Exercise 35. **Ground state in quantum mechanics.** The spatial density distribution of the electron in a hydrogen atom is given in terms of the density $\rho : \Omega \to [0, \infty]$, where for simplicity we let $\Omega = B_R(0) \subset \mathbb{R}^3$. The wave function $\psi : \Omega \to \mathbb{C} \simeq \mathbb{R}^2$ defines the density $\rho$ via $\rho(x) = |\psi(x)|^2$. The ground state is the minimizer of the total energy $I(\psi) = I_{\text{kin}}(\psi) + I_{\text{Coul}}(\psi)$ under the constraint $\int_\Omega |\psi(x)|^2 \, dx = \int_\Omega \rho(x) \, dx = 1$.

The kinetic energy $I_{\text{kin}}(\psi) = \int_\Omega \mu |\nabla \psi(x)|^2 \, dx$ and the Coulomb-interaction energy $I_{\text{Coul}}(\psi) = \int_\Omega -\gamma |x| |\psi(x)|^2 \, dx$ are given via positive physical constants $\mu$ and $\gamma$.

(a) Show that a ground state exists for all $R \in [0, \infty]$.

*Hint:* Show and use $(x \mapsto 1/|x|) \in L^2(\Omega)$ and $\|u\|_{L^4} \leq C \|u\|_{L^2}^{1/4} \|u\|_{H^1}^{3/4}$.

(b) Show that there is a real and nonnegative ground state, i.e. $\psi(x) = \text{Re} \psi(x) \geq 0$ a.e.

(c) For the physical case $R = \infty$ show that a solution of the form $\psi(x) = \alpha e^{-\beta |x|}$ satisfies the Euler–Lagrange equations.

Exercise 36. **Variational inequality.** We consider a string which is fixed rigidly on the left end and which is elastically supported on the right end under the constraint that the support is in a given interval.

Characterize the minimizer of $I(u) = \int_0^1 \left\{ \frac{1}{2} u''(x)^2 - h(x) u \right\} \, dx + \frac{k}{2} (u(1) - u_0)^2$ under the constraint $u(1) \in [-1, 2]$, where $h \in L^2(0,1)$, $k > 0$ and $u_0 \in \mathbb{R}$ are given.

(a) Give a functions space and a constraint set, such that always a minimizer exists. Is it unique?

(b) Solve the constraint minimization problem for the case $h = 0$ explicitly, and characterize the set of $(k, u_0)$ for which one of the two constraints is active.