Exercise Sheet 3

Exercise 10. Euler-Lagrange equation. Consider the domain $\Omega \in \mathbb{R}^2$, the set $M = \mathrm{C}^2(\overline{\Omega}, \mathbb{R})$, and the functional $I(u) : M \to \mathbb{R}$ defined via

$$I(u) = \int_\Omega \left[\frac{1}{2} \left(\nabla u(x) \cdot \begin{pmatrix} 9 & 5 \\ 3 & 2 \end{pmatrix} \nabla u(x) + \frac{\pi^2}{17} u^2 \right) - \frac{1}{3} u^3 \right] \mathrm{d}x + \int_{\partial\Omega} 9 \sin u \, \mathrm{d}a.$$

Derive the associated EULER–LAGRANGE equation including boundary condition.

Exercise 11. Noether's theorem for rotationally invariant systems. The density $f \in \mathrm{C}^2([\alpha, \beta] \times \mathbb{R}^m \times \mathbb{R}^m; \mathbb{R})$ with $m \geq 2$ defines the functional $I(u) = \int_\alpha^\beta f(t, u(t), \dot{u}(t)) \, \mathrm{d}t$. For the rotation matrix $R_\varphi \in \mathbb{R}^{m \times m}$ with

$$R_\varphi(y_1, y_2, y_3, ..., y_m)^\top = (\cos\varphi \, y_1 - \sin\varphi \, y_2, \sin\varphi \, y_1 + \cos\varphi \, y_2, y_3, ..., y_m)^\top, \ \varphi \in \mathbb{R}, \ y \in \mathbb{R}^m,$$

the density f satisfies the rotational symmetry $f(t, R_\varphi u, R_\varphi A) = f(t, u, A)$ for all t, u, A, φ.

(a) Show that along solutions $u : [\alpha, \beta] \to \mathbb{R}^m$ of the EULER–LAGRANGE equation we have conservation of the moment of momentum (Drehimpulserhaltung):

$$\frac{\mathrm{d}}{\mathrm{d}t} [u_1(t) \partial_{A_2} f(t, u(t), \dot{u}(t)) - u_2(t) \partial_{A_1} f(t, u(t), \dot{u}(t))] = 0.$$

(*Hint: Calculate first* $\frac{\mathrm{d}}{\mathrm{d}\varphi} f(t, R_\varphi u, R_\varphi A)\big|_{\varphi=0}$.)

(b) Now consider $R_\varphi = \mathrm{e}^{\varphi B} \in \mathbb{R}^{m \times m}$ for a general $B \in \mathbb{R}^{m \times m}$ with $B = -B^\top$ and assume the symmetry $f(t, R_\varphi u, R_\varphi A) = f(t, u, A)$. Which quantity $J(u, \dot{u})$ is now conserved?

Exercise 12. Weak and strong local minimizers. Consider $M = \mathrm{C}^1([a, b]; \mathbb{R})$, functions $g, h \in \mathrm{C}^2(\mathbb{R}; \mathbb{R})$, and the functional $I : M \to \mathbb{R}$ defined via

$$I(u) = \int_a^b \left\{ g(u'(x)) + h(u(x)) \right\} \mathrm{d}x.$$

(a) Derive the associated EULER–LAGRANGE equation. Which conditions guarantee that critical points of the form $\overline{u}(x) = u^0 = \text{const}$ exist?

(b) Assume that $\overline{u}(x) = u^0 = \text{const}$ is a critical point of I. Show that the conditions $h''(u^*) > 0$ and $g''(0) > 0$ are sufficient to imply that $\overline{u}$ is a strict weak local minimizer.

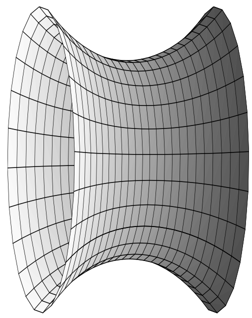
(c) Assume now that $g(A) \geq 0 = g(0)$ for all $A \in \mathbb{R}^{1 \times 1}$ and that u^0 is a local minimizer of h. Show that $\overline{u}$ is a strong local minimizer. What additional conditions imply that $\overline{u}$ is a global minimizer?

(please turn)

Exercise 13. Minimal surface of revolution. Consider $I : M \to \mathbb{R}$ with

$$I(u) = \int_0^\ell 2\pi u(x)\sqrt{1+u'(x)^2}\,\mathrm{d}x, \;\; M = \{\, u \in \mathrm{C}^1([0,\ell]) \mid u(x) \geq 0, \; u(0) = r_0, \; u(\ell) = r_\ell \,\}.$$

Solutions of the EULER–LAGRANGE equation have the form $u(x) = U(c,d,x) = c\cosh\left(\frac{x-d}{c}\right)$.



(a) Consider the case $r_0 = r_\ell$ and show that we may choose $d = \ell/2$. Consider c as free parameter, which determines ℓ_c and thus the solution u_c and $I(u_c)$. Discuss the number of solutions for different values of ℓ. For these solutions plot (using a computer!) the value of $I(u_c)$ in dependence of ℓ_c and a (multi-valued) parametric plot giving $I(u)$ in dependence of ℓ. (of the curve $c \mapsto (I(u_c), \ell_c)$).

(b) Show numerically that there is a number $k_{\mathrm{crit}} \in [1.2, 1.6]$ such that $\max \ell_c = k_{\mathrm{crit}} r_0$. Compare k_{crit} to our experimental value $13.5\mathrm{cm}/9\mathrm{cm} = 1.5$ obtained by two volunteers on 30.10.2019.

(c) Consider arbitrary $r_0 > 0$ and $r_\ell > 0$. Derive the estimate $i(r_0, r_1, \ell) := \inf\{\, I(u) \mid u \in M \,\} \leq \pi(r_0^2 + r_\ell^2)$ via suitable sequences. Compare with (a).

(d) Provide a good lower bound for $i(r_0, r_0, \ell)$ by using $u_{\mathrm{m}} = \min\{\, u(x) \mid x \in [0,\ell] \,\}$ and the estimate $\sqrt{1+u'^2} \geq \max\{1, |u'|\}$. (*Hint: Use* $|u'|\,\mathrm{d}x = |\mathrm{d}u|$ *and minimize w.r.t.* u_{m}.)