Problem sheet 3

Problem 3.1. Let $(\mathcal{G}, o) \in \mathscr{G}$ be a rooted graph and $r > 0$. Show that the ball $\mathcal{B} = \{\mathcal{H} \in \mathcal{G} \colon d(\mathcal{G}, \mathcal{H}) < r\}$ is both, open and closed.

Problem 3.2. Consider the map $\varphi \colon \mathscr{G} \to \mathscr{G} \wedge \mathscr{G} \wedge k$, $(\mathcal{G}, o) \mapsto (\mathcal{G}, o) \wedge k$ for a fixed $k \in \mathbb{N}$. Show that φ is continuous.

Problem 3.3. Fix some $(\mathcal{H}, o_{\mathcal{H}}) \in \mathscr{G}$ and consider the map $h \colon \mathscr{G} \mapsto \{0, 1\}$, $(\mathcal{G}, o) \mapsto \mathbb{1}_{\{(\mathcal{G},o)\wedge k=(\mathcal{H},\wr_{\mathcal{H}})\}}$ for some $k \in \mathbb{N}$. Show that h is a continuous bounded map.

Problem 3.4. We identify with $(\mathbb{Z}, 0)$ the rooted graph, which has root 0 and all nearest neighbour edges, i.e. $\mathcal{E}(\mathbb{Z}) = \{\{i, i+1\} \colon i \in \mathbb{Z}\}$. Let $\mathcal{G}_n^{(1)}$ be the line of length n, i.e. $\mathcal{V}(\mathcal{G}_n^{(1)}) = [n]$ and $\mathcal{E}(\mathcal{G}_n^{(1)}) = \{\{1, 2\}, \{2, 3\}, \dots, \{n-1, n\}\}$. Let further $\mathcal{G}_n^{(2)}$ be the cycle of length n, i.e. $\mathcal{V}(\mathcal{G}_n^{(2)}) = [n]$ and $\mathcal{E}(\mathcal{G}_n^{(2)}) = \{\{1, 2\}, \{2, 3\}, \dots, \{n-1, n\}, \{n, 1\}\}$. Show that both, $\mathcal{G}_n^{(1)}$ and $\mathcal{G}_n^{(2)}$ converge weakly locally to $(\mathbb{Z}, 0)$, as $n \to \infty$.

Problem 3.5. Let η be a Poisson process with intensity ν on $\mathscr{S}$. Show that, for every measurable function $f \colon \mathscr{S} \to [0, \infty)$, we have

$$\mathbb{E}\Big[\sum_{x \in \eta} f(x)\Big] = \int_{\mathscr{S}} f(x) \nu(\mathrm{d}x).$$