Problem sheet 4

Problem 4.1. Show that the function $h\colon \mathscr{G} \to [0,1]$ defined as $h(\mathcal{G}, o) = |\mathcal{G}|^{-1}\mathbb{1}_{\{|\mathcal{G}|<\infty\}}$ is continuous in the weak local topology.

Problem 4.2. Consider the Stochastik block model on the vertices $\mathcal{V}_n = \mathcal{V}_n^{(1)} \cup \mathcal{V}_n^{(2)}$, where $\mathcal{V}_n = [n]$ and $\mathcal{V}_n^{(1)}$ contains of all even labels and $\mathcal{V}_n^{(2)}$ of all uneven labels. We connect each vertex of the same type independently with probability a/n and each pair of different types with probability b/n. Identify the Poisson BGW tree $\mathcal{T}$ that describes the weak local limit and identify, for which a, b the tree has positive probability of survival.

Problem 4.3 (Theorem 4.6)**.** Prove Theorem 4.6 of the lecture notes: Let $(\mathcal{G}_n)$ be the inhomogeneous random graph associated with the kernels (κ_n) that converge graphically to κ. Then the empirical degree-degree distribution converges to the probability measure ν_2 given by

$$\nu_2(k,l) = \frac{1}{c}\int \frac{\lambda(x)^{k-1}}{(k-1)!}e^{-\lambda(x)}\Big(\int \frac{\lambda(y)^{l-1}}{(l-1)!}e^{-\lambda(y)}\kappa(x,y)\mu(\mathrm{d}y)\Big)\mu(\mathrm{d}x),$$

where $c = \int\int \kappa(x,y)\mu(\mathrm{d}y)\mu(\mathrm{d}x)$.

Problem 4.4 (Global clustering)**.** In the lecture, we studied the *local clustering coefficient*, describing the probability that two neighbours of a uniformly chosen vertex are neighbours themselves and thus form a triangle. In contrast the *global clustering coefficient* is defined as

$$\boldsymbol{c}_n^{\mathrm{glob}} = \frac{\sum_{v\in\mathcal{G}_n}\Delta(\mathcal{G}_n, v)}{\frac{1}{2}\sum_{v\in\mathcal{G}_n}\mathcal{D}(v)(\mathcal{D}(v)-1)},$$

i.e. the overall proportion of wedges that complete a triangle. Assume $(\mathcal{G}_n)_n$ converges weakly locally to $(\mathcal{G}, o) \in \mathscr{G}$ and that the sequence $(\mathcal{D}(O_n)^2)_n$ is uniformly integrable. Show that

$$\boldsymbol{c}_n^{\mathrm{glob}} \longrightarrow \frac{\mathbb{E}[\Delta(\mathcal{G}, o)]}{\frac{1}{2}\mathbb{E}[\mathcal{D}(o)(\mathcal{D}(o)-1)]},$$

in probability, as $n \to \infty$.

Problem 4.5 (Proposition 4.11)**.** Finish the proof of Proposition 4.11: Let $(\mathcal{G}_n)$ be a sequence of random graphs converging weakly locally to a random rooted graph $(\mathcal{G}, o)$. Then, in probability,

$$\frac{\mathcal{K}_n}{n} \longrightarrow \mathbb{E}\Big[\tfrac{1}{|\mathcal{G}|}\Big] \geq 0.$$

Problem 4.6. Let $(\mathcal{G}_n)_n$ be a sequence of random graphs converging weakly locally to a rooted graph $(\mathcal{G}, o)$, and $\mathcal{V}_n^{(2)}$ be the set of all vertices of degree at least two. Show that

$$\frac{n}{|\mathcal{V}_n^{(2)}|} \longrightarrow \frac{1}{\mathbb{P}(\mathcal{D}(o)\geq 2)} \tag{1}$$

in probability. Using (1), show that $\frac{1}{n}\sum_{v\in\mathcal{V}_n^{(2)}}\boldsymbol{c}_n^{\mathrm{loc}}(v)\longrightarrow\mathbb{E}\Big[\frac{\Delta(\mathcal{G},o))}{\frac{1}{2}\mathcal{D}(o)(\mathcal{D}(o)-1)}\mathbb{1}_{\{\mathcal{D}(o)\geq 2\}}\Big]$ in probability implies

$$\boldsymbol{c}_n^{\mathrm{av}}\longrightarrow\mathbb{E}\Big[\frac{\Delta(\mathcal{G},o)}{\frac{1}{2}\mathcal{D}(o)(\mathcal{D}(o)-1)}\,\Big|\,\mathcal{D}(o)\geq 2\Big],$$

in probability.