Exercise Sheet 9

Exercise 27. Non-compactness of embeddings. For a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ we consider the embeddings

$$1 < p < d: \quad \mathrm{W}^{1,p}(\Omega) \to \mathrm{L}^{p^*}(\Omega) \qquad \text{and} \qquad d < q < \infty: \quad \mathrm{W}^{1,q}(\Omega) \to \mathrm{C}^{1-d/q}(\overline{\Omega}).$$

Construct suitable sequences $(u_k)_{k\in\mathbb{N}}$ that prove that the embeddings are not compact. (*Hint: Try $u_k(x) = k^\alpha \varphi(k^\beta x)$ for some function φ.*)

Exercise 28. Optimal p^∂ for the trace mapping. For a bounded Lipschitz domain $\Omega = \Gamma\times]0,1[\subset \mathbb{R}^d$ and $p \in]1,d[$ we set $p^\partial = p(d-1)/(d-p) \in]p,\infty[$. Show that the trace mapping from $\mathrm{W}^{1,p}(\Omega)$ into $\mathrm{L}^{p^\partial}(\Gamma\times\{0\})$ exists as a bounded linear operator.
(*Hint: For $u \in \mathrm{C}^1(\overline{\Omega})$ define $w(y,s) = (1-s)\,|v(x,s)|^{p^\partial}$ and write $w(y,0)$ via na integral over $s \in [0,1]$.*)