Partial Differential Equations
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Partial Differential Equations
Exercise Sheet 4

Exercise 13: Burgers equation (named after Johannes Martinus Burgers, 1895–1981):

$$\partial_t u + u\partial_x u = 0\,.$$

We consider the piecewise constant functions

$$\widetilde{u}(t,x) = \begin{cases} u_+ & \text{for } x > ct, \\ u_- & \text{for } x < ct, \end{cases} \qquad \widehat{u}(t,x) = \begin{cases} u_1 & \text{for } x < c_1 t, \\ u_2 & \text{for } c_1 t < x < c_2 t, \\ u_3 & \text{for } x > c_2 t \end{cases}$$

(a) Show that $\widetilde{u}$ is a weak solution if and only if $2c = u_+ + u_-$. Under what conditions does the shock satisfy the entropy condition, i.e. the characteristics merge at the jump in forward time?

(b) Given u_1 and u_3 (and hence $\widehat{u}(0,x)$) find all u_2, c_1, and c_2 such that $c_1 < c_2$ and that $\widehat{u}$ is a weak solution for $t > 0$. When do the jumps satisfy the entropy condition?

(c) Construct a piecewise constant solution $u^{(1)}$ satisfying $u^{(1)}(0,x) = f_1(x)$ with $f_1(x) = 6$ for $x < -1$, $f_1(x) = 4$ for $|x| < 1$, and $f_1(x) = 2$ for $x > 1$.

(d) Give a second initial condition f_2 such that the solution $u^{(2)}$ satisfies $u^{(2)}(t,x) = u^{(1)}(t,x)$ for $t \geq 4$ and $x \in \mathbb{R}$ (i.e. entropy solutions are non-unique for negative time).

Exercise 14. Divergence form versus quasilinear form.

Quasilinear form: (QF) $A(x,u,\mathrm{D}u) : \mathrm{D}^2 u = b(x,u,\mathrm{D}u)$
Divergence from: (DF) $\operatorname{div} \mathbf{a}(x,u,\mathrm{D}u) = \widetilde{b}(x,u,\mathrm{D}u)$

(a) Assume that A does not depend on $\mathrm{D}u$. Show that (QF) can be rewritten as (DF). [Hint: Search for $\mathbf{a}(x,u,\xi)$ which is linear in ξ.]

(b) For $d = 2$ and $\mathbf{a}(x,u,\xi) = (\mathbf{a}_1(\xi_1,\xi_2), \mathbf{a}_2(\xi_1,\xi_2))^\mathsf{T}$ derive the associated A in (QF).

(c) Provide an elliptic example of (QF) that cannot be written as (DF). [Hint: It suffices to consider $A(\xi)$ in diagonal form.]

(d) For $f \in \mathrm{C}^2(\mathbb{R}^d,\mathbb{R})$ let $\mathbf{a}(\xi) = \mathrm{D}f(\xi) \in \mathbb{R}^d$ in (DF) and calculate the associated A for (QF). Show that (QF) is elliptic if f is uniformly convex, i.e. $\mathrm{D}^2 f(\xi) \geq c\mathrm{I}$ with $c > 0$ in the sense of positive definite matrices.

(please turn over)

Exercise 15. General jump conditions: Let $\mathbf{a} : \Omega\times\mathbb{R} \to \mathbb{R}^d$ and $b : \Omega\times\mathbb{R} \to \mathbb{R}$ define the quasilinear PDE in *conservation form*, also known as *divergence form*:

$$\begin{aligned}&\big(\operatorname{div}\mathbf{a}(\cdot, u(\cdot))\big)(x) = b(x, u(x)) \text{ for } x \in \Omega \subset \mathbb{R}^d,\\ &\text{with } \big(\operatorname{div}\mathbf{a}(\cdot, u(\cdot))\big)(x) := \textstyle\sum_{j=1}^d \partial_{x_j}\alpha_j(x) \text{ where } \alpha_j(x) = \mathbf{a}_j(x, u(x))\end{aligned}$$

(a) Write the equation in the quasilinear form $\widetilde{a}(x, u)\cdot\nabla u = \widetilde{b}(x, u)$ by assuming that everything is sufficiently smooth.

(b) A function $u \in \mathrm{L}^\infty(\Omega)$ is called *weak solution*, if

$$\int_\Omega \Big(\mathbf{a}(x, u(x))\cdot\nabla\phi(x) + b(x, u(x))\phi(x)\Big)\,\mathrm{d}x = 0 \quad \text{for all } \phi \in \mathrm{C}_\mathrm{c}^\infty(\Omega).$$

Show that classical solutions are weak solutions.

(c) Let $\mathcal{C}$ be a smooth hypersurface separating Ω into the two pieces Ω_+ and Ω_-. Let $u : \Omega \to \mathbb{R}$ be such that the restrictions $u_\pm = u|_{\Omega_\pm}$ have extensions lying in $\mathrm{C}^1(\overline{\Omega}_\pm)$. Derive the *jump relations*

$$\Big(\mathbf{a}(y, u_+(y)) - \mathbf{a}(y, u_-(y))\Big)\cdot\nu(y) \;=\; 0 \;\text{ for all } y \in \mathcal{C},$$

where ν is the normal vector to $\mathcal{C}$. (Hint: First show that in Ω_+ and Ω_- we have classical solutions. Then do integration by parts in Ω_+ and Ω_- separately and compare the boundary terms.)

(c) Discuss the jump relations in the special two-dimensional case

$$\partial_t\big(\alpha(x, u)\big) + \partial_x\big(\beta(x, u)\big) = b(t, x, u),$$

if $\mathcal{C}$ is given in the form $x = s(t)$.

Exercise 16. Wave equation: We consider the Cauchy problem

$$u_{tt} = u_{xx} + bu \text{ for } (t, x) \in \mathbb{R}^2, \quad u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x),$$

where b is a real parameter.

(a) Use a power-series expansion to find the solution of the Cauchy problem for $u_0(x) = \mathrm{e}^{\alpha x}$ and $u_1 \equiv 0$, where $\alpha \in \mathbb{C}$.

(b) Use the linearity to find the solution for $u_0(x) = \frac{1}{h}\big(\mathrm{e}^{(\alpha+h)x} - \mathrm{e}^{\alpha x}\big)$ and $u_1 \equiv 0$. Consider and justify the limit $h \to 0$ to find the solution v for the Cauchy data $v_0(x) = x\sin(kx)$ and $v_1 \equiv 0$ mit $k \in \mathbb{R}$.

(c) Return to the solution u from Part (a). Show that $w = u_t$ is again a solution. What are the Cauchy data for w at $t = 0$? Give the solution for the Cauchy data $u_0 \equiv 0$ and $u_1(x) = \mathrm{e}^{\alpha x}$.