Exercise 40. Quasiconvexity implies weak lower semicontinuity. For \( p > q \geq 1 \) consider a quasiconvex function \( f : \mathbb{R}^{m \times d} \to \mathbb{R} \) such that

\[
\exists C > 0 \ \forall A, B \in \mathbb{R}^{m \times d} : \quad |f(A) - f(B)| \leq C(1 + |A| + |B|)^{q-1}|A-B|.
\]

Define the functional \( I_A(u) = \int_{\Omega} f(A + \nabla u(x)) \, dx \).

(a) Show that \( I_A(u) \geq I_A(0) \) for all \( u \in W_0^{1,q}(\Omega; \mathbb{R}^m) \) (which is the closure of \( C^\infty_c(\Omega; \mathbb{R}^m) \)).

(b) It can be used without proof that the cut-off function

\[
\chi_{\varepsilon} : \Omega \to [0,1] : x \mapsto \min\{1, \max\{0, (\text{dist}(x, \partial \Omega) - \varepsilon)/\varepsilon\}\}
\]

lies in \( W^{1,\infty}_0(\Omega; \mathbb{R}^m) \) and satisfies \( \|\nabla \chi_{\varepsilon}\|_{L^\infty} = 1/\varepsilon \). Show that for all \( u \in W^{1,p}(\Omega; \mathbb{R}^m) \) we have \( \chi_{\varepsilon} u \in W_0^{1,p}(\Omega; \mathbb{R}^d) \) and

\[
\|\nabla(\chi_{\varepsilon} u) - \nabla u\|_{L^p} \leq C\varepsilon^{1/r}(\|\nabla u\|_{L^p} + \frac{1}{\varepsilon}\|u\|_{L^p})
\]

for a suitable constant \( C \), where \( r \in [1, \infty] \) is given by \( 1/q = 1/p + 1/r \).

(c) For \( u_k \to 0 \) in \( W^{1,p}(\Omega; \mathbb{R}^m) \) show the estimate \( \lim \inf_{k \to \infty} I_A(u_k) \geq I_A(0) \). \( \text{(Hint: Recall } p > q \text{ and consider } \chi_{\varepsilon_k} u_k \text{ for a good sequence } \varepsilon_k \to 0.\)\)

(d) Why is the assumption \( p > q \) bad for the direct method in the calculus of variations?

Exercise 41. Counterexample concerning Reshetnyak’s theorem. Take \( m = d = p = 2 \) and \( \Omega = [-1,1]^2 \) and the sequence

\[
u^k(x_1, x_2) = \frac{1}{\sqrt{k}} (1-|x_2|)^k \sin(kx_1), \cos(kx_1))
\]

We will show that \( \nabla \nu^k \to 0 \) in \( L^2(\Omega) \) but \( \det(\nabla \nu^k) \not\to 0 \) in \( L^1(\Omega) \).

(a) Show that \( \nu^k \to 0 \) in \( H^1(\Omega; \mathbb{R}^2) \).

(b) Prove that \( \int_{\Omega} \det(\nabla \nu^k) \varphi \, dx \to 0 \) for all \( \varphi \in C_c(\Omega) \).

(c) Show that \( \det(\nabla \nu^k) \) does not converge weakly to \( 0 \) in \( L^1(\Omega) \). \( \text{(Hint: Consider suitable } \varphi \in L^\infty(\Omega) \text{ in (b).)} \)

Exercise 42. Cofactor matrix and adjugate matrix.

(Auf deutsch: Kofaktormatrix und adjunkte Matrix)

For a quadratic matrix \( A \in \mathbb{R}^{d \times d} \) define \( \text{cof} A \in \mathbb{R}^{d \times d} \) such that \( \text{cof}(A)_{ij} = (-1)^{i+j}M_{ij} \), where \( M_{ij} \) is the determinant of the \( (d-1) \times (d-1) \) matrix obtained after deleting column \( i \) and row \( j \). Moreover, \( \text{adj}(A) = \text{cof}(A)^T \).

(a) For \( f(A) = \det A \) show \( Df(A)[B] = \text{cof}(A) : B = \text{tr}(\text{adj}(A)B) \).

(b) Prove the formula \( \text{cof}(A)A^T = \text{adj}(A)A = \det(A)I \). Relate this to Cramer’s rule and to Euler’s formula \( qf(A) = (Df(A), A) \) for \( q \)-homogeneous functions.

(c) For \( d = 2 \) and \( d = 3 \) show \( \det(A+B) = \det A + \text{cof}(A) : B + \det B \) and \( \det(A+B) = \det A + \text{cof}(A) : B + A : \text{cof}(B) + \det B \), respectively.