Consider a Lipschitz function $\tilde{u}: \mathbb{R}^d \rightarrow \mathbb{R}^m$, i.e. for $L = \text{Lip}(\tilde{u})$ we have
$$\forall x, y \in \mathbb{R}^d : |\tilde{u}(x) - \tilde{u}(y)| \leq L|x - y|.$$ (a) Take the Dirac sequence $\psi_\delta$ from the lectures and define
$$u_\delta = \tilde{u} * \psi_\delta : x \mapsto \int_{\mathbb{R}^d} \tilde{u}(y)\psi_\delta(x - y)\,dy.$$ Show that $\text{Lip}(u_\delta) \leq L$ and $\|\tilde{u} - u_\delta\|_{C^0} \leq L\delta$.
(b) For any $\lambda, \mu, \delta$, take the Dirac sequence $\psi_\delta$ and consider $w \in C^1(\mathbb{R}^d; \mathbb{R}^m)$ establish the identity
$$\text{Lip}_{B_R(x_0)}(w) = \sup\{ \|\nabla w(y)\|_{\mathbb{R}^m} : y \in B_R(x_0) \},$$ where the expression in left-hand side indicates the smallest Lipschitz constant of $w|_{B_R(x_0)}$.
(Hint: For estimating $w(x) - w(y)$ consider $w$ on the connecting line.)
(c) Conclude $\|\nabla u_\delta\|_{C^0} \leq L = \text{Lip}(\tilde{u})$.

Exercise 15. Second variation Consider the functional $I : C^1(\overline{\Omega}; \mathbb{R}^m) \rightarrow \mathbb{R}$ with $I(u) = \int_\Omega f(x, u(x), \nabla u(x))\,dx$, where $f \in C^2(\Omega \times \mathbb{R}^m \times \mathbb{R}^{m \times d})$. For $\gamma_1, \gamma_2 > 0$ assume the estimates
$$\int_\Omega \partial^2_A f(x, u_0(x), \nabla u_0(x))|\nabla w, \nabla w|\,dx \geq \gamma_1 \int_\Omega |\nabla w|^2\,dx,$$ (Eq.1)
$$D^2 I(u_0)[w, w] \geq \gamma_2 \int_\Omega |w|^2\,dx.$$ (Eq.2)
(a) Use (Eq.1) and suitable estimates for $\partial_A \partial_u f$ and $\partial_u^2 f$ to find $C^*$ such that
$$D^2 I(u_0)[w, w] \geq \gamma_1/2 \int_\Omega |\nabla w|^2\,dx - C^*|w|^2\,dx \text{ for all } w.$$ (b) Combine (Eq.2) and (Eq.1) to find $\gamma_3 > 0$, such that
$$D^2 I(u_0)[w, w] \geq \gamma_3 \int_\Omega |\nabla w|^2 + |w|^2\,dx \text{ for all } w \in C^1(\overline{\Omega}; \mathbb{R}^m).$$

Exercise 16. Anisotropic elasticity theory. The functional $I : C^1(\overline{\Omega}; \mathbb{R}^d) \rightarrow \mathbb{R}$; $u \mapsto \int_\Omega f(\nabla u)\,dx$ is defined via
$$f(A) = \frac{\lambda}{2}(\text{spur}A)^2 + \frac{\mu}{4} |A + A^T|^2 + \frac{\delta}{2} A_{11}^2.$$ (a) Establish the formula $\partial^2_A f(A)[B, B] = 2f(B)$ for all $A, B \in \mathbb{R}^{d \times d}$.
(b) For which $\lambda, \mu, \delta \in \mathbb{R}$ do we have $f(A) \geq 0$ for all $A \in \mathbb{R}^{d \times d}$ (which is equivalent to convexity)? Try first to solve the case $d = 2$.
(Hint: For testing the positivity, it essentially suffices to consider diagonal matrices.)
(c) For which $\lambda, \mu, \delta \in \mathbb{R}$ does $f$ satisfy the LEGENDRE–HADAMARD condition? Try first to solve the case $d = 2$.
(Hint: Write $\partial^2_A f(x, u, A)[b \otimes \eta, b \otimes \eta] \geq 0$ in the form $K(\eta b \cdot b \geq 0$ with $K(\eta) \in \mathbb{R}^{d \times d.}$)