Problem sheet 3

Problem 3.1. Let $(\mathcal{G}, o) \in \mathscr{G}$ be a rooted graph and $r > 0$. Show that the ball $\mathcal{B} = \{\mathcal{H} \in \mathcal{G} \colon d(\mathcal{G}, \mathcal{H}) < r\}$ is both, open and closed.

Proof. We start with openness. We show that for every $(\mathcal{H}, o') \in \mathcal{B}$, there exists $\varepsilon > 0$ such that the ball

$$\mathcal{B}_\varepsilon := \{(\mathcal{H}', o'') \in \mathcal{G} : d((\mathcal{H}, o'), (\mathcal{H}', o'')) \leq \varepsilon\}$$

is contained in $\mathcal{B}$. Fix $(\mathcal{H}, o') \in \mathcal{B}$. Then, by the definition, we have $d\left((\mathcal{G}, o), (\mathcal{H}, o')\right) < r$. Set now $R_0 := \min\left\{R \in \mathbb{N} : \frac{1}{1+R} < r\right\}$; then $(\mathcal{G}, o) \wedge R_0 = (\mathcal{H}, o') \wedge R_0$. Define $\varepsilon := \frac{1}{1+R_0}$. Then for any $(\mathcal{H}', o'') \in \mathcal{B}_\varepsilon$ we have $(\mathcal{H}, o') \wedge R_0 = (\mathcal{H}', o'') \wedge R_0$, and hence, by transitivity, $(\mathcal{G}, o) \wedge R_0 = (\mathcal{H}', o'') \wedge R_0$, which implies $(\mathcal{H}', o'') \in \mathcal{B}$, being $\varepsilon < r$.
We now show closeness. To do so, we show that the complement set of $\mathcal{B}$ is open. Consider now $(\mathcal{H}, o') \notin \mathcal{B}$, i.e.,

$$d\left((\mathcal{G}, o), (\mathcal{H}, o')\right) \geq r.$$

We show that there exists $\varepsilon > 0$ such that

$$\mathcal{B}_\varepsilon \subset \mathcal{G} \setminus \mathcal{B} = \{\mathcal{H} \in \mathcal{G} \colon d(\mathcal{G}, \mathcal{H}) \geq r\}.$$

Let $R_0 := \max\left\{R \in \mathbb{N} : \frac{1}{1+R} \geq r\right\}$. Then, by construction, $(\mathcal{G}, o) \wedge R_0 \neq (\mathcal{H}, o') \wedge R_0$. Now, set $\varepsilon := \frac{1}{1+R_0}$. By definition, any $(\mathcal{H}', o'') \in \mathcal{B}_\varepsilon$ is such that $(\mathcal{H}', o'') \wedge R_0 = (\mathcal{H}, o') \wedge R_0 \neq (\mathcal{G}, o) \wedge R_0 \Rightarrow d\left((\mathcal{G}, o), (\mathcal{H}', o'')\right) \geq \varepsilon \geq r \Rightarrow (\mathcal{H}', o'') \in \mathcal{G} \setminus \mathcal{B}$. □

Problem 3.2. Consider the map $\varphi \colon \mathscr{G} \to \mathscr{G} \wedge k$, $(\mathcal{G}, o) \mapsto (\mathcal{G}, o) \wedge k$ for a fixed $k \in \mathbb{N}$. Show that φ is continuous.

Proof. Let $\varepsilon > 0$ and choose $\delta < 1/(k+1)$. Let $(\mathcal{G}, o), (\mathcal{H}, r)$ be two rooted graphs such that $d((\mathcal{G}, o), (\mathcal{H}, r)) = 1/(N+1) < \delta$. Hence, $N > (1/\delta) - 1$, where N denotes, as usual, the maximal depth, for which $\mathcal{G}$ and $\mathcal{H}$ agree. By choice of δ, in particular, $N > k$ and therefore $(\mathcal{G}, o) \wedge k = (\mathcal{H}, r) \wedge k$. Thus,

$$|\varphi(\mathcal{G}, o) - \varphi(\mathcal{H}, r)| = 0 < \varepsilon,$$

implying continuity. □

Problem 3.3. Fix some $(\mathcal{H}, o_{\mathcal{H}}) \in \mathscr{G}$ and consider the map $h \colon \mathscr{G} \mapsto \{0, 1\}$, $(\mathcal{G}, o) \mapsto \mathbb{1}_{\{(\mathcal{G}, o) \wedge k = (\mathcal{H}, o_{\mathcal{H}})\}}$ for some $k \in \mathbb{N}$. Show that h is a continuous bounded map.

Proof. For every $\varepsilon > 0$, we show that there exists $\delta > 0$ such that for all $(\mathcal{G}', o') \in \mathscr{G}$,

$$d\left((\mathcal{G}, o), (\mathcal{G}', o')\right) < \delta \quad \Rightarrow \quad |h(\mathcal{G}, o) - h(\mathcal{G}', o')| < \varepsilon.$$

1. Case $h(\mathcal{G}, o) = 1$, hence $(\mathcal{G}, o) \wedge k = (\mathcal{H}, o_{\mathcal{H}})$. Choose $\delta < \frac{1}{k+1}$. Then, by definition of the distance $d\left((\mathcal{G}, o), (\mathcal{G}', o')\right) < \frac{1}{1+k} \Rightarrow (\mathcal{G}, o) \wedge k = (\mathcal{G}', o') \wedge k$. Therefore, for all $(\mathcal{G}', o')$ with $d\left((\mathcal{G}, o), (\mathcal{G}', o')\right) < \delta$, we have

$$(\mathcal{G}', o') \wedge k = (\mathcal{G}, o) \wedge k = (\mathcal{H}, o_{\mathcal{H}}) \Rightarrow h(\mathcal{G}', o') = 1.$$

2. Case $h(\mathcal{G}, o) = 0$, hence $(\mathcal{G}, o) \wedge k \neq (\mathcal{H}, o_{\mathcal{H}})$. Using the same $\delta < \frac{1}{k+1}$ as before, for all $(\mathcal{G}', o')$ with $d\left((\mathcal{G}, o), (\mathcal{G}', o')\right) < \delta$, we have

$$(\mathcal{G}', o') \wedge k = (\mathcal{G}, o) \wedge k \neq (\mathcal{H}, o_{\mathcal{H}}) \quad \Rightarrow \quad h(\mathcal{G}', o') = 0.$$

   So again,

$$|h(\mathcal{G}', o') - h(\mathcal{G}, o)| = 0 < \varepsilon.$$

This shows coninuity. □

Problem 3.4. We identify with $(\mathbb{Z}, 0)$ the rooted graph, which has root 0 and all nearest neighbour edges, i.e. $\mathcal{E}(\mathbb{Z}) = \{\{i, i+1\} : i \in \mathbb{Z}\}$. Let $\mathcal{G}_n^{(1)}$ be the line of length n, i.e. $\mathcal{V}(\mathcal{G}_n^{(1)}) = [n]$ and $\mathcal{E}(\mathcal{G}_n^{(1)}) = \{\{1,2\}, \{2,3\}, \dots, \{n-1, n\}\}$. Let further $\mathcal{G}_n^{(2)}$ be the cycle of length n, i.e. $\mathcal{V}(\mathcal{G}_n^{(2)}) = [n]$ and $\mathcal{E}(\mathcal{G}_n^{(2)}) = \{\{1,2\}, \{2,3\}, \dots, \{n-1, n\}, \{n, 1\}\}$. Show that both, $\mathcal{G}_n^{(1)}$ and $\mathcal{G}_n^{(2)}$ converge weakly locally to $(\mathbb{Z}, 0)$, as $n \to \infty$.

Proof. Fix $k \in \mathbb{N}$. Choose a vertex uniformly at random from $[n]$, call it U and consider

$$\mathbb{P}(U \in \{k+1, \dots, n-k-1\}) = \frac{n-2k-2}{n} \longrightarrow 1,$$

assuming $n > 2k+2$. On this event, the graph $(\mathcal{G}_n^{(1)}, U) \cap k$ consists of the vertices $\{U-k, \dots, U, \dots, U+k\}$ and the edges

$$\{U-k, U-k+1\}, \{U-k+1, U-k+2\}, \dots, \{U-1, U\}, \{U, U+1\}, \dots, \{U+k-1, U+k\},$$

as the boundary points are not seen. After shifting each label by U, thus centring in 0, this coincides with the graph $(\mathbb{Z}, 0) \wedge k$. Hence,

$$\begin{aligned}\mathbb{P}((\mathcal{G}^{(1)}, U) \wedge k = (\mathbb{Z}, 0)) &\geq \mathbb{P}\Big((\mathcal{G}^{(1)}, U) \wedge k = (\mathbb{Z}, 0), U \in \{k+1, \dots, n-k-1\}\Big) \\ &= 1\mathbb{P}(U \in \{k+1, \dots, n-k-1\}) \longrightarrow 1.\end{aligned}$$

The same applies verbatim for $\mathcal{G}^{(2)}$. □

Problem 3.5. Let η be a Poisson process with intensity ν on $\mathscr{S}$. Show that, for every measurable function $f : \mathscr{S} \to [0, \infty)$, we have

$$\mathbb{E}\Big[\sum_{x \in \eta} f(x)\Big] = \int_{\mathscr{S}} f(x) \nu(\mathrm{d}x).$$

Proof. We first consider the case $\nu(\mathscr{S}) < \infty$. In that case we have

$$\mathbb{E}\sum_{x\in\eta} f(x) = \mathbb{E}\sum_{i=1}^{N} f(X_i) = \sum_{k=0}^{\infty}\mathbb{P}(N=k)\sum_{i=1}^{k}\mathbb{E}[f(X_i)\mid N=k].$$

Now recall, that N is Poisson distributed with parameter $\nu(\mathscr{S})$ and $X_1,\dots,X_k$ are, given $N=k$, i.i.d. with respect to the normalised intensity measure $\nu(\mathrm{d}x)/\nu(\mathscr{S})$. Therefore, we have

$$\begin{aligned}\sum_{k=0}^{\infty}\mathbb{P}(N=k)\sum_{i=1}^{k}\mathbb{E}[f(X_i)\mid N=k] &= \sum_{k=0}^{\infty}k\mathbb{P}(N=k)\int_{\mathscr{S}} f(x)\nu(\mathrm{d}x)/\nu(\mathscr{S})\\ &= \underbrace{\mathbb{E}[N]}_{=\nu(\mathscr{S})}\int_{\mathscr{S}} f(x)\nu(\mathrm{d}x)/\nu(\mathscr{S})\\ &= \int_{\mathscr{S}} f(x)\nu(\mathrm{d}x)/\nu(\mathscr{S}).\end{aligned}$$

Next, we turn to the general case where $\nu(\mathscr{S})=\infty$ is possible. We first show the equality for $f(x)=\mathbb{1}_{x\in\mathcal{A}}$ for a Borel set $\mathcal{A}$. Then,

$$\mathbb{E}\sum_{x\in\eta}\mathbb{1}_{\{x\in\mathcal{A}\}} = \mathbb{E}\Big[\sharp\{\mathbf{x}\in\eta\colon x\in\mathcal{A}\}\Big] = \nu(\mathcal{A}) = \int \mathbb{1}_{x\in\mathcal{A}}\nu(\mathrm{d}x),$$

by the properties of the Poisson point process. By the monotone class theorem, the equality is valid for all bounded measurable functions. For any measurable function f, we approximate f with the sequence of bounded functions $f_n = f\wedge k$, for which the equation applies. The proof then finishes with monotone convergence. □