Problem sheet 2

Problem 2.1. Let $\mathcal{G}_n$ be an Erdős-Rényi graph on n vertices, where each edge is present independently with probability c/n, $c > 0$. Let $\mathcal{X}_n$ be the number of triangles in $\mathcal{G}_n$, where we call $\{u, v, w\} \subset \mathcal{V}_n$ a triangle if $u \sim v, v \sim w$ and $w \sim u$. Find the limiting distribution of $\mathcal{X}_n$, as $n \to \infty$.

Problem 2.2. Show that the space $\mathscr{G}$ of rooted graphs is *not* compact.

Problem 2.3 (Poisson Thinning)**.** Let N be a Poisson distribution with parameter $\lambda > 0$. Let X be a multinomial distribution with N trials and probabilities $p_1, \dots, p_m$, i.e., we perform N independent experiments with m potential outcomes and outcome j has probability p_j. Let N_j denote the number of outcomes j. Show that, $N_1, \dots, N_m$ are independently Poisson distributed with respective parameters $\lambda p_1, \dots, \lambda p_m$.

Problem 2.4 (Uniform model and Erdős-Rényi)**.** Fix $m \in \mathbb{N}$ and consider the space $\mathscr{G}_{n,m} := \{\mathcal{G} \colon |\mathcal{V}(G)| = n, |\mathcal{E}(\mathcal{G})| = m\}$, the space of all graphs on n vertices and m edges. Consider the uniform measure on $\mathscr{G}_{n,m}$, i.e.,

$$\mathbb{P}_{n,m}(\mathcal{G}) := \frac{1}{\binom{\binom{n}{2}}{m}}, \qquad \text{for any } \mathcal{G} \in \mathscr{G}_{n,m}.$$

The corresponding random graph drawn with this distribution is denoted by $\mathcal{G}(n, m)$. Let $\mathcal{G}(n, p)$ the Erdős-Rényi random graph on n vertices, where each edge is present independently with probability $p \in (0, 1)$. Show that $\mathcal{G}(n, p)$, conditioned on having exactly m edges, is distributed as $\mathcal{G}(n, m)$.