Exercise Sheet 5

Exercises 14 to 16 are still to be discussed on November 21, 2019.

Exercise 17. Quasiconvexity. The original definition of quasiconvexity for a continuous function $F : \mathbb{R}^{m\times d} \to \mathbb{R}$ reads

$$\forall\, \widetilde{w} \in \mathrm{PC}^1_0(\overline{\Omega};\mathbb{R}^m) : \quad \int_\Omega F\big(A + \nabla\widetilde{w}(y)\big)\,\mathrm{d}y \geq |\Omega|\,F(A)$$

involves as domain $\Omega = B_1(0) \subset \mathbb{R}^d$.

(a) Show that in the definition of quasiconvexity any open bounded domain $\Omega \subset \mathbb{R}^d$ can be used without changing the definition. (*Hint: Show and use that for two open and bounded sets Ω and $\widetilde{\Omega}$ we always have $x_*{+}r\widetilde{\Omega} \subset \Omega$ for suitable $x_* \in \mathbb{R}^d$ and $r > 0$.*)

(b) Considering $\Omega = Q := \,]0,1[^d \subset \mathbb{R}^d$ we may look at periodic functions $\psi \in \mathrm{PC}^1_{\mathrm{per}}(\mathbb{R}^d;\mathbb{R}^m)$, i.e. $\psi \in \mathrm{PC}^1(\mathbb{R}^d;\mathbb{R}^m)$ with $\psi(m{+}y)$ for $m \in \mathbb{Z}^d$ and $y \in \mathbb{R}^d$. Show that quasiconvexity in A is equivalent to

$$\forall\, \psi \in \mathrm{PC}^1_{\mathrm{per}}(\mathbb{R}^d;\mathbb{R}^m) : \quad \int_Q F(A{+}\nabla\psi(x))\,\mathrm{d}x \geq F(A).$$