Exercise 10. Euler-Lagrange equation. Consider the domain $\Omega \subset \mathbb{R}^2$, the set $M = C^2(\Omega, \mathbb{R})$, and the functional $I(u) : M \to \mathbb{R}$ defined via
\[
I(u) = \int_{\Omega} \left[ \frac{1}{2} \left( \nabla u(x) \cdot \begin{pmatrix} 9 & 5 \\ 3 & 2 \end{pmatrix} \nabla u(x) + \frac{\pi^2}{17} u^2 \right) - \frac{1}{3} u^3 \right] \, dx + \int_{\partial \Omega} 9 \sin u \, da.
\]
Derive the associated Euler–Lagrange equation including boundary condition.

Exercise 11. Noether’s theorem for rotationally invariant systems. The density $f \in C^2([\alpha, \beta] \times \mathbb{R}^m \times \mathbb{R}^m; \mathbb{R})$ with $m \geq 2$ defines the functional $I(u) = \int_{\alpha}^{\beta} f(t, u(t), \dot{u}(t)) \, dt$. For the rotation matrix $R_\varphi \in \mathbb{R}^{m \times m}$ with
\[
R_\varphi(y_1, y_2, y_3, \ldots, y_m)^\top = \begin{pmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{pmatrix}, \quad \varphi \in \mathbb{R}, \; y \in \mathbb{R}^m,
\]
the density $f$ satisfies the rotational symmetry $f(t, R_\varphi u, R_\varphi A) = f(t, u, A)$ for all $t, u, A, \varphi$.
(a) Show that along solutions $u : [\alpha, \beta] \to \mathbb{R}^m$ of the Euler–Lagrange equation we have conservation of the moment of momentum (Drehimpulserhaltung):
\[
\frac{d}{dt} \left[ u_1(t) \partial_{A_2} f(t, u(t), \dot{u}(t)) - u_2(t) \partial_{A_1} f(t, u(t), \dot{u}(t)) \right] = 0.
\]
(Hint: Calculate first $\frac{d}{d\varphi} f(t, R_\varphi u, R_\varphi A)\big|_{\varphi=0}$)
(b) Now consider $R_\varphi = e^{iB} \in \mathbb{R}^{m \times m}$ for a general $B \in \mathbb{R}^{m \times m}$ with $B = -B^\top$ and assume the symmetry $f(t, R_\varphi u, R_\varphi A) = f(t, u, A)$. Which quantity $J(u, \dot{u})$ is now conserved?

Exercise 12. Weak and strong local minimizers. Consider $M = C^1([a, b]; \mathbb{R})$, functions $g, h \in C^2(\mathbb{R}; \mathbb{R})$, and the functional $I : M \to \mathbb{R}$ defined via
\[
I(u) = \int_{a}^{b} \left\{ g(u'(x)) + h(u(x)) \right\} \, dx.
\]
(a) Derive the associated Euler–Lagrange equation. Which conditions guarantee that critical points of the form $\overline{u}(x) = u^0 = \text{const}$ exist?
(b) Assume that $\overline{u}(x) = u^0 = \text{const}$ is a critical point of $I$. Show that the conditions $h''(u^*) > 0$ and $g''(0) > 0$ are sufficient to imply that $\overline{u}$ is a strict weak local minimizer.
(c) Assume now that $g(A) \geq 0 = g(0)$ for all $A \in \mathbb{R}^{1 \times 1}$ and that $u^0$ is a local minimizer of $h$. Show that $\overline{u}$ is a strong local minimizer. What additional conditions imply that $\overline{u}$ is a global minimizer?

(please turn)
Exercise 13. Minimal surface of revolution. Consider $I : M \rightarrow \mathbb{R}$ with

$$I(u) = \int_0^\ell 2\pi u(x) \sqrt{1+u'(x)^2} \, dx, \quad M = \{ u \in C^1([0, \ell]) \mid u(x) \geq 0, \, u(0) = r_0, \, u(\ell) = r_\ell \}.$$ 

Solutions of the EULER–LAGRANGE equation have the form $u(x) = U(c, d, x) = c \cosh \left( \frac{x-d}{c} \right)$.

(a) Consider the case $r_0 = r_\ell$ and show that we may choose $d = \ell/2$. Consider $c$ as free parameter, which determines $\ell_c$ and thus the solution $u_c$ and $I(u_c)$. Discuss the number of solutions for different values of $\ell$. For these solutions plot (using a computer!) the value of $I(u_c)$ in dependence of $\ell_c$ and a (multi-valued) parametric plot giving $I(u)$ in dependence of $\ell$. (of the curve $c \mapsto (I(u_c), \ell_c)$).

(b) Show numerically that there is a number $k_{\text{crit}} \in [1.2, 1.6]$ such that $\max \ell_c = k_{\text{crit}} r_0$. Compare $k_{\text{crit}}$ to our experimental value $13.5\, \text{cm}/9\, \text{cm} = 1.5$ obtained by two volunteers on 30.10.2019.

(c) Consider arbitrary $r_0 > 0$ and $r_\ell > 0$. Derive the estimate $i(r_0, r_1, \ell) := \inf \{ I(u) \mid u \in M \} \leq \pi (r_0^2 + r_\ell^2)$ via suitable sequences. Compare with (a).

(d) Provide a good lower bound for $i(r_0, r_0, \ell)$ by using $u_m = \min \{ u(x) \mid x \in [0, \ell] \}$ and the estimate $\sqrt{1+u'^2} \geq \max \{ 1, |u'| \}$. (Hint: Use $|u'| \, dx = |du|$ and minimize w.r.t. $u_m$.)