Partial Differential Equations
Exercise Sheet 10

Exercise 35. Parabolic problem on a half-space (reflection principle). We consider the half-space $\Omega = \mathbb{R}^{d-1}\times\,]0,\infty[\,\subset \mathbb{R}^d$ and $u_0 \in \mathrm{L}^1(\Omega)$.

(a) Construct a function $H_\mathrm{N} :]0,\infty[\times\Omega\times\Omega \to \mathbb{R}$ such that $u_\mathrm{N}(t,x) = \int_\Omega H_\mathrm{N}(t,x,y)u_0(y)\,\mathrm{d}y$ provides a solution of the initial-boundary-value problem

$$\begin{aligned} &\partial_t u = \Delta u \quad \text{ for } (t,x) \in\,]0,\infty[\,\times\Omega,\\ &\text{(IC)} \quad u(0,x) = u_0(x) \quad \text{ for } x \in \Omega,\\ &\text{(BC)} \quad \nabla u(t,x)\cdot\nu = 0 \quad \text{ for } t>0 \text{ and } x \in \partial\Omega. \end{aligned} \tag{HE}$$

Hint: Extend the initial condition u_0 symmetrically to all of $\mathbb{R}^d$ and show that the chosen symmetry is maintained by the full-space solution.

(b) Find similarly a function H_D such that $u_\mathrm{D}(t,x) = \int_\Omega H_\mathrm{D}(t,x,y)u_0(y)\,\mathrm{d}y$ solves the heat equation (HE), where the Neumann boundary conditions (BC) are replaced by the Dirichlet boundary conditions $u(t,x) = 0$ for $x \in \partial\Omega$.

(c) Consider now the case $u_0(x) \geq 0$. Show that $u_N(t,x) \geq u_\mathrm{D}(t,x) \geq 0$ and that $\int_\Omega u_\mathrm{N}(t,x)\,\mathrm{d}x = \int_\Omega u_0(x)\,\mathrm{d}x$ and $\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u_D(t,x)\,\mathrm{d}x \leq 0$.

Exercise 36. Convolutions.

(a) Give a short and self-contained proof (not relying on the result given in the lecture) of the following convolution estimate:

$$\forall\, f \in L^1(\mathbb{R}^d)\ \forall\, g \in \mathrm{L}^p(\mathbb{R}^d):\quad \|f{*}g\|_{\mathrm{L}^p(\mathbb{R}^d)} \leq \|f\|_{L^1(\mathbb{R}^d)}\|g\|_{L^p(\mathbb{R}^d)}.$$

(b) Using the convolution result from the lecture course show that for $p \geq q \geq 1$ there exists a constant $C_{d,p,q} > 0$ such that the solutions $u(t,\cdot) = \widetilde{H}_d(t,\cdot){*}u_0$ of the heat equation satisfy

$$\|u(t,\cdot)\|_{\mathrm{L}^p(\mathbb{R}^d)} \leq \frac{C_{d,p,q}}{t^{\alpha(d,p,q)}}\|u_0\|_{\mathrm{L}^q(\mathbb{R}^d)} \text{ for all } t>0, \text{ where } \alpha(d,p,q) = \frac{d}{2}\Big(\frac{1}{q}-\frac{1}{p}\Big).$$

(please turn over)

Exercise 37. Non-uniqueness for the heat equation. We want to show that the equation $u_t = u_{xx}$ has a solution $u \in \mathrm{C}^\infty(\mathbb{R}\times\mathbb{R})$ such that $u(t,x) = 0$ for $t \le 0$ and $x \in \mathbb{R}$, while $u(t,0) \ne 0$ for $t > 0$. We construct u in the form

$$u(t,x) = \sum_{k=0}^{\infty} h_k(t) x^k, \text{ where } h_1 \equiv 0,\ h_0 = g \in \mathrm{C}^\infty(\mathbb{R}) \text{ with } g(t) = 0 \text{ for } t \le 0.$$

(a) Show by formal calculations that u solves the heat equation if $h_k' = (k+2)(k+1)h_{k+2}$. Using $h_1 \equiv 0$ and $h_0 = g$ show that the solution has to have the form

$$u(t,x) = \sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k)!}\, x^{2k}.$$

(b) To show that the formula in (a) is convergent for $t > 0$ we choose the special function g with $g(t) = \mathrm{e}^{-1/t^2}$. Show the estimate

$$|g^{(k)}(t)| \le \frac{k!}{(Ct)^k}\, \mathrm{e}^{-1/(2t^2)}, \quad k \in \mathbb{N}_0, \quad t > 0.$$

Hint: Use that g has a holomorphic extension into $\{\, t \in \mathbb{C} \mid \operatorname{Re} t > 0 \,\}$ and that the derivatives satisfy $g^{(k)}(t) = \frac{k!}{2\pi \mathrm{i}} \oint_\Gamma \frac{g(\tau)}{(\tau - t)^{k+1}}\, \mathrm{d}\tau$.

(c) Prove that u defined in (a) is a C^∞-solution of the heat equation and explain why u is not given in the form $u(t,\cdot) = \widetilde{H}_d(t,\cdot) * u_0$.

Dates for the teaching evaluation: June 17–28, 2019
(token = name-year of Abel Prize winner).

Dates for the oral exams:
July 22 – 24, 2019 and September 30 – October 2, 2019.