Partial Differential Equations
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Partial Differential Equations
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Quasilinear form versus divergence form, non-characteristic Cauchy data,
elliptic, parabolic, hyperbolic

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D'Alembert formula for $u_{tt} = u_{xx}$, elliptic equation via power series, parabolic equation

3.3 Theorem of Cauchy–Kovalevskaya

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Hölder regularity $u \in \mathrm{C}^{2+\alpha}(\mathbb{R}^d)$ for $\Delta u = f \in \mathrm{C}^{\alpha}_{\mathrm{c}}(\mathbb{R}^d)$

4.3 Lax–Milgram theory

Hilbert space theory, weak convergence, cONS, Sobolev spaces,
existence and uniqueness for general bilinear forms,
weak solutions for Dirichlet and Neumann problems, Poincaré and Friedrich's inequality

4.4 Spectral theory

Rellich's embedding theorem, spectrum of compact symmetric operators,
cONS obtained from eigenpairs for symmetric bilinear forms $B(\phi_k, v) = \lambda_k \langle \phi_k, v \rangle$
solutions of general symmetric elliptic equations by eigenvalue expansion

5. Parabolic equations

5.1 The heat kernel

Fourier transform in $\mathbb{R}^d$, convolution with heat kernel,
constant-of-variations formula for inhomogeneous equation

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... via heat kernel $\|\mathrm{D}_x^\alpha u(t)\|_{\mathrm{L}^p} \leq C_{|\alpha|} t^{-|\alpha|/2} \|u(0)\|_{\mathrm{L}^p}$ and via eigenfunction expansion

5.3 A general existence and uniqueness result

Weak solutions for general non-symmetric parabolic problems
Existence and uniqueness of solutions via finite-dimensional approximation:
1. Approximation, 2. A priori estimate, 3. Extraction of converging subsequeunce,
4. Identification of limit point as (weak) solution, 5. uniqueness.

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6.1 Classical solution formulas in $\mathbb{R}^d$

D'Alembert's formula, spherical means, soluton formulas for $d = 3$ and $d = 2$, Huygens principle

6.2 Solution via Fourier transform

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Energy conservation, weak solutions, and approximation