Partielle Differentialgleichungen
Sommersemester 2019
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Partial Differential Equations
Exercise Sheet 1

Exercise 1: The Cauchy-Riemann equations (CRe). The CRe for $(u,v)^\top \in \mathrm{C}^1(\mathbb{R}^2;\mathbb{R}^2)$ are given as $\partial_x u - \partial_y v = 0$ and $\partial_y u + \partial_x v = 0$.

(a) For which $\alpha, \beta \in \mathbb{R}$ does the couple $\begin{pmatrix} u \\ v \end{pmatrix} : (x,y) \mapsto \begin{pmatrix} x{+}\alpha y - 3x^2y{+}y^3 \\ x{+}y + x^3{+}\beta xy^2 \end{pmatrix}$ satisfy CRe?

(b) Give all solutions $(u,v)^\top \in \mathrm{C}^1(\mathbb{R}^2;\mathbb{R}^2)$ of CRe satisfying $u(x,0) = \mathrm{e}^x$ and $v(x,0) = 2\sin x$. (Hint: You may use Part (c)!)

(c) Assume $u, v \in \mathrm{C}^1(\mathbb{R}^2)$. Show that CRe is equivalent to the complex differentiability of the function $f : \mathbb{C} \to \mathbb{C}$ defined via $f(x{+}\mathrm{i}y) = u(x,y) + \mathrm{i}\,v(x,y)$.

Exercise 2: Integration by parts. Let $\Omega \subset \mathbb{R}^d$ denote a bounded domain with piecewise C^1 smooth boundary $\partial\Omega$ and outward unit normal vector $n : \partial\Omega \to \mathbb{S}^{d-1} \subset \mathbb{R}^d$.

(a) For $\alpha \in \mathrm{C}^1(\overline{\Omega})$ and $v \in \mathrm{C}^1(\overline{\Omega};\mathbb{R}^d)$ prove the integration-by-part formula

$$\int_\Omega \alpha \,\mathrm{div}\, v \,\mathrm{d}x = \int_{\partial\Omega} \alpha v \cdot n \,\mathrm{d}a - \int_\Omega \nabla\alpha \cdot v \,\mathrm{d}x.$$

(Hint: Use Gauß' divergence theorem.)

(b) Prove that any $u, v \in \mathrm{C}^2(\overline{\Omega})$ satisfy

$$\int_\Omega v\Delta u \,\mathrm{d}x = -\int_\Omega \nabla v \cdot \nabla u \,\mathrm{d}x + \int_{\partial\Omega} v\nabla u \cdot n \,\mathrm{d}a,$$
$$\int_\Omega (v\Delta u - u\Delta v) \,\mathrm{d}x = \int_{\partial\Omega} (v\nabla u - u\nabla v) \cdot n \,\mathrm{d}a.$$

(please turn over)

Exercise 3. Separation of variables: Consider the three PDEs

$$\text{(i) } u_{tt} = u_{xx}, \qquad \text{(ii) } u_{tt} + u_{xx} = 0, \qquad \text{(iii) } u_t = u_{xx}.$$

(a) For all three PDEs construct all possible solutions with *separated variables*, i.e. they have the form $u(t,x) = V(t)W(x)$ for some real-valued, twice differentiable functions V and W. (Hint: After inserting the ansatz into the PDE, separate the variables to the different sides of the equality. Argue then that both sides must be constant.)

(b) For the heat equation (iii) find all solutions of (a) satisfying additionally $W(0) = 0 = W(1)$. From this, construct a solution of (iii) that additionally satisfies the boundary conditions $u(t,0) = u(t,1) = 0$ and the initial condition $u(0,x) = \frac{\pi^2}{17}\sin(\pi x) - 1.2\sin(4\pi x)$.

(c) For the wave equation (i) find all solution of (a) satisfying additionally $W(0) = 0 = W(\ell)$, where $\ell > 0$ is given. Show that all these solutions satisfy $u(t+2n\ell, x) = u(t,x)$ for $n \in \mathbb{Z}$.

Exercise 4. Schrödinger equation: A free quantum-mechanical particle is governed by the complex-valued Schrödinger equation without potential. The wave function $\Psi : \mathbb{R}\times\mathbb{R}^d \to \mathbb{C}$ has to satisfy

$$\mathrm{i}\frac{\partial}{\partial t}\Psi = \Delta\Psi. \tag{SE}$$

(a) We look for solutions Ψ of (SE) in the form

$$\Psi(t,x) = \gamma \mathrm{e}^{-a(t)|x|^2 + b(t)\cdot x + c(t)} \quad \text{for } t \in \mathbb{R} \text{ and } x \in \mathbb{R}^d.$$

Derive ordinary differential equations $\dot a = f(a,b,c)$, $\dot b = g(a,b,c)$, $\dot c = h(a,b,c)$, for $a : \mathbb{R} \to \mathbb{C}$, $b : \mathbb{R} \to \mathbb{C}^d$, and $c : \mathbb{R} \to \mathbb{C}$ such that this Ψ solves (SE). (Hint: For a one obtains a complex Bernoulli equation.)

(b) Provide the general solution for the ODE system in (a).

(c) Show that $\mathrm{Re}(a(0)) > 0$ implies $\mathrm{Re}(a(t)) > 0$ for all $t \in \mathbb{R}$. Then, the function $\rho(t,\cdot) = |\Psi(t,\cdot)|^2 \geq 0$ is a Gaußian distribution which has the physical interpretation of the density distribution of the quantum particle's position in $\mathbb{R}^d$ at time $t \in \mathbb{R}$. The variance $\sigma(t)$ and the mean $Y(t)$ of the Gaußian are given by

$$\sigma(t) = \frac{1}{2\sqrt{\mathrm{Re}(a(t))}} > 0 \quad \text{and} \quad Y(t) = \frac{1}{2\,\mathrm{Re}(a(t))}\mathrm{Re}(b(t)) \in \mathbb{R}^d.$$

Discuss the behavior of these functions with respect to $t \in \mathbb{R}$.

Criteria for the "exercise credit points" (Übungsscheinkriterien):

- participation at the exercises,
- 35 % solved exercises (as mark on the voting sheet)
- two presentations (first and second half of term) of solutions on the blackboard.