Problem sheet 1

Problem 1.1 (WLLN for Bernoulli)**.** Let $X_{i,j}^{(n)}$ be independent Bernoulli random variables with expectation $p_{i,j}^{(n)}$, such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i,j=1}^{n} p_{i,j}^{(n)} = c.$$

Show that, in probability,

$$\frac{1}{n} \sum_{i,j=1}^{n} X_{i,j}^{(n)} \longrightarrow c.$$

Proof. We first calculate for the variance

$$\mathrm{Var}(\sum_{i,j} X_{i,j}^{(n)}) = \sum_{i,j} p_{i,j}^{(n)}(1 - p_{i,j}^{(n)}) \leq \sum_{i,j} p_{i,j}^{(n)} = O(n).$$

By assumption, for each $\varepsilon > 0$, there exists N such that

$$|\tfrac{1}{n} \sum_{i,j} p_{i,j}^{(n)} - c| \leq \varepsilon/2$$

for all $n \geq N$. Thus,

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{P}(|\sum_{i,j} X_{i,j}^{(n)} - c| \geq n\varepsilon) &\leq \lim_{n\to\infty} \mathbb{P}(|\sum_{i,j} X_{i,j}^{(n)} - \sum_{i,j} p_{i,j}^{(n)}| + |\sum_{i,j} p^{(n)} - c| \geq n\varepsilon) \\
&\leq \lim_{n\to\infty} \mathbb{P}(|\sum_{i,j} X_{i,j}^{(n)} - \sum_{i,j} p_{i,j}^{(n)}| \geq n\varepsilon/2) \\
&\leq \lim_{n\to\infty} \frac{4\,\mathrm{Var}(\sum_{i,j} X_{i,j}^{(n)})}{n^2\varepsilon^2} \\
&= 0,
\end{aligned}$$

using Chebyshev's inequality. □

Problem 1.2 (Coupling)**.**

(a) Let X and Y be two real random variables with distribution function F_X and F_Y, such that $F_X(x) \leq F_Y(x)$ for all $x \in \mathbb{R}$. Show that there exists a probability space, on which X and Y can jointly be defined such that $X \geq Y$ almost surely. We call this *coupling*.

(b) Let $(\mathcal{G}_n)$ and $(\overline{\mathcal{G}}_n)$ be two sequences of inhomogeneous random graphs with kernels κ_n and $\overline{\kappa}_n$, respectively. Assume that $\kappa_n(x_i, x_j) \leq \overline{\kappa}_n(x_i, x_j)$ for all $n \in \mathbb{N}$ and $x_i, x_j \in \mathscr{S}$. Show that both graph sequences can be coupled in a way that $\mathcal{E}_n \subset \overline{\mathcal{E}}_n$, i.e., each edge appearing in $\mathcal{G}_n$ also appears in $\overline{\mathcal{G}}_n$.

Proof. We start with (a). Let U be a random variable uniformly distributed on $(0,1)$. Let us define the generalised inverse of the distribution functions as F_X^{-1} and F_Y^{-1}, respectively. We set $X = F_X^{-1}(U)$ and $Y = F_Y^{-1}(U)$. Clearly, $X \sim F_X$ and $Y \sim F_Y$. Further, for each x, we have $X \le x \Leftrightarrow U \le F_X(x)$. As $F_X(x) \le F_Y(x)$, this implies $U \le F_Y(x) \Leftrightarrow Y \le x$. Summarising, $X \le x \Rightarrow Y \le x$ for all x and thus $Y \le X$.
To show (b), we again use uniform random variables. More precisely, let $(U_{\{i,j\}} \colon i,j \in \mathbb{N})$ be a family of i.i.d. uniforms. We then define graph $\mathcal{G}_n$ as a functional of the random variables $(U_{\{i,j\}} \colon i,j \in [n])$, that maps these to the graph with vertex set $[n]$ and edge set

$$\mathcal{E}_n := \{\{i,j\} \colon U_{\{i,j\}} \le \kappa_n(x_i,x_j)/n\}.$$

Clearly, $\mathcal{G}_n$ has the law of the inhomogeneous random graph associated with kernel κ. We define the graph $\overline{\kappa}$ in the same way. Then, $\{i,j\} \in \mathcal{E}_n$ implies $U_{i,j} \le \kappa_n(x_i,x_j)/n \le \overline{\kappa}_n(x_i,x_j)/n$, hence $\{i,j\} \in \overline{\mathcal{E}}_n$. $\square$

Problem 1.3. Let Λ be a positive random variable on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Let X be a random variable that is, conditioned on Λ, Poisson distributed with mean Λ. Put differently, X is mixed-Poisson distributed with mixing distribution $\mathbb{P} \circ \Lambda^{-1}$

(a) Assume that Λ has a density f satisfying

$$cx^{-\tau} \le f(x) \le Cx^{-\tau}, \qquad \text{for all } x \ge A,$$

where $\tau > 2$ and $0 < c < C < \infty$ and $A > 0$ is some bound. Show that there exist constants $0 < c' < C' < \infty$ such that

$$c'x^{-\tau} \le \mathbb{P}(X = x) \le C'x^{-\tau}.$$

<u>Hint:</u> The Γ-function is defined as $\Gamma(k) = \int_0^\infty x^{k-1}e^{-x}\mathrm{d}x$ and it has the following useful property $\frac{\Gamma(k-\tau)}{\Gamma(k)} \sim k^{-\tau}$, as $k \to \infty$.

(b) Assume that there are $\lambda_1, \dots, \lambda_m > 0$ such that $\mathbb{P}(\Lambda \in \{\lambda_1, \dots, \lambda_m\}) = 1$. Show that X is light-tailed.

Proof. We first show (a). Let us write $\mathcal{D} = \{y \colon f(y) > 0\} \subset (0,\infty)$. Let us for simplicity assume $\mathcal{D} = (A,\infty)$ for some $A \ge 0$. By definition, we have

$$\begin{aligned}
\mathbb{P}(X = k) &= \int_{\mathcal{D}} \frac{\lambda^k}{k!} e^{-\lambda} f(\lambda)\,\mathrm{d}\lambda = \frac{1}{\Gamma(k+1)} \int_A^\infty \lambda^k e^{-\lambda} f(\lambda)\,\mathrm{d}\lambda \\
&\le \frac{C}{\Gamma(k+1)} \int_0^\infty \lambda^{k+1-\tau-1} e^{-\lambda}\,\mathrm{d}\lambda = C\frac{\Gamma(k+1-\tau)}{\Gamma(k+1)} \\
&\le C'k^{-\tau},
\end{aligned}$$

where the constant C' exists as $\frac{\Gamma(k+1-\tau)}{\Gamma(k+1)} \sim k^{-\tau}$ and additionally $\mathbb{P}(X = k) \le 1$. For the opposite direction, we have by similar argumentation

$$\mathbb{P}(X = k) \ge c\frac{\Gamma(k+1-\tau)}{\Gamma(k+1)} - \frac{c}{k!} \int_0^A \lambda^{k-\tau} e^{-\lambda}\,\mathrm{d}\lambda,$$

Since A is finite and fixed, the integral on the right-hand side can be at most of the same order than $\Gamma(k+1-\tau)$. If it is of strictly smaller order, we have

$$\frac{1}{k!}\int_0^A \lambda^{k-\tau}e^{-\lambda} = o(k^{-\tau}).$$

If it is of the same order, we have

$$\frac{\Gamma(k+1-\tau)}{\Gamma(k+1)} = O\Big(\frac{1}{k!}\int_0^A \lambda^{k-\tau}e^{-\lambda}\Big),$$

where the limiting constant must be smaller than 1. In both scenarios, we find a very small constant $c'' < c$ such that

$$c\frac{1}{k!}\int_0^A \lambda^{k-\tau}e^{-\lambda}\mathrm{d}\lambda \geq c''\frac{\Gamma(k+1-\tau)}{\Gamma(k+1)},$$

from which we infer the lower bound

$$\mathbb{P}(X=k) \geq c'k^{-\tau},$$

for a significantly small constant c'.
In order to show (b), we show that the moment generation function of X exists on the positive half line $(0,\infty)$. To this end, fix $t>0$. Recall, that a Poisson(λ) random variable Y has the moment generating function

$$\mathbb{E}e^{tY} = e^{\lambda(e^t-1)}$$

Then, by definition,

$$\begin{aligned}\mathbb{E}[e^{tX}] &= \sum_{j=1}^m \mathbb{P}(\Lambda=\lambda_j)\mathbb{E}[e^{-tX} \mid \Lambda=\lambda_j] = \sum_{j=1}^m \mathbb{P}(\Lambda=\lambda_j)e^{\lambda_j(e^t-1)}\\ &\leq e^{\lambda_{\max}(e^t-1)}\sum_{j=1}^m \mathbb{P}(\Lambda=\lambda_j) = e^{\lambda_{\max}(e^t-1)} < \infty,\end{aligned}$$

where $\lambda_{\max} := \max\{\lambda_1,\dots,\lambda_m\} < \infty$. $\square$

Problem 1.4. Prove or falsify the following statements:

(a) $\log^p n = o(n^{1/p})$, as $n\to\infty$, for any $p>1$.

(b) $3^n = O(2^n)$, as $n\to\infty$.

(c) $\sum_{i=0}^n i! \asymp n!$, as $n\to\infty$.

(d) Let $X^{(n)}$ be a binomial random variable with n trials and success probability $p^{(n)} = 9/n^2$, then $\mathbb{E}X^{(n)} = O(1/n)$.

(e) $\sin(1/x) \sim 1/x$, as $x\to\infty$.

Proof. (a) For any $p > 1$, we have with l'Hospital

$$\lim_{x\to\infty} \frac{(\log x)^p}{x^{1/p}} = \lim_{x\to\infty} \frac{p(\log x)^{p-1})/x}{-(1-1/p)x^{1/p-1}} = c \lim_{x\to\infty} \frac{(\log x)^{p-1}}{x^{1/p}}.$$

Repeating this p times then yields $\log^p(x)/x^{1/p} \to 0$, thus proving Statement (a).
(b) As $(3/2)^n \to \infty$, Statement (b) is wrong.
(c) Clearly, we have $\sum_{i=0}^n i! \geq n!$. Moreover,

$$\sum_{i=0}^{n} n! = n! + \sum_{i=0}^{n-1} i! \leq n! + (n-1)! \sum_{i=0}^{n-1} 1 = 2n!.$$

Therefore, $n! \leq \sum_{i=0}^n i! \leq 2n!$, and thus $n! \asymp \sum^n i!$.
(d) We have $\mathbb{E}X^{(n)} = np^{(n)} = 9/n = O(1/n)$.
(e) We write $y = 1/x$ and consider $y \to 0$. Writing $\sin(y)$ in its series representation, we have

$$\frac{\sin(y)}{y} = \frac{1}{y}\sum_{n=0}^{\infty}(-1)^n \frac{y^{2n+1}}{(2n+1)!} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} \underbrace{y^{2n}}_{\to 0 \text{ as } y \to \mathbf{0}} \to 1,$$

this proves Statement (e). $\square$