# Exercise Sheet 10

**Exercise 29. A fundamental lemma for Sobolev functions.** Consider a bounded, Lipschitz domain $\Omega \subset \mathbb{R}^d$ and $u \in \mathrm{W}^{1,p}(\Omega)$ for some $p \in [1, \infty]$. Assume that $\nabla u = 0$ in $\mathrm{L}^p(\Omega)$. Show that there exists a constant $c \in \mathbb{R}$ such that $u(x) = c$ a.e. in $\Omega$.
(*Hint: Smoothing might be needed and connecting curves between points in* $\Omega$.)

**Exercise 30. Coercivity of a degenerately convex functional.** Consider $\Omega = B_1(0) \subset \mathbb{R}^d$, $\alpha > 0$, $q \in \,]1, \infty[$, and

$$I(u) = \int_\Omega |x|^\alpha |\nabla u(x)|^q \,\mathrm{d}x.$$

(a) For $p \in \,]1, q]$ show that $I : \mathrm{W}^{1,p}_0(\Omega) \to [0, \infty]$ is strictly convex.
(b) Use a Hölder estimate to derive the coercivity of $I$ on $\mathrm{W}^{1,p}_0(\Omega)$ if $p\alpha < d(q-p)$ holds.
(c) Finally consider the case $d = 1$ and the set $M = \{\, u \in \mathrm{W}^{1,p}(\Omega) \mid u(\pm 1) = \pm 1 \,\}$. Study the existence of a minimizer of $I$ on $M$ for all cases of $\alpha$, $p$, and $q$. If possible, give the minimizer explicitly.

**Exercise 31. Weak and strong continuity.** Consider $f \in \mathrm{C}^0(\overline{\Omega} \times \mathbb{R}^m)$ which satisfies

$$\exists\, C > 0,\ p \in [1, \infty[\,,\ h \in \mathrm{L}^1(\Omega)\ \forall\, (x, u) \in \Omega \times \mathbb{R}^m : \quad |f(x, u)| \le C(h(x) + |u|^p).$$

(a) Show that the functional $I : \mathrm{L}^p(\Omega; \mathbb{R}^m) \to \mathbb{R};\ u \mapsto \int_\Omega f(x, u(x))\,\mathrm{d}x$ is strongly continuous.
(b) Show that weak continuity of $I$ implies that $f(x, \cdot)$ is affine, i.e. there exist $a \in \mathrm{C}^0(\overline{\Omega})$ and $b \in \mathrm{C}^0(\overline{\Omega}; \mathbb{R}^m)$ such that $f(x, u) = a(x) + b(x) \cdot u$.
(*Hint: Look at functions* $u$ *rapidly oscillating between two values* $w_0$ *and* $w_1$ *such that the weak limit takes the value* $w_\theta := (1-\theta)w_0 + \theta w_1$.)

**Special Christmas Problem*:** For $N \in \mathbb{N}$ and $\gamma \in \mathbb{R}$ consider on $\mathrm{H}^2(\mathbb{S}) = \mathrm{W}^{2,2}(\mathbb{S})$ with $\mathbb{S} = \mathbb{R}/_{2\pi\mathbb{Z}}$ (unit circle of length $2\pi$) the functional

$$J(u) = \int_{\mathbb{S}} \Big\{ \frac{1}{2}\big(u''(x) + N^2 u(x)\big)^2 + \frac{N^2\gamma}{3} u(x)^3 + \frac{1}{4} u(x)^4 \Big\}\,\mathrm{d}x.$$

We extend $J$ to a functional $I : H := \mathrm{L}^2(\mathbb{S}) \to \mathbb{R}_\infty$ by setting it $\infty$ outside of $\mathrm{H}^2(\mathbb{S})$.
(a) Show that $I$ is weakly lower semicontinuous on $H$ and coercive for each fixed $N$.
(b) We are interested in equicoercivity with respect to $N \in \mathbb{N}$. Construct $\gamma_1 > 0$ such that for $|\gamma| < \gamma_1$ there exists $C_\gamma$ with the following property:

$$\forall\, N \in \mathbb{N}\ \forall\, u \in H : \quad I(u) \ge \frac{1}{C_\gamma}\|u\|^2 - C_\gamma. \qquad (**)$$

(c) Show that there exists $\gamma_2 > \gamma_1$ such that (**) does not hold for $\gamma > \gamma_2$.
(d)* Try find the optimal values for $\gamma_1 = \gamma_2$.

* **The best reasonably complete solution will be rewarded with a book coupon of 50 Euro. Deadline for submission is February 9, 2020.**