Exercise Sheet 7

**Definition:** The epigraph of a function \( I : X \to \mathbb{R}_\infty \) is defined as

\[
\text{epi}(I) := \{ (u, \alpha) \in X \times \mathbb{R} \mid I(u) \leq \alpha \} \subset X \times \mathbb{R}.
\]

**Exercise 21. Estimates via affine functions for convex functions.** Consider a proper, convex, and lower semicontinuous functional \( I : X \to \mathbb{R}_\infty \).

(a) Show that for all \( u \) with \( I(u) < \infty \) and all \( \varepsilon > 0 \) there exists \( \xi \in X^* \) such that \( I(u+v) \geq I(u) - \varepsilon + \langle \xi, v \rangle \) for all \( v \in X \). (*Hint: Use epi(I) and separate it in \( X \times \mathbb{R} \) from a suitable set.*)

(b) Show that for all \( u \) with \( I(u) = \infty \) and all \( M \in \mathbb{R} \) there exists \( \xi_M \in X^* \) such that \( I(u+v) \geq M + \langle \xi_M, v \rangle \) for all \( v \in X \). (*Hint: Work in \( X \times \mathbb{R} \) and construct a line segment connecting \( (u, M) \) and \( (u_1, I(u_1) - 1) \) that does not intersect epi(I).)

(c) Conclude from (a) and (b) (without using sublevels) that \( I \) is weakly lower semicontinuous.

**Exercise 22. Bounded convex functions are Lipschitz continuous.** Let \( I : X \to \mathbb{R}_\infty \) be proper, convex, and lsc. Assume further that

\[
\exists M, K \in \mathbb{R} \forall u \in B_R(u_*) : \quad K \leq I(u) \leq M.
\]

Show that \( I \) restricted to \( B_r(u_*) \) with \( r \in ]0, R[ \) is Lipschitz continuous with a Lipschitz constant that only depends on \( M-K \) and \( r/R \).

**Exercise 23. Continuity points of convex functionals.** For a proper, lower semicontinuous convex functional \( I : X \to \mathbb{R}_\infty \) on a Banach space \( X \) the domain is defined via

\[
\text{dom}(I) := \{ u \in X \mid I(u) < \infty \} \neq \emptyset.
\]

(a) Show that for \( u_1 \in \text{dom}(I) \) the following conditions are equivalent:

(i) \( \exists \delta > 0 : \sup \{ I(u) \mid u \in B_\delta(u_1) \} < \infty \);

(ii) \( I \) is continuous in \( u_1 \).

(b) Show that \( I \) is continuous on \( A := \text{int}(\text{dom}(I)) \), if \( I \) is continuous at one \( u_1 \in A \).

(c) Assume that \( I \) is continuous at one \( u_1 \in A \). Find a supporting hyperplane for all \( u \in A \), i.e. there exists \( \beta \in X^* \) such that \( I(u+v) \geq I(u) + \langle \beta, v \rangle \) for all \( v \in X \).

(*Hint: Use the “open epigraph” \( \{ (u, \alpha) \in X \times \mathbb{R} \mid u \in A, I(u) < \alpha \} \).)

**Exercise 24. Sobolev embeddings.** Let \( \Omega = B_1(0) \subset \mathbb{R}^d \).

(a) Consider the function \( u : \Omega \to \mathbb{R} \) with \( u(x) = |x|^\alpha \) for \( x \neq 0 \) and \( u(0) = 0 \). For which \( p \) do we have \( u \in L^p(\Omega) \) and for which \( u \in W^{1,p}(\Omega) \)?

(b) Consider the function \( u(x) = (1 - \log |x|)^\beta \) with \( \beta \in \mathbb{R} \). For which \( \beta \) and \( p \in [1, \infty] \) do we have \( u \in L^p(\Omega) \) and for which \( u \in W^{1,p}(\Omega) \)?

(c) For the case \( d \geq 2 \) give a function \( u \in W^{1,d}(\Omega) \setminus L^\infty(\Omega) \).