonconvexities in the functions $W$ and $D$. For instance in elastoplasticity $z$ will contain a plastic tensor in $\mathbb{R}^{d\times d}$ as well as hardening variables (Sections 7.1 and 7.2), in shape-memory materials $z \in \{\, z \in [0,1]^m \mid \sum_1^m z^{(j)} = 1 \,\}$ contains the portions of the phases (Section 7.3), in ferromagnetic materials $z \in \mathbb{S}^{d-1}$ is the magnetization (Section 7.4), and in damage problems $z \in [0,1]$ denotes the proportion of intact material ([MR04a]) . There are also applications, where $z$ is not defined on all of $\Omega$ but only on hypersurfaces like in delamination (Section 7.5) and in crack propagation (Section 7.6).

## 7.1 Linearized elastoplasticity

The theory of linearized elastoplasticity has been the major driving force for the theory of rate-independent systems since the mid 1970s. So we give the main structure of the theory, but refer to the huge list of references for further details, see, e.g., [HR99] for a recent monograph.

A bounded body $\Omega \subset \mathbb{R}^d$ is subject to small deformations which are described by the displacement $u : \Omega \to \mathbb{R}^d$. As internal variables we have $z = (\varepsilon_{\rm pl}, q)$, where $\varepsilon_{\rm pl} \in \mathbb{R}^{d\times d}_0 = \{\, A \in \mathbb{R}^{d\times d} \mid \mathrm{tr}\, A = 0,\ A = A^{\mathsf{T}} \,\}$ is the plastic strain and $q \in \mathbb{R}^n$ denotes hardening variables. The stored-energy functional has the form

$$\mathcal{E}(t, u, \varepsilon_{\rm pl}, q) = \int_\Omega \tfrac{1}{2}\mathbb{A}(\varepsilon(u){-}\varepsilon_{\rm pl}){:}(\varepsilon(u){-}\varepsilon_{\rm pl}) + Q(\varepsilon_{\rm pl}, q)\,\mathrm{d}x - \langle \ell(t), u\rangle,$$

where $\mathbb{A}$ is the (fourth-order) elastic tensor and $Q : \mathbb{R}^{d\times d}_0 {\times} \mathbb{R}^n \to [0,\infty)$ is a quadratic form which describes hardening effects.

The dissipation functional $\Psi$ takes the form $\Psi(\dot{\varepsilon_{\rm pl}}, \dot q) = \int_\Omega \psi(\dot{\varepsilon_{\rm pl}}(x), \dot q(x))\,\mathrm{d}x$, where $\psi : \mathbb{R}^{d\times d}_0 {\times} \mathbb{R}^n \to [0,\infty)$ is convex, 1-homogeneous and coercive.

Under the additional assumption that there is enough hardening, i.e., the quadratic form $Q$ is coercive, it is quite standard to apply the existence results of Section 2.3. For this we choose

$$Y = \mathrm{H}^1_{\Gamma_{\rm Dir}}(\Omega;\mathbb{R}^d) {\times} \mathrm{L}^2(\Omega;\mathbb{R}^{d\times d}_0) {\times} \mathrm{L}^2(\Omega;\mathbb{R}^n).$$

However, in most plasticity models the hardening is weaker. In particular, we have $Q(\varepsilon_{\rm pl}, q) = Q(0, q)$, such that the energy is not coercive in the variables $\varepsilon(u)$ and $\varepsilon_{\rm pl}$, but only in their difference $\varepsilon(u){-}\varepsilon_{\rm pl}$. In this situation, the dissipation can be used to control $\varepsilon_{\rm pl}$ as well, but more careful bookkeeping is necessary, see [Joh76, Suq81, HR95, AC00]. To explain the general approach we restrict ourselves to the case of von Mises plasticity, where $q$ is a scalar hardening variable and the dissipation potential takes the form

$$\psi(\dot{\varepsilon_{\rm pl}}, \dot q) = s_2|\dot{\varepsilon_{\rm pl}}| \text{ for } \dot q \geq s_1|\dot{\varepsilon_{\rm pl}}| \quad \text{and} \quad \psi(\dot{\varepsilon_{\rm pl}}, \dot q) = \infty \text{ else}.$$

The elastic domain is $C_* = \partial\psi(0) = \{\, (\sigma, r) \in \mathbb{R}_0^{d\times d}\times\mathbb{R} \mid r \leq 0,\ |\sigma|+s_1 r \leq s_2 \,\}$. The energy density is given as above with $Q(\varepsilon_{\text{pl}}, q) = \frac{s_3}{2}q^2$. Here all the constants $s_j$ are strictly positive. The dissipation distance $\mathcal{D}$ is given via $\mathcal{D}((\varepsilon_{\text{pl}}^0, q^0), (\varepsilon_{\text{pl}}^1, q^1)) = \Psi((\varepsilon_{\text{pl}}^1, q^1)-(\varepsilon_{\text{pl}}^0, q^0))$.

The arising difficulty is that neither the dissipation functional $\Psi$ nor the stored energy density are coercive in the sense assumed in the abstract Section 3. However, the sum of stored and dissipated energies is coercive, namely for each $(\varepsilon_{\text{pl}}^0, q^0)$ the mapping

$$\mathcal{K}_{(\varepsilon_{\text{pl}}^0, q^0)} : (u, \varepsilon_{\text{pl}}, q) \mapsto \mathcal{E}(t, u, \varepsilon_{\text{pl}}, q) + \Psi((\varepsilon_{\text{pl}}, q)-(\varepsilon_{\text{pl}}^0, q^0))$$

satisfies either $\mathcal{K}_{(\varepsilon_{\text{pl}}^0, q^0)}(u, \varepsilon_{\text{pl}}, q) = \infty$ or

$$\mathcal{K}_{(\varepsilon_{\text{pl}}^0, q^0)}(u, \varepsilon_{\text{pl}}, q) \geq \tfrac{a}{2}\|\varepsilon(u)-\varepsilon_{\text{pl}}\|_{\mathrm{L}^2}^2 - \|\ell(t)\|_{\mathrm{H}^{-1}}\|u\|_{\mathrm{H}^1} + \tfrac{s_3}{2}\|q\|_{\mathrm{L}^2}^2 + s_2\|\varepsilon_{\text{pl}}-\varepsilon_{\text{pl}}^0\|_{\mathrm{L}^1}$$

if $s_1|\varepsilon_{\text{pl}}(x)-\varepsilon_{\text{pl}}^0(x)| \leq |q(x)-q^0(x)|$ for a.e. $x \in \Omega$. Using this pointwise constraint and Korn's inequality on $\mathrm{H}^1_{\Gamma_{\text{Dir}}}(\Omega; \mathbb{R}^d)$, it is then easy to find constants $c, C^0 > 0$ such that

$$K_{(\varepsilon_{\text{pl}}^0, q^0)}(u, \varepsilon_{\text{pl}}, q) \leq c(\|u\|_{\mathrm{H}^1}^2 + \|\varepsilon_{\text{pl}}\|_{\mathrm{L}^2}^2 + \|q\|_{\mathrm{L}^2}^2) - C^0.$$

Here $C^0$ depends only on $(\varepsilon_{\text{pl}}^0, q^0)$ and $\ell(t)$.

Thus, we choose the underlying space $\mathcal{Y} = Y = \mathrm{H}^1(\Omega; \mathbb{R}^d)\times\mathrm{L}^2(\Omega; \mathbb{R}_0^{d\times d})\times\mathrm{L}^2(\Omega)$, which makes $\mathcal{E}$ and $\Psi$ weakly lower semi-continuous due to convexity. Fixing an initial datum $(u^0, \varepsilon_{\text{pl}}^0, q^0) \in Y$, the abstract condition (A2) can be replaced by

$$\begin{aligned} \exists\, c_E^{(0)}, c_E^{(1)} > 0 : \quad & \mathcal{E}(t, \varepsilon_{\text{pl}}, q) < \infty \ \text{ and } \ \Psi((\varepsilon_{\text{pl}}, q)-(\varepsilon_{\text{pl}}^0, q^0)) < \infty \\ & \Longrightarrow \quad |\partial_t\mathcal{E}(t, u, \varepsilon_{\text{pl}}, q)| \leq c_E^{(1)}(c_E^{(0)}+\mathcal{E}(t, u, \varepsilon_{\text{pl}}, q)). \end{aligned}$$

Moreover, since $\partial_t\mathcal{E}(t, u, \varepsilon_{\text{pl}}, q) = -\langle\dot{\ell}(t), u\rangle$ is linear in $u$, it is weakly continuous. Finally, we can use the quadratic nature of $\mathcal{E}$ to show that the stable sets are convex and strongly closed, and hence weakly closed. Thus, existence can be deduced from Theorem 5.2 using the assumption (A7).

The case without any hardening, i.e., $s_3 = 0$, is also called perfect plasticity. In this case no a priori bounds in $\mathrm{L}^2$-spaces are possible, but the dissipation provides $\mathrm{L}^1$-bounds. Since this space is not weakly closed, one is led to consider $\varepsilon_{\text{pl}}$ as a bounded measure and $u$ in the set of ***bounded deformations***, where $\varepsilon(u)$ is a bounded measure, see [TS80, Suq81, ER03, DDM04].

For the further rich literature in linearized elastoplasticity we refer to [Alb98, HR99] and the references therein.

## 7.2 Finite-strain elastoplasticity

While the theory of linearized elastoplasticity is well developed in terms of existence and uniqueness results and also provides reliable and efficient finite-element discretizations, there is a big lack of theory for the case of finite-strain elastoplasticity.

The reason is that the linearized theory is based on the additive decomposition

$$\varepsilon = \tfrac{1}{2}(\mathrm{D}u + \mathrm{D}u^{\mathsf{T}}) = \varepsilon_{\text{elast}} + \varepsilon_{\text{plast}},$$

which is well suited for methods in linear functional analysis, whereas the finite-strain theory is based on the ***multiplicative decomposition***

$$F = \mathrm{D}y = F_{\mathrm{elast}} P \quad \text{with } P = F_{\mathrm{plast}}. \tag{7.2}$$

The main feature here is that the nonlinearities arise from the multiplicative group of invertible matrices. (In finite-strain elasticity this is also called geometrically nonlinear elasticity. ) The main open question is to understand the interaction of functional analytical tools, mainly based on linear function spaces, and these algebraic nonlinearities.

At the present stage there are only very little results in this direction. In [Nef03] a local existence result for a viscously regularized director model is obtained. For the rate-independent setting there is a series of negative result, in the sense that it is shown that the incremental problem (IP) studied in Section 3 does not have solutions in general, see [OR99, ORS00, CHM02, HH03, Mie03a]. This mathematical difficulty is also observed in experiments where the dislocations accumulate on interfaces which have microstructure. Here we want to address some results in the positive direction, namely where it is possible to establish existence of solutions for (IP) for an arbitrary number of steps, see [Mie02, Mie03a]. However, the limit for the timestep tending to 0 is not yet understood.

To be more specific, let $y : \Omega \to \mathbb{R}^d$ be the deformation of the body $\Omega \subset \mathbb{R}^d$. The energy $\mathcal{E}$ stored in a deformed body depends only on the elastic part $F_{\mathrm{elast}} = \mathrm{D}yP^{-1}$ of the deformation tensor and on suitable hardening parameters $q \in \mathbb{R}^n$. But it is not allowed to depend on the plastic part $P = F_{\mathrm{plast}}$, which is contained in $\mathrm{SL}(\mathbb{R}^d) = \{\, P \mid \det P = 1 \,\}$ or another Lie group $\mathfrak{G}$ contained in $\mathrm{GL}_+(\mathbb{R}^d) = \{\, G \in \mathbb{R}^{d\times d} \mid \det G > 0 \,\}$. The energy functional takes the form

$$\mathcal{E}(t, y, (P, p)) = \int_\Omega W(x, \mathrm{D}y(x)P(x)^{-1}, p(x))\,\mathrm{d}x - \langle \ell(t), y\rangle$$

with external loading $\langle \ell(t), y\rangle = \int_\Omega f_{\mathrm{ext}}(t,x)\cdot y(x)\,\mathrm{d}x + \int_\Gamma g_{\mathrm{ext}}(t,x)\cdot y(x)\,\mathrm{d}a$.

To model the plastic effects one prescribes either a plastic flow law or, equivalently, a dissipation potential $\psi : \Omega\times\mathrm{T}(\mathfrak{G}\times\mathbb{R}^m) \to [0,\infty]$, which generates the global dissipation distance $D(x,\cdot,\cdot)$ on $\mathfrak{G}\times\mathbb{R}^m$. Thus, the second ingredient of the material model is the dissipation distance between two internal states $z_j = (P_j, p_j) : \Omega \to \mathrm{SL}(d)\times\mathbb{R}^m$:

$$\mathcal{D}(z_1, z_2) = \int_\Omega D(x, (P_1(x), p_1(x)),\ (P_2(x), p_2(x)))\,\mathrm{d}x.$$

The main assumption in plasticity theory is that the actual plastic tensor $P$ does not appear in the constitutive laws. Changes of $P$ can only influence the forces through the hardening variable. Thus, the dissipation potential $\psi$ and the dissipation distance $D$ have to satisfy ***plastic invariance***, namely

$$\begin{aligned}&\psi(x, (P,p), (\dot P, \dot p)) = \psi(x, (\mathbf{1}, p), (\dot P\, P^{-1}, \dot p)),\\ &D(x, (P_1,p_1), (P_2,p_2)) = D(x, (\mathbf{1}, p_1), (P_2 P_1^{-1}, p_2)).\end{aligned}$$

This symmetry leads naturally to a logarithmic behavior of $D$ which contradicts any convexity properties.

Allowing for finite strains we are forced to abolish convexity of the stored-energy density $W$. Instead it has to be polyconvex or quasiconvex and frame indifferent, see [Bal77]. These notions work well together with the philosophy that $F = \mathrm{D}y$ is an element of $\mathrm{GL}_+(\mathbb{R}^d)$, i.e., we set $W(F) = \infty$ for $\det F \leq 0$.

The associated incremental problem (IP) has the form

$$(y_k, P_k, p_k) \in \operatorname{Arg\,min}\{\, \mathcal{E}(t_k, y, P, p) + \mathcal{D}((P_{k-1}, p_{k-1}), (P, p)) \mid y \in \mathcal{F},\ (P, q) \in \mathcal{Z}\,\},$$

where the spaces $\mathcal{F}$ and $\mathcal{Z}$ still need to be specified. Typical choices are $\mathcal{F} = y_{\mathrm{Dir}} + \mathrm{W}^{1,q_y}_{\Gamma_{\mathrm{Dir}}}(\Omega; \mathbb{R}^d)$ and $\mathcal{Z}$ is a subset of $\mathrm{L}^{q_P}(\Omega; \mathbb{R}^{d\times d}) \times \mathrm{L}^{q_p}(\Omega; \mathbb{R}^m)$. Under suitable assumptions on $W$ and $D$ it is then possible to prove coercivity, but there is no hope to proof weak lower semi-continuity in the variable $P$.

The crucial observation in [CHM02, Mie03a, Mie04b] is that weak lower semi-continuity is by far not needed to prove existence of minimizers for (IP). The point is that $z = (P, p)$ appears only pointwise under the integral. Thus, the minimization can be done pointwise in $x \in \Omega$. This leads to the ***condensed energy density*** $W^{\mathrm{cond}}$ as defined in Section 3.4:

$$W^{\mathrm{cond}}(P_{\mathrm{old}}, p_{\mathrm{old}}; x, F) = \min\{\, W(x, FP^{-1}, p) + D(x, (P_{\mathrm{old}}, p_{\mathrm{old}}), (P, p)) \mid (P, p) \in \mathfrak{G} \times \mathbb{R}^m \,\}.$$

Plastic invariance is inherited in the form

$$W^{\mathrm{cond}}(P_{\mathrm{old}}, p_{\mathrm{old}}; x, F) = W^{\mathrm{cond}}(\mathbf{1}, p_{\mathrm{old}}; x, F\, P_{\mathrm{old}}^{-1}).$$

In [Mie04b] it is shown that under the usual technical assumptions on $W$ and $D$ the solvability of (IP) can be shown:

$$\begin{aligned} &\text{(a)} \quad W^{\mathrm{cond}}((\mathbf{1}, p_*); x, \cdot) : \mathbb{R}^{d\times d} \to \mathbb{R}_\infty := \mathbb{R} \cup \{\infty\} \text{ is polyconvex,} \\ &\text{(b)} \quad W^{\mathrm{cond}}((\mathbf{1}, p_*); F) \geq c|F|^{q_F} - C \text{ and } D((\mathbf{1}, p_*), (P, p)) \geq c|P|^{q_P} - C, \\ &\text{(c)} \quad \tfrac{1}{q_F} + \tfrac{1}{q_P} \leq \tfrac{1}{q_y} < \tfrac{1}{d}. \end{aligned} \tag{7.3}$$

The major problem with these assumptions is that in practice only $W$ and $\psi$ are given and $W^{\mathrm{cond}}$ and $D$ are defined implicitly. In particular, the condition (7.3)(a) is difficult to check. By now, only one nontrivial model is known, which is only two-dimensional, see [Mie04b, Sect. 4]. On the one-dimensional setting with $\mathfrak{G} = \mathrm{GL}_+(\mathbb{R}^1) = (0, \infty)$ the solvability of (IP) is rather straightforward, since polyconvexity equals convexity. Even the convergence of the solutions of (IP) to solutions of (S) & (E) can be shown, see [Mie04b, Sect. 5+6].

In microscopical models for finite-strain elastoplasticity it is often desirable to introduce regularizing (also called non-local) terms, which generate a small length scale which stops formation of microstructure. One typical term of this kind is $(\mathrm{curl} P)P^{\mathsf{T}}$, which is also called the dislocation-density tensor $(\mathrm{curl} P)P^{\mathsf{T}}$. In [MM04b] it is shown that the functional

$$(y, P) \mapsto \int_\Omega W(\mathrm{D}yP^{-1}) + D(P) + \tfrac{\kappa}{r}|(\mathrm{curl} P)P^{\mathsf{T}}|^r \,\mathrm{d}x$$

is weakly lower semi-continuous on $\mathrm{W}^{1,q_y}(\Omega; \mathbb{R}^d) \times \mathrm{L}^{q_P}(\Omega; \mathrm{SL}(\mathbb{R}^d))$, if both, $W$ and $D$, are polyconvex. Based on this result, existence for (IP) is established.

## 7.3 Phase transformations in shape-memory alloys

Over the last decade the shape-memory effect became important in many mechanical and medical applications. Thus, the need of good models for simulation and optimization of such materials arises. The energetic formulation (S) & (E) was in fact developed for models of phase transformations induced by stress or strain, rather than by temperature changes, cf. [MT99, MTL02]. For such isothermal cases the energetic formulation is a tool which can model the static behavior quite well, but the dynamical effects are modelled only crudely. The dissipation potential is fitted phenomenologically to obtain the desired hysteresis loops.

We assume that in each microscopic point $x \in \Omega$ the shape-memory material is free to choose one of $m$ crystallographic phases, denoted by $\{e_1, \ldots, e_m\} \in \mathbb{R}^m$, and that the elastic energy density $W$ is then given by $W(\mathrm{D}\varphi, e_j)$. If the model is made on the mesoscopic level, then the internal variables are phase portions $z^{(j)} \in [0,1]$ for the $j$-th phase. We set

$$Z = \{\, z \in [0,1]^p \subset \mathbb{R}^m \mid \textstyle\sum_1^m z^{(j)} = 1 \,\} \text{ and } \mathcal{Z} = \mathrm{L}^1(\Omega; Z) \subset \mathrm{L}^1(\Omega; \mathbb{R}^m).$$

The material properties are given via a ***mixture function*** (also called cross-quasiconvexification) $W : \mathbb{R}^{d\times d} \times Z \to [0, \infty]$, see [MTL02, GMH02]. The dissipation can be shown to have the form $D(z_0, z_1) = \psi(z_1 - z_0)$ with $\psi(v) = \max\{\, \sigma_m \cdot v \mid m = 1, \ldots, M \,\} \geq C_\psi |v|$, where $\sigma_m \in \mathbb{R}^p$ are thermodynamically conjugated threshold values. The derivation of this model is in fact a special case of the relaxation described in Section 5.6.

In the case of no regularization term, i.e., $\kappa = 0$ in (7.1), we are unable to prove existence results for this model in its full generality. The mixture function $W$ is constructed as a relaxation, which means that the associated energy functional is weakly lower semi-continuous on $\mathrm{W}^{1,p}(\Omega, \mathbb{R}^d) \times \mathrm{L}^1(\Omega, Z)$. Clearly $\mathcal{D}$ is convex and weakly lower semi-continuous. Thus, our abstract conditions (A1) to (A5) can be satisfied easily. However, condition (A6) (weak continuity of $\mathcal{D}$) does not hold. The weak continuity of $\partial_t \mathcal{E}$ in condition (A7) also holds, but the weak closedness of the stable sets $\mathcal{S}(t)$ seems to be wrong in general. Hence, in general we are able to prove existence of solutions for the incremental problem (IP), but the convergence of the piecewise constant interpolants to solutions of the energetic formulation (S) & (E) is still open. However, the case with only two phases $(m = 2)$ has been treated in [MTL02] under the additional assumption that the elastic behavior is linear and both phases have the same elastic tensor. The missing closedness of the stable sets is shown via an explicit representation of the set in terms of a pseudo-differential operator of order 0 and a finite number of quadratic inequalities which have to hold pointwise. Then, a careful analysis using H-measures (cf. [Tar90]) shows that the non convex sets $\mathcal{S}(t)$ are weakly closed.

The situation is much better if a regularizing term $\frac{\kappa}{r}|\mathrm{D}z|^r$, with $r \geq 1$ and $\kappa > 0$, is added to the stored energy. In this case the underlying space can be chosen as $\mathcal{F} \times \mathcal{Z}$, with $\mathcal{F} = \mathrm{W}^{1,p}_{\Gamma_{\mathrm{Dir}}}(\Omega, \mathbb{R}^d)$ and $\mathcal{Z} = \mathrm{W}^{1,r}(\Omega, \mathbb{R}^m) \cap \mathrm{L}^1(\Omega, Z)$ equipped with the weak topologies in both cases. Conditions (A1) to (A5) remain valid but now (A6) also holds since $\mathrm{W}^{1,r}(\Omega)$ is compactly embedded into $\mathrm{L}^1(\Omega)$ and $\mathcal{D}$ is strongly continuous on $\mathrm{L}^1(\Omega, Z)$. Such regularizations are used in [Rou02, AGR03, MR03, FM04].

In [Mai05] a microscopic model without phase mixtures is considered, i.e., we assume $z \in Z_{\mathrm{p}} := \{e_1, \ldots, e_m\} \subset \mathbb{R}^m$, where $e_j$ is the $j$-th unit vector. The subscript $_{\mathrm{p}}$ stands for "pure" phases, and the functions $z \in \mathcal{Z}_{\rm p} = \{ z \in {\rm BV}(\Omega, \mathbb{R}^m) \mid z(x) \in Z_{\rm p}$ a.e. in $\Omega \}$ are like characteristic functions which indicate exactly one phase at each material point. The dissipation is assumed as above, but now the elastic energy contains an additional term measuring the surface area of the interfaces between the different regions:

$$\mathcal{E}(t, \varphi, z) = \int_\Omega W({\rm D}\varphi, z)\,{\rm d}x + \kappa \int_\Omega |{\rm D}z| - \langle \ell_{\rm ext}(t), \varphi\rangle,$$

where $\kappa$ is a positive constant and the total variation of $z$ over $\Omega$ is $\int_\Omega |{\rm D}z|$, which equals $\sqrt{2}$ times the area of all interfaces. Thus, the underlying space $\mathcal{Y} = \mathcal{F}\times\mathcal{Z}_{\rm p}$ is defined with $\mathcal{F}$ as above and $\mathcal{Z}_{\rm p}$ is equipped with the weak$^*$ topology, i.e., the measure ${\rm D}z$ is tested with functions in ${\rm C}^0(\overline{\Omega})$. Again we have a compact embedding of ${\rm BV}(\Omega)$ into ${\rm L}^1(\Omega)$, which shows that $\mathcal{Z}_{\rm p}$ is a weakly$^*$ closed and $\mathcal{D}$ is weakly$^*$ continuous on $\mathcal{Z}_{\rm p}$. We refer to [Mai05] for details.

A totally different approach to shape-memory materials is given in [AP02]. This model is isotropic and uses the linearized strain tensor $\varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^{\sf T})$. The internal variable is the mesoscopically averaged transformation strain, namely

$$z \in Z = \mathbb{R}^{d\times d}_0 = \{ A \in \mathbb{R}^{d\times d} \mid {\rm tr}\, A = 0 \}.$$

For a given fixed temperature the stored-energy density $W$ takes the form

$$W(\varepsilon, z) = \tfrac{1}{2}\mathbb{A}(\varepsilon - z):(\varepsilon - z) + H(z) \;\text{ with } H(r) = c_1|z| + c_2|z|^2 + I_{\{|z|\le c_3\}}(z).$$

The dissipation potential is given simply by $\psi(\dot z) = c_4|\dot z|$. Thus, the functionals

$$\mathcal{E}(t, u, z) = \int_\Omega W(\varepsilon(u), z)\,{\rm d}x - \langle \ell(t), u\rangle \;\text{ and }\; \mathcal{D}(z_0, z_1) = \int_\Omega c_4|z_1(x) - z_0(x)|\,{\rm d}x$$

are convex and weakly lower semi-continuous on the Hilbert space $\mathcal{Y} = {\rm H}^1_{\Gamma_{\rm Dir}}(\Omega;\mathbb{R}^d)\times{\rm L}^2(\Omega; Z)$. (Throughout the constants $c_j$ are positive.) Moreover, $\mathcal{E}(t,\cdot)$ is uniformly convex, which implies that the incremental problem (IP) has unique solutions and that these solutions satisfy an a priori Lipschitz bound in time, see Section 3.5. However, our theory does not provide existence of solutions. For the existence proof in the convex case we lack the necessary smoothness (see Theorem 4.3) whereas for the nonsmooth case we lack the suitable compactness (see (A6) or (A7) in Theorem 5.2).

However, a slight modification and a regularization make the problem accessible for our general theory, see [AMS04]. We replace the special function $H$ by a smooth version $H_\rho$ which satisfies

$$H \in {\rm C}^3(Z, R), H \text{ uniformly convex and } \exists\, c, C > 0\; \forall\, z \in Z: \; c|z|^2 \le H(z) \le C|z|^2.$$

Moreover, we require that for $z$ fixed $H_\rho(z) \to H(z)$ if $\rho \searrow 0$. As an example we may choose

$$H_\rho(z) = c_1\big(\sqrt{\rho^2 + |z|^2} - \rho\big) + \big(c_2 + \tfrac{1}{\rho}h(\tfrac{1}{\rho}(|z|^2 - c_3^2))\big)|z|^2, \text{ where } h(s) = \tfrac{{\rm e}^s}{1+{\rm e}^s}.$$

Moreover, we add a spatial regularization and obtain the energy functional

$$\mathcal{E}_\rho(t, u, z) = \int_\Omega \tfrac{1}{2}\mathbb{A}(\varepsilon(u) - z):(\varepsilon(u) - z) + H_\rho(z) + \tfrac{\rho c_5}{2}|{\rm D}z|^2\,{\rm d}x.$$

Now the suitable function space is $\mathcal{Y} = \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega;\mathbb{R}^d)\times\mathrm{H}^1(\Omega;Z)$ which, on the one hand, makes $\mathcal{D}$ weakly continuous (cf. (A6)) and, on the other hand, makes $\mathcal{E}_\rho$ uniformly convex and $\mathrm{C}^3$. Thus, now both abstract existence results are applicable. In particular, one obtains a unique solution for the associated energetic formulation (S) & (E) and the solutions of the incremental problem (IP) converge strongly like $\sqrt{\text{stepsize}}$.

Note that the smoothness of $H_\rho$ is not enough to guarantee smoothness if $c_5 = 0$. Then, the suitable space for uniform convexity is again $\mathrm{H}^1\times\mathrm{L}^2$ as above. However, a functional

$$\mathcal{H} : \mathrm{L}^2(\Omega, Z) \to \mathbb{R};\ z \mapsto \int_\Omega h_*(z(x))\,\mathrm{d}x$$

is $\mathrm{C}^3$ if and only if $h_* : Z \to \mathbb{R}$ is a quadratic functional.

## 7.4 Models in ferromagnetism

Hysteretic effects in ferromagnetism were one of the driving forces in the development of hysteresis operators like the Preisach operator as a superposition of many relay operators, see [Vis94, I.4 & IV]. These models are mainly used for mean field models, which replace the continuum by a system with a finite number of degrees of freedom. Here we want to propose a continuum model with infinitely many degrees of freedom. Thus, we use simpler hysteresis operators which will also generate complicated hysteretic behavior because of spatial variations of the internal variable.

In ferromagnetism we are interested in the interplay between elastic effects and magnetic effects, sometimes also called magnetostriction since magnetic fields may deform a body. The models described here and studied in [EM04a] lie inbetween the parabolic and hyperbolic models considered in [Vis94, Vis04] and the purely static models in [DeS93, DJ02]. Thus, our models describe the statics as good as the latter works but our dynamics are not as good as in the former ones.

For simplicity, we assume small strains and use the linearized strain tensor $\varepsilon(u)$. The internal variable is the magnetization $z \in Z = \mathbb{S}^{d-1} = \{\, z \in \mathbb{R}^d \mid |z| = 1 \,\}$. More standard notations for the magnetization are the symbols $m$ and $M$, but we stay with $z$ to remain consistent with the other parts of this paper. The functional of the stored energy takes the form

$$\mathcal{E}(t,u,z) = \int_\Omega W(\varepsilon(u),z) + \tfrac{\kappa^2}{2}|\mathrm{D}z|^2\,\mathrm{d}x + \int_{\mathbb{R}^d} \tfrac{\mu_0}{2}|\nabla\phi_z|^2\,\mathrm{d}x - \langle \ell_{\text{mech}}(t), u\rangle - \langle \mu_0 H_{\text{ext}}(t), z\rangle,$$

where $\ell_{\text{mech}}(t)$ denotes the mechanical loading, $H_{\text{ext}}(t)$ is the external magnetic field satisfying $\operatorname{div} H_{\text{ext}} = 0$. The stored energy density $W$ contains information on the interaction between the elastic behavior and the magnetic directions and $\kappa > 0$ is the exchange length which gives the thickness of the domain walls.

The potential $\phi_z$ describes the field induced by the magnetization inside the body, i.e., the magnetic flux is $B = \mu_0(H{+}\mathbb{E}_\Omega z)$ with $H = H_{\text{ext}} - \nabla\phi_z$. Here $\mathbb{E}_\Omega$ denotes the operator which extends a function on $\Omega$ by $0$ to all of $\mathbb{R}^d$. Thus, $\operatorname{div} B = 0$ yields the definition of $\phi_z$ as a solution of

$$\operatorname{div}(-\nabla\phi_z{+}\mathbb{E}_\Omega z) = 0 \text{ on } \mathbb{R}^d.$$

Of course, $\phi_z$ is defined only up to a constant, but $\widehat{\mathcal{G}} : \mathrm{L}^2(\Omega,\mathbb{R}^d) \to \mathrm{L}^2(\mathbb{R}^d,\mathbb{R}^d);\ z \mapsto \nabla\phi_z$, is a bounded linear operator. Moreover, $\mathcal{G} : z \mapsto (\widehat{\mathcal{G}}z)\big|_\Omega$ is an orthogonal projection on $\mathrm{L}^2(\Omega,\mathbb{R}^d)$ satisfying $\int_{\mathbb{R}^d} |\nabla\phi_z|^2\,\mathrm{d}x = \int_\Omega z\cdot(\mathcal{G}z)\,\mathrm{d}x$, see [DeS93].

In addition to the stored energy we define a dissipation distance via a metric on $Z = \mathbb{S}^{d-1}$. The simplest distance which respects the geometry is

$$D(z_0, z_1) = \tfrac{\delta}{\pi} \arccos(z_0 \cdot z_1) \quad \text{giving} \quad \mathcal{D}(z_0, z_1) = \textstyle\int_\Omega D(z_0(x), z_1(x))\, \mathrm{d}x.$$

Using these functionals with a suitable $W$ and the space

$$\mathcal{Y} = \mathrm{H}^1(\Omega, \mathbb{R}^d) \times \{\, z \in \mathrm{H}^1(\Omega, \mathbb{R}^d) \mid |z(x)| = 1 \text{ a.e.} \,\},$$

it is shown in [EM04a] that for $\kappa > 0$ the energetic formulation (S) & (E) has a solution. Again the crucial compactness condition (A6) is satisfied.

In [Vis04] the bulk energy $W$ and the exchange energy are neglected (i.e., $\kappa = 0$). Moreover, the models are considered to be mesoscopic and $z$ is considered to be a mesoscopic average satisfying $z \in Z = \{\, z \in \mathbb{R}^d \mid |z| \leq 1 \,\}$. Thus, the energy reduces to

$$\mathcal{E}_0(t, z) = \textstyle\int_{\mathbb{R}^d} \frac{\mu_0}{2} |\nabla \phi_z|^2 \, \mathrm{d}x + \int_\Omega I_Z(z(x))\, \mathrm{d}x - \langle \mu_0 H_{\text{ext}}, z \rangle.$$

The dissipation is taken to be in the form $\psi(z, \dot z) = \widehat{\psi}(\dot z)$, since $Z$ is a closed convex subset of $\mathbb{R}^d$. With these assumptions the problem is convex and using the formula $\mathrm{D}\mathcal{E}_0(t, z)[\widetilde z] = \mu_0(\mathcal{G}z - H_{\text{ext}}) = -\mu_0 H|_\Omega$ we find the subdifferential formulation which is equivalent to (S) & (E), namely

$$0 \in \partial \widehat{\psi}(\dot z(t, x)) + \partial I_Z(z) - \mu_0(H_{\text{ext}} - \nabla \phi_z) \text{ for a.a. } (t, x) \in [0, T] \times \Omega.$$

In [Vis04] a “scalar relay” is used which corresponds to the choice $\widehat{\psi}(\dot z) = \delta_0 \dot z \cdot \theta + (\delta_0 + \delta_1) |\dot z \cdot \theta|$ with $\delta_0, \delta_1 > 0$ and a given vector $\theta \in \mathbb{S}^{d-1}$. Existence and uniqueness results for parabolic versions (not rate-independent) of this problem are then established.

## 7.5 A delamination problem

In this section we provide a simple model for rate-independent delamination and refer to [KMR04] for a better model and the detailed analysis. Thus, we remove all unnecessary distractions and focus the attention on the interplay of the different continuity properties in the suitable topologies.

The body $\Omega \subset \mathbb{R}^d$ consists of several pieces which are glued together at certain interfaces. The model is based on the assumption that sufficiently strong forces can destroy the glue. To be precise, we assume that $\mathrm{int}(\mathrm{cl}(\Omega))$ differs from $\Omega$ by a finite set of sufficiently smooth hypersurfaces $\Gamma_j$, $j = 1, \ldots, n$, along which parts of the body are glued together. This means that with $\Gamma := \bigcup_{j=1}^n \Gamma_j$ we have $\mathrm{int}(\mathrm{cl}(\Omega)) = \Omega \cup \Gamma$ and $\Omega \cap \Gamma = \emptyset$. The two sides of the body are glued together along these surfaces with a glue that is softer than the material itself. Upon loading, some parts of the glue may break and thus lose their effectiveness. The remaining fraction of the glue which is still effective is denoted by the internal state function $z : \Gamma \to [0, 1]$.

We let $\mathcal{Z} = \{\, z : \Gamma \to [0, 1] \mid z \text{ measurable} \,\} \subset \mathrm{L}^1(\Gamma)$. The dissipation distance $\widetilde{\mathcal{D}}(z_0, z_1)$ is proportional to the amount of glue that is broken from state $z_0$ to state $z_1$:

$$\widetilde{\mathcal{D}}(z_0, z_1) = \textstyle\int_\Gamma \psi_{\text{delam}}(z_1(y) - z_0(y))\, \mathrm{d}a(y) \quad \text{with } \psi_{\text{delam}}(v) = -\kappa v \text{ for } v \leq 0 \text{ and } +\infty \text{ else.}$$

Here we explicitly forbid the healing of the glue by setting $\psi_{\text{delam}}(v)$ equal to $\infty$ for $v > 0$.

The energy is given by the elastic energy in the body, the elastic energy in the glue, and the potential of the external loadings:

$$\mathcal{E}(t, \varphi, z) = \int_\Omega W(\mathrm{D}\varphi)\,\mathrm{d}x + \int_\Gamma z(y)Q(y, [\![\varphi]\!]_\Gamma(y))\,\mathrm{d}a(y) - \langle \ell_{\text{ext}}(t), \varphi \rangle,$$

where for $y \in \Gamma$ the vector $[\![\varphi]\!]_\Gamma(y)$ denotes the jump of the deformation $\varphi$ across the interface $\Gamma$ and $Q(y, \cdot)$ is the potential defining the elastic properties of the glue.

For simplicity we assume further that $W$ is coercive and provides linearized elasticity and that $Q$ is quadratic as well. Then there is a unique minimizer $\varphi = \Phi(t, z) \in \mathcal{F} := \{\, \phi \in \mathrm{H}^1(\Omega, \mathbb{R}^d) \,|\, \phi|_{\Gamma_{\text{Dir}}} = \varphi_{\text{Dir}} \,\}$ of $\mathcal{E}(t, \cdot, z)$. We let $\mathcal{Y} = \mathrm{H}^1_{\Gamma_{\text{Dir}}}(\Omega, \mathbb{R}^d) \times \mathcal{Z}$ be equipped with the weak topology of $\mathrm{H}^1_{\Gamma_{\text{Dir}}}(\Omega, \mathbb{R}^d) \times \mathrm{L}^1(\Gamma)$.

Note that $\mathcal{E}$ is not convex since the integral over $\Gamma$ is trilinear. Nevertheless $\mathcal{E}$ is weakly lower semi-continuous, since the compactness of the embedding $\mathrm{H}^{1/2}(\Gamma) \subset \mathrm{L}^2(\Gamma)$ implies $Q([\![\varphi_k]\!]_\Gamma) \to Q([\![\varphi^*]\!]_\Gamma)$ strongly in $\mathrm{L}^1(\Gamma)$ if $\varphi \rightharpoonup \varphi^*$ in $\mathrm{H}^1(\Omega)$. Moreover, $z_k \rightharpoonup z^*$ in $\mathrm{L}^1(\Gamma)$ and $\|z_z\|_\infty \leq 1$ implies $z_k \overset{*}{\rightharpoonup} z^*$ in $\mathrm{L}^\infty(\Gamma)$. Thus, we conclude $\int_\Gamma z_k Q([\![\varphi_k]\!]_\Gamma)\,\mathrm{d}a \to \int_\Gamma z^* Q([\![\varphi^*]\!]_\Gamma)\,\mathrm{d}a$, as it is desired for lower semi-continuity.

Thus, it is not difficult to satisfy the assumptions (A1)–(A5). Since (A6) (weak continuity of $\mathcal{D}$) does not hold, we have to show that the stable sets $\mathcal{S}(t)$ are weakly closed. Note that each element $(\varphi, z)$ in $\mathcal{S}(t)$ satisfies $\varphi = \Phi(t, z)$, since the elastic problem for fixed $z$ is strictly convex. Moreover, it can be shown that the mapping $\Phi(t, \cdot)$ is compact, in the sense that $z_k \rightharpoonup z^*$ in $\mathcal{Z}$ implies $\Phi(t, z_k) \to \Phi(t, z)$ in $\mathrm{H}^1(\Omega)$. We refer to [KMR04, Lem.2.1] for the proof of this delicate continuity result. Finally, an application of Proposition 5.11 and the usage of the special form of $\psi_{\text{delam}}$ establish the weak closedness of $\mathcal{S}(t)$. Thus, assumption (A7) holds and existence of solutions follows according to Theorem 5.2, namely, for each stable initial state $z_0$ and each loading $\ell_{\text{ext}} \in \mathrm{C}^{\text{Lip}}([0, T], \mathrm{H}^{-1}(\Omega))$ the energetic formulation (S) & (E) of the delamination problem has a solution $(\varphi, z)$ with $\varphi \in \mathrm{L}^\infty([0, T], \mathrm{H}^1(\Omega, \mathbb{R}^d))$ and $z \in \mathrm{BV}_{\widetilde{\mathcal{D}}}([0, T], \mathcal{Z})$. Recall that the condition on $z$ is equivalent to the monotonicity $z(s) \geq z(t)$ on $\Gamma$ for $s < t$ and that then $\mathrm{Diss}_{\widetilde{\mathcal{D}}}(z; [s, t]) = \widetilde{\mathcal{D}}(z(s), z(t))$.

In [KMR04] also numerical simulations are given which display the different contributions in the energy balance (E).

## 7.6 Crack propagation in brittle materials

In a series of papers starting with [FM93, FM98, Bul98, Bul00, BFM00] and culminating with [DT02, Cha03, FL03, DFT04] the following fracture model is developed and analyzed. We follow the notation of the latter paper and will show that the approach taken there is exactly that of the energetic formulation (S) & (E), if the notions are reinterpreted correctly. This problem is technically very difficult and needs certain adjustments to the abstract theory which we will not discuss here. Nevertheless the main strategy of the proof is exactly as it was described in Sections 3 and 5, namely by using the incremental problem in the form of a minimization problem and then passing to the limit as the stepsize of the discretization goes to 0. We will present here only a simplified version which omits certain finer details. But this enables us to compare these result with our abstract theory without too much effort. For full details we refer to [DFT04].

The model deals with nonlinear elasticity such that the energy density $W$ is given as a quasiconvex function of the displacement gradient $A = \mathrm{D}u$ with suitable growth restrictions, namely

$$\exists\, p>1\ \exists\, c, C>0\ \forall\, A \in \mathbb{R}^{m\times d}:\ c|A|^p - C \le W(A) \le C|A|^p + C.$$

The time-dependent exterior forces are assumed to have a potential $F$ which is nonlinear and coercive. This is needed because pieces which are broken off by cracks all around could "fall to infinity". The assumption is

$$\exists\, c>0\ \exists\, f \in \mathrm{L}^1(\Omega)\ \forall\, u \in \mathbb{R}^m:\ -F(t,x,u) \ge c|u|^q - f(x)$$

for a suitable $q$ related to $p$ and $d$.

The internal variables are the cracks themselves. A crack is considered to be a subset $\Gamma \subset \overline{\Omega}$ which satisfies $\mathcal{H}^{d-1}(\Gamma) < \infty$, where $\mathcal{H}^{d-1}$ denotes the surface measure or the $(d-1)$-dimensional Hausdorff measure. (More precisely, a crack $\Gamma$ is the equivalence class of all $\widetilde{\Gamma}$ which satisfy $\mathcal{H}^{d-1}(\Gamma\backslash\widetilde{\Gamma})+\mathcal{H}^{d-1}(\widetilde{\Gamma}\backslash\Gamma) = 0$. All inclusions $\subset$ are also meant to be up to sets $N$ with $\mathcal{H}^{d-1}(N) = 0$.) The state space $\mathcal{Y}$ is then given as

$$\mathcal{Y} = \Big\{\ (u,\Gamma)\ \ \Big|\ \ \begin{array}{l}\Gamma \subset \overline{\Omega},\ \mathcal{H}^{d-1}(\Gamma) < \infty,\ \Gamma \text{ rectifiable},\\ u \in \mathrm{GSBV}(\Omega,\mathbb{R}^m),\ u|_{\Gamma_{\mathrm{Dir}}} = u_{\mathrm{Dir}},\ J(u) \subset \Gamma\ \Big\},\end{array}$$

where $\mathrm{GSBV}(\Omega,\mathbb{R}^m)$ is the set of generalized special functions of bounded variations and $J(u)$ denotes the jump set of such functions. This space is equipped with the following weak convergence:

$$(u_k,\Gamma_k) \xrightarrow{\mathcal{Y}} (u,\Gamma) \quad\Longleftrightarrow\quad \left\{\begin{array}{l} u_k \to u \text{ a.e. in } \Omega,\ \mathrm{D}u_k \rightharpoonup \mathrm{D}u \text{ in } \mathrm{L}^p(\Omega,\mathbb{R}^{m\times d}),\\ \sup\{\,\mathcal{H}^{d-1}(J(u_k)) \mid k \in \mathbb{N}\,\} < \infty,\ \Gamma_k \xrightarrow{\sigma_p} \Gamma.\end{array}\right.$$

See [DFT04, Def. 4.1] for the exact definition of $\sigma_p$ convergence of sets.

The functional for the stored energy (in our convention) is given as

$$\mathcal{E}(t,u,\Gamma) = \int_\Omega W(\mathrm{D}u(x)) - F(t,x,u(x))\,\mathrm{d}x,$$

which does not depend directly on $\Gamma$, since $\Gamma$ has Lebesgue measure 0. The dissipation is associated with the crack propagation. For simplicity we take

$$\mathcal{D}(\Gamma_0,\Gamma_1) = \left\{\begin{array}{cl} \kappa\mathcal{H}^{d-1}(\Gamma_1\backslash\Gamma_0) & \text{for } \Gamma_0 \subset \Gamma_1,\\ \infty & \text{else,}\end{array}\right.$$

but more general $x$-dependent and anisotropic surface measures can be used.

In [DFT04] the notations are somewhat different. They use total energy $\mathbb{E}(t)(u,\Gamma) = \mathcal{E}(t,u,\Gamma) + \mathcal{D}(\emptyset,\Gamma)$ to write the energetic formulation in the following way:

(a) ***global stability:*** for every $t \in [0,T]$ the pair $u(t),\Gamma(t)$ is a minimum energy configuration, i.e., $\mathbb{E}(t)(u(t),\Gamma(t)) \le \mathbb{E}(t)(\widetilde{u},\widetilde{\Gamma})$ for all $(\widetilde{u},\widetilde{\Gamma}) \in \mathcal{Y}$ with $\Gamma(t) \subset \widetilde{\Gamma}$;

(b) ***irreversibility:*** $\Gamma(s)$ is contained in $\Gamma(t)$ for $0 \le s \le t \le T$;

(c) ***energy balance:*** the increment in stored energy plus the energy spent in crack increase equals the work of the external forces, i.e.,

$$\mathbb{E}(t)(u(t),\Gamma(t)) = \mathbb{E}(s)(u(s),\Gamma(s)) - \int_s^t \int_\Omega \partial_\tau F(\tau,x,u(\tau,x))\,\mathrm{d}x\,\mathrm{d}\tau.$$

Using the above definition of $\mathcal{D}$ it is easy to see that (a), (b) and (c) are equivalent to our energetic formulation (S) & (E). Thus, the existence results there are also existence results for (S) & (E).

Like in the delamination case, it is possible to show that the functionals $\mathcal{E}$ and $\mathcal{D}$ are lower semi-continuous with compact sublevels, since $\mathcal{E}$ is a volume integral and $\mathcal{D}$ is a surface integral. However, the analysis is much deeper, since here the crack $\Gamma$ is not prescribed a priori. Thus, already the lower semi-continuity property is nontrivial. The most difficult part is the proof of the closedness of the stable set which relies on the so-called "jump transfer in GSBV" ([DFT04, Thm. 5.3]), which supplies assumption (5.9) of our abstract Proposition 5.11.

It should be noted that the analysis in [DFT04] provides several new tools. In fact, they form a significant part of the basis for the abstract existence result in Theorem 5.2 (see also [FM04]). In particular, the $t$-dependent choice of subsequences for the $u$-component (see Step 2), the approximation of Lebesgue integrals by suitable Riemann sums in the proof of Proposition 5.7 and, most importantly, the following concrete version of Proposition 5.6:

$$\left.\begin{array}{r}(u_k,\Gamma_k) \xrightarrow{\mathcal{Y}} (u,\Gamma)\\ \mathcal{E}(t,u_k,\Gamma_k) \to \mathcal{E}(t,u,\Gamma)\end{array}\right\} \quad\Longrightarrow\quad \mathrm{D}_u\mathcal{E}(t,u_k,\Gamma_k) \rightharpoonup \mathrm{D}_u\mathcal{E}(t,u,\Gamma) \ \text{ in } \ \big(\mathrm{W}^{1,p}(\Omega,\mathbb{R}^m)\big)^*.$$

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