Exercise 32. Minimizers and Euler-Lagrange equations. Let $\Omega \subset \mathbb{R}^d$ be a bounded, Lipschitz domain and $p \in [1, \infty[$. For $h \in L^\infty(\Omega)$ and integers $n, m \in \mathbb{N}$ consider the functional
\[
I(u) = \int_\Omega \left\{ \frac{1}{p} |\nabla u|^p + \frac{1}{2m} u^{2n} + \cos(u^m) - uh \right\} \, dx
\]
on the space $X = W^{1,p}(\Omega)$.
(a) Discuss the existence of global minimizers $u_*$.
(b) Give sufficient conditions such that $I$ is Gateaux differentiable on all of $X$. Give conditions such that $D^{I}(u)[\varphi]$ exists for all $\varphi \in C^1(\overline{\Omega})$.
(c) Using extra conditions for $u_*$ give conditions such that $D^{I}(u_*)[\varphi] = 0$ holds for $v \in X$ and give conditions such that $I(u_*)[\varphi] = 0$.

Exercise 33. Continuity of Gateaux derivative implies Fréchet derivative. Consider a functional $I : X \to R$ that is continuously Gateaux differentiable, i.e. $u \mapsto D^G I(u)$ is a norm-norm continuous mapping from $X$ to $X^*$. Conclude that $I$ is also Fréchet differentiable with $D^{G} I(u) = D^G I(u)$.
(Hint: For Gateaux differentiable functionals $J$ show first (*) $|J(u_1) - J(u_0)| \leq \ell \| u_1 - u_0 \|$ with $\ell = \sup \{ \| D^G J((1-\theta)u_0 + \theta u_1) \|_{X^*} | \theta \in [0,1] \}$. Then, for fixed $u$ consider the functional $R(h) = I(u+h) - I(u) - D^G I(u)[h]$.)

Exercise 34. Continuity of Gateaux derivative. Let $\Omega \subset \mathbb{R}^d$ be a bounded, Lipschitz domain and $r, p, q \in [1, \infty[$ such that $W^{1,p}(\Omega) \subset L^q(\Omega)$.
(a) Consider a function $g \in C^1(\overline{\Omega} \times \mathbb{R}^m; \mathbb{R}^k)$ satisfying
\[
\exists C > 0 \ \forall (x,u) \in \overline{\Omega} \times \mathbb{R}^m : \ |g(x,u)| \leq C (1 + |u|)^{r/q}.
\]
We define the Nemitskii operator $G : L^r(\Omega; \mathbb{R}^m) \to L^q(\Omega; \mathbb{R}^k)$ via $G(u)(x) = g(x, u(x))$. Show that $G$ is norm-norm continuous.
(b) Consider the functional $I(u) = \int_\Omega f(x, u(x), \nabla u(x)) \, dx$ where $f \in C^1(\overline{\Omega} \times \mathbb{R}^m \times \mathbb{R}^{m \times d})$ satisfies
\[
|f(x,u,A)| + |\partial_u f(x,u,A)|^{\prime} + |\partial_A f(x,u,A)|^{\prime} \leq C (1 + |u|^q + |A|^p).
\]
Show that the Gateaux derivate $u \mapsto D^G I(u)$ given by
\[
D^G I(u)[v] = \int_\Omega \{ \partial_u f(x,u(x),\nabla u(x)) \cdot v(x) + \partial_A f(x,u(x),\nabla u(x)) : \nabla v(x) \} \, dx
\]
is norm-norm continuous from $W^{1,p}(\Omega; \mathbb{R}^m)$ to $W^{1,p}(\Omega; \mathbb{R}^m)^*$.
(c) Conclude that $I$ in (b) is even continuously Fréchet differentiable.