PARTICLE SYSTEMS WITH CONTACT INTERACTION

Lecture notes

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Contents

CHAPTER 1

INTRODUCTION AND MOTIVATION

Broadly speaking, *interacting particle systems* (IPS) are continuous-time Markov processes defined on the vertices of a (countably infinite) graph. Each vertex is in one of two states, 0 or 1, and the update rates depend on the states of nearby vertices. IPS are often used as toy models for complex stochastic phenomena such as the spread of information or infection, opinion formation, and magnetisation in ferromagnets. Their appeal lies in the fact that they are easy to define and belong to the simplest classes of models that nevertheless exhibit interesting behaviour, such as phase transitions from microscopic local interactions to macroscopic collective effects. At the same time, they are often surprisingly difficult to analyse rigorously, and their study has led to important developments in modern probability theory. Moreover, although IPS may at first seem overly simplistic, they often capture phenomena of genuine interest remarkably well and therefore permit meaningful qualitative mathematical analysis.

In general, an IPS on a graph $G = (V, E)$ is a process $(\xi_t)_{t\geq 0}$ with state space $\mathcal{S} = \{0,1\}^V$, where $\xi_t(x) = 1$ indicates that site x is occupied at time t. Depending on the model, particles may jump, branch, or die according to the configuration in a neighbourhood of x, as determined by the edge set E. In this course, we focus on IPS with *contact interaction*. One of the most prominent example is the *contact process*, introduced by Harris [Har74], which provides a simple model for the spread of an infection.

Roughly speaking, the dynamics are as follows. Each vertex of G is either healthy (state 0) or infected (state 1). Each infected vertex transmits the infection to each of its neighbours at rate $\lambda > 0$, and recovers at rate 1. More informally, each infected vertex carries an independent Poisson clock of rate λ along each incident edge; when such a clock rings, the infection is transmitted to the vertex at the other end of the edge. In addition, each infected vertex has an independent recovery clock of rate 1; when this clock rings, the vertex immediately becomes healthy again. See Figure 1.1.

The aim of this course is to define and study the model rigorously when the underlying graph is the nearest-neighbour lattice $\mathbb{Z}^d$. We are particularly interested in the existence of

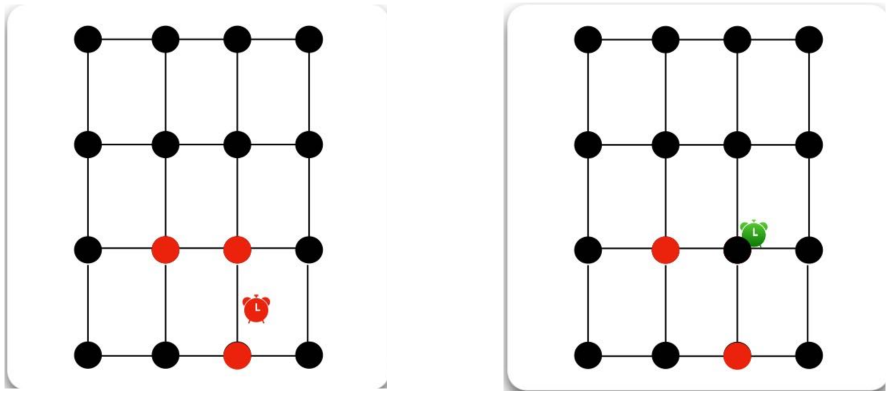

Figure 1.1: The contact process

a phase transition in the infection parameter λ, at which the system changes from almost sure extinction of the infection to the possibility of survival for all time. Moreover, we shall study the amount of time for which the origin is infected in the supercritical regime, which may give rise to a second phase transition.

There are two standard ways to define the contact process: via its Markov generator, or via the so-called *Harris graphical representation*. The first approach is, in some sense, more general and can be used to treat many interacting particle systems in a unified framework [Lig05]. However, it is somewhat less intuitive and relies on more analytic machinery. In this course, we shall therefore follow the second approach, which is more closely tailored to the contact process, more intuitive, and particularly well suited to a detailed study of the model. It formalises the informal description via Poisson clocks above. It also has the advantage of being robust under generalisations to random environments, for instance when the underlying graph itself is random. The only ingredient required for its definition is the Poisson process that we recall in Section 2.2. The contact process is closely related to an oriented form of *percolation* that we study first in Section 2.3. Chapter 3 is then entirely devoted to the contact process, its graphical representation, basic properties, and ultimately its phase transitions.

CHAPTER 2

TOOLS

This chapter is devoted to important tools that we need to define and study the contact process. We start with the *FKG inequality*, a famous correlation inequality. Afterwards, we recall the *Poisson process* that will help us to properly model infection and recovery times. Finally, we introduce *oriented percolation*, a percolation model that can be seen to contain a temporal dimension, thereby yielding a simple way to describe infection paths. This model plays a crucial role later on as it can be coupled with the contact process.

2.1 FKG inequality

The FKG inequality, named after Fortuin, Kasteleyn, and Ginibre [FKG71] is a famous correlation inequality. Loosely speaking, it states that two events that both profit from many occupied/infected vertices are more likely to both occur in the same realisation than in two independent copies. Let V be a countable (vertex) set and $\mathcal{S} = \{0,1\}^V$. We endow $\mathcal{S}$ with the partial ordering, writing $\omega \preceq \omega'$ if $\omega(x) \leq \omega'(x)$ for all $x \in V$. A function $f\colon \mathcal{S} \to \mathbb{R}$ is called *increasing* if $\omega \preceq \omega'$ implies $f(\omega) \leq f(\omega')$. An event $A \subset \mathcal{S}$ is called increasing if the corresponding indicator $\mathbb{1}_A$ is. Let $\mathbf{P}_p$ be the Bernoulli product measure on $\mathcal{S}$ with parameter p. That is, for each $x \in V$, the projection $\pi_x\colon \omega \mapsto \omega(x)$ is independently Bernoulli(p) distributed under $\mathbf{P}_p$.

Lemma 2.1 (FKG inequality)**.** Let $f, g\colon \mathcal{S} \to \mathbb{R}$ be two increasing functions. Then

$$\mathbf{E}_p[fg] \geq \mathbf{E}_p[f]\mathbf{E}_p[g]. \tag{FKG}$$

Put differently, increasing functions are positively correlated.

Remark 2.2.

(i) If $f = \mathbb{1}_A$ and $g = \mathbb{1}_B$, then the FKG inequality states $\mathbf{P}_p(A \cap B) \geq \mathbf{P}_p(A)\mathbf{P}_p(B)$.

(ii) If we call a function f *decreasing* if $-f$ is increasing, then (FKG) also applies to two decreasing functions. Furthermore, if f is increasing and g is decreasing, we have (FKG) with the reverse inequality.

Since $\mathcal{S}$ is only partially ordered, we have to find a way to compare $f(\omega)$ and $f(\omega')$ for $\omega \neq \omega' \in \mathcal{S}$. To this end, define $\omega \vee \omega'$ as the point-wise maximum of ω and ω', i.e. $[\omega \vee \omega'](x) = \max\{\omega(x), \omega'(x)\}$ for all $x \in V$. Similarly, $\omega \wedge \omega'$ denotes the point-wise minimum. Observe that $\omega \wedge \omega' \preceq \omega, \omega' \preceq \omega \vee \omega'$. We need the following lemma.

Lemma 2.3. Assume that $\mathcal{S}$ is finite. Consider the functions $a, b, c, d\colon \mathcal{S} \to [0, \infty)$ and assume that they satisfy

$$a(\omega)b(\omega') \leq c(\omega \wedge \omega')d(\omega \vee \omega'), \qquad \text{for all } \omega, \omega' \in \mathcal{S}.$$

Then,

$$\sum_{\omega,\omega'} a(\omega)b(\omega') \leq \sum_{\omega,\omega'} c(\omega)d(\omega')$$

Proof. Since $\mathcal{S}$ is finite, we may assume $V = \{1, \ldots, n\}$ and write $\mathcal{S} = \{0,1\}^n$. The proof works by induction on $|V| = n$. We start with $|V| = 1$ in which case $\mathcal{S} = \{0,1\}$ and the statement follows directly by checking all the cases. Now assume that the statement holds true for $|V| = n-1$. We decompose $\omega = (x, \varphi) \in \mathcal{S}$ with $x \in \{0,1\}^{n-1}$ and $\varphi \in \{0,1\}$. Define $a_\varphi(x) = a(x, \varphi)$ and b_φ, c_φ, and d_φ in the obvious way. For all $x, y \in \{0,1\}^{n-1}$ and $\varphi, \phi \in \{0,1\}$ fixed, we have by assumption on a, b, c, d,

$$a_\varphi(x)b_\phi(y) \leq c_{\varphi\wedge\phi}(x \wedge y)d_{\varphi\vee\phi}(x \vee y).$$

Summing over φ and ϕ then implies

$$(a_0(x) + a_1(x))(b_0(y) + b_1(y)) \leq (c_0(x \wedge y) + c_1(x \wedge y))(d_0(x \vee y) + d_1(x \vee y)).$$

The induction hypothesis applies to the functions $(a_0 + a_1)$ etc. and we finally infer

$$\begin{aligned}\sum_{\omega,\omega' \in \mathcal{S}} a(\omega)b(\omega') &= \sum_{x,y \in \{0,1\}^{n-1}} (a_0(x) + a_1(x))(b_0(y) + b_1(y)) \\ &\leq \sum_{x,y \in \{0,1\}^{n-1}} (c_0(x) + c_1(x))(d_0(y) + d_1(y)) \\ &= \sum_{\omega,\omega' \in \mathcal{S}} c(\omega)d(\omega').\end{aligned}$$

□

Proof of Lemma 2.1. To keep notation concise, we omit the index p in all calculations. We first assume that V is finite. Without loss of generality, we may further assume that $f = \mathbb{1}_A$ and $g = \mathbb{1}_B$ since the result then follows from the monotone class theorem. Therefore, we need to show that

$$\mathbf{P}(A \cap B) \geq \mathbf{P}(A)\mathbf{P}(B).$$

Note that $\omega \in A$ and $\omega' \in B$ implies $\omega \vee \omega' \in A \cap B$ since A and B are increasing. Moreover,

$$\mathbf{P}(\omega \vee \omega')\mathbf{P}(\omega \wedge \omega') = \mathbf{P}(\omega)\mathbf{P}(\omega'). \tag{2.1}$$

Next define the functions

$$\begin{aligned} a(\omega) &= \mathbb{1}_A(\omega)\mathbf{P}(\omega), & b(\omega) &= \mathbb{1}_B(\omega)\mathbf{P}(\omega), \\ c(\omega) &= \mathbf{P}(\omega), & d(\omega) &= \mathbb{1}_{A\cap B}(\omega)\mathbf{P}(\omega). \end{aligned}$$

Consider

$$a(\omega)b(\omega') = \mathbb{1}_A(\omega)\mathbf{P}(\omega)\mathbb{1}_B(\omega')\mathbf{P}(\omega').$$

Then this is either zero or $\omega \in A$ and $\omega' \in B$, hence $\omega \vee \omega' \in A \cap B$. In that case

$$\begin{aligned} a(\omega)b(\omega') &\le \mathbb{1}_{A\cap B}(\omega \wedge \omega')\mathbf{P}(\omega)\mathbf{P}(\omega') \\ &= \mathbb{1}_{A\cap B}(\omega \wedge \omega')\mathbf{P}(\omega \wedge \omega')\mathbf{P}(\omega \vee \omega') \\ &= c(\omega \wedge \omega')d(\omega \vee \omega'), \end{aligned}$$

using (2.1). In the other case, when the left-hand side is zero, the same inequality trivially applies. Therefore, we infer from Lemma 2.3

$$\sum_{\omega,\omega'} a(\omega)b(\omega') \le \sum_{\omega,\omega'} c(\omega)d(\omega').$$

The proof for finite $\mathcal{S}$ concludes with the two observations

$$\sum_{\omega,\omega'} a(\omega)b(\omega) = \Big(\sum_{\omega} \mathbb{1}_A(\omega)\mathbf{P}(\omega)\Big)\Big(\sum_{\omega'} \mathbb{1}_B(\omega')\mathbf{P}(\omega')\Big) = \mathbf{P}(A)\mathbf{P}(B)$$

and

$$\sum_{\omega,\omega'} c(\omega)d(\omega') = \Big(\sum_{\omega'} \mathbb{1}_{A\cap B}(\omega')\mathbf{P}(\omega')\Big)\Big(\sum_{\omega} \mathbf{P}(\omega)\Big) = \mathbf{P}(A\cap B).$$

The proof for infinite $\mathcal{S}$ follows by approximating f and g by functions of bounded support and monotone convergence. □

Corollary 2.4. Let μ be a probability measure on $\mathcal{S}$ that satisfies

$$\mu(\omega \wedge \omega')\mu(\omega \vee \omega') \ge \mu(\omega)\mu(\omega').$$

Then, μ satisfies (FKG).

Exercise 1. Confirm (2.1). Conclude the proof for infinite state space $\mathcal{S}$.

2.2 The Poisson process

The Poisson process is the central object that we use to model transition and recovery times. More generally, the Poisson process models the occurrence of events after exponential waiting times. Let $(W_i)_{i\in\mathbb{N}}$ be a sequence of i.i.d. exponential random variables with parameter

$\lambda \in (0, \infty)$ and let

$$T_n := \sum_{i=1}^{n} W_i, \quad n \in \mathbb{N}, \qquad \text{and } T_0 = 0.$$

The *Poisson process with rate* λ is defined as

$$\mathcal{N}_t = \sup\{n \in \mathbb{N}_0 \colon T_n \leq t\}, \quad t \geq 0.$$

In words, W_i is the waiting time between the $(i-1)^{\text{th}}$ and i^{th} event, T_n is the occurrence time of the nth event and $\mathcal{N}_t$ is the number of events that have occurred by time t. The times $(T_n)_n$ are also called the *jump times* of $\mathcal{N}$. Equivalently, we can write

$$\mathcal{N}_t = \sum_{n \in \mathbb{N}} \mathbb{1}_{\{T_n \leq t\}}.$$

Clearly, $\mathcal{N}_0 = 0$ and for any $t > 0$ and $k \in \mathbb{N}_0$, we have

$$\{\mathcal{N}_t \geq k\} = \{T_k \leq t\}.$$

Proposition 2.5 (Properties of the Poisson process.)**.** Let $\mathcal{N} = (\mathcal{N}_t)_{t \geq 0}$ be a Poisson process of rate $\lambda \in (0, \infty)$. Then,

(i) $\mathcal{N}_0 = 0$ almost surely,

(ii) $\mathcal{N}$ has independent increments. That is, for all $n \in \mathbb{N}$ and times $0 \leq t_0 < t_1 < \cdots < t_n$, the random variables $\mathcal{N}_{t_0}, \mathcal{N}_{t_1} - \mathcal{N}_{t_0}, \ldots, \mathcal{N}_{t_n} - \mathcal{N}_{t_{n-1}}$ are independent.

(iii) $\mathcal{N}$ has stationary increments. That is, for all $s, t \geq 0$ the random variable $\mathcal{N}_{t+s} - \mathcal{N}_s$ is Poisson distributed with parameter λt.

(iv) $\mathcal{N}$ has càdlàg paths.

We will first prove the following remarkable property of the Poisson process.

Lemma 2.6. Let $\mathcal{N}$ be a Poisson process of intensity λ. Then, for each $s > 0$, the process $\overline{\mathcal{N}}$ defined by

$$\overline{\mathcal{N}}_t = \mathcal{N}_{t+s} - \mathcal{N}_s$$

is again a Poisson process of intensity λ, *independent* of $(\mathcal{N}_u \colon 0 \leq u \leq s)$.

Proof. Recall the memory-less property of the exponential distribution, i.e.

$$\mathbb{P}(W_1 > t + s \mid W_1 > s) = \mathbb{P}(W_1 > t) = e^{-\lambda t} \quad \text{for } s, t \geq 0. \tag{2.2}$$

Now, for fixed s, we have

$$\{\mathcal{N}_s = n\} = \{T_n \leq s < T_{n+1}\} = \{T_n \leq s\} \cup \{W_{n+1} > s - T_n\}.$$

On the event $\mathcal{N}_s = n$, we can write the jump times of $\overline{\mathcal{N}}$ via

$$\overline{T}_1 = W_{n+1} - (s - T_n) = T_{n+1} - s \quad \text{and} \quad \overline{T}_m = T_{n+m} - s, \ m \geq 2.$$

By definition, the waiting times $W_{n+1}, W_{n+2}, \ldots$ are independent of the past jump times $T_1, \ldots, T_n$, therefore using (2.2), we have

$$\begin{aligned}
\mathbb{P}(\overline{T}_1 > t \mid \mathcal{N}_s = n) &= \mathbb{P}(W_{n+1} > t + (s - T_n) \mid W_{n+1} > s - T_n, T_n \leq s) \\
&= \frac{\mathbb{P}(W_{n+1} > t + (s - T_n), T_n \leq s \mid W_{n+1} > s - T_n)}{\mathbb{P}(T_n \leq s)} \\
&= \frac{1}{\mathbb{P}(T_n \leq s)} \int_{[0,s]} \mathbb{P}(W_{n+1} > t + (s - x) \mid W_{n+1} > s - x) \mathbb{P}_{T_n}(\mathrm{d}x) \\
&= e^{-\lambda t}.
\end{aligned}$$

Hence, on $\{\mathcal{N}_s = n\}$, the first waiting time $\overline{W}_1 = \overline{T}_1$ is exponentially distributed. Moreover, $\overline{W}_2 = \overline{T}_2 - \overline{T}_1 = W_{n+2}$ follows the same exponential distribution and is independent of $\overline{W}_1$. More precisely, the waiting times $(\overline{W}_m)_{m \in \mathbb{N}}$ form an i.i.d. sequence of exponential random variables. Since this does not depend on s or n, this concludes the proof. □

Proof of Proposition 2.5. Property (i) and (iv) are immediate from the definition. Property (ii) follows directly from Lemma 2.6. It thus remains to prove stationary Poisson increments. To this end, we note that the n-th jump time T_n is Gamma distributed with parameters n, λ; i.e. T_n has density

$$f_n(x) = \frac{\lambda^n}{(n-1)!} x^{n-1} e^{-\lambda x} \mathbb{1}_{[0,\infty)}(x). \tag{2.3}$$

By Lemma 2.6 it suffices to consider $t > 0$ and $s = 0$. Straightforward calculation and applying integration by parts yields for all $n \in \mathbb{N}_0$,

$$\begin{aligned}
\mathbb{P}(\mathcal{N}_t \geq n+1) = \mathbb{P}(T_{n+1} \leq t) &= \frac{\lambda^{n+1}}{n!} \int_0^t x^n e^{-\lambda x} \mathrm{d}x \\
&= -\frac{(\lambda t)^n}{n!} e^{-\lambda t} + \mathbb{P}(T_n \leq t) = -\frac{(\lambda t)^n}{n!} e^{-\lambda t} + \mathbb{P}(\mathcal{N}_t \geq n).
\end{aligned}$$

Since

$$\mathbb{P}(\mathcal{N}_t = n) = \mathbb{P}(\mathcal{N}_t \geq n) - \mathbb{P}(\mathcal{N}_t \geq n+1) = \frac{(\lambda t)^n}{n!} e^{-\lambda t},$$

the proof concludes. □

Exercise 2. Prove Equations (2.2) and (2.3).

Exercise 3. Let $\mathcal{N} = (\mathcal{N}_t)_{t \geq 0}$ be a stochastic process with the Properties (i) to (iv) of Proposition 2.5. Show that $\mathcal{N}$ is a Poisson process of intensity λ.

Let us next discuss some direct implications of the properties in Proposition 2.5. We start with a strong law of large numbers.

Corollary 2.7. Let $\mathcal{N}$ be a Poisson process of intensity λ. Then

$$\lim_{t \to \infty} \frac{\mathcal{N}_t}{t} = \lambda, \quad \text{almost surely.}$$

Proof. From Proposition 2.5 we infer that, almost surely, $\mathcal{N}_t < \infty$ for all t and $\mathcal{N}_t \to \infty$ for $t \to \infty$. Hence, by the classical law of large numbers for the sum of independent random variables

$$\frac{T_{\mathcal{N}_t}}{\mathcal{N}_t} \longrightarrow \frac{1}{\lambda}, \qquad \text{almost surely.}$$

For any $t > 0$, we have $T_{\mathcal{N}_t} \le t < T_{\mathcal{N}_t+1}$ and therefore

$$\frac{T_{\mathcal{N}_t}}{\mathcal{N}_t} \le \frac{t}{\mathcal{N}_t} < \frac{T_{\mathcal{N}_t+1}}{\mathcal{N}_t+1}\frac{\mathcal{N}_t+1}{\mathcal{N}_t}.$$

The proof concludes by sending $t \to \infty$. □

Next, we show that the Poisson process is a Markov process.

Corollary 2.8. Let $\mathcal{N}$ be a Poisson process of intensity $\lambda > 0$. Then, $\mathcal{N}$ is a time-homogeneous Markov process with transition probabilities given by

$$p_t(n,j) = \begin{cases} e^{-\lambda t}\frac{(\lambda t)^{j-n}}{(j-n)!}, & \text{if } j \ge n, \\ 0, & \text{if } j < n. \end{cases}$$

That is, for all $s, t \ge 0$ and $j \in \mathbb{N}$, we have

$$\mathbb{P}(\mathcal{N}_{t+s} = j \mid \mathcal{N}_u \colon 0 \le u \le s) = p_t(\mathcal{N}_s, j).$$

Exercise 4. Prove Corollary 2.8.

Finally, let us have a look at the remaining waiting time to the next jump in the random interval $[T_{\mathcal{N}_t}, T_{\mathcal{N}_t+1}]$ at time t. Define

$$R_t = T_{\mathcal{N}_t+1} - t$$

the *remaining waiting time* until the next jump; by

$$C_t = t - T_{\mathcal{N}_t}$$

the *current waiting time*; and by

$$L_t = R_t + C_t = T_{\mathcal{N}_t+1} - T_{\mathcal{N}_t},$$

the *total length* of the interval.

Corollary 2.9 (Waiting time paradox)**.** The remaining waiting time R_t is exponentially distributed with parameter λ.

Proof. Since increments are stationary and independent

$$\mathbb{P}(R_t > r) = \mathbb{P}(\mathcal{N}_{t+r} - \mathcal{N}_t = 0) = e^{-\lambda r}.$$

□

Exercise 5. Find the distribution of C_t and show that it is independent of R_t. Calculate the distribution of L_t.

2.2.1 Thinning and superposition

The independence entailed in the Poisson process results in two remarkable properties: If a Poisson process is independently thinned (i.e. some of the jump times are removed), the remaining jump times still forms a Poisson process with reduced intensity. Vice versa, the sum of two independent Poisson processes is again a Poisson process. We need the following lemma.

Lemma 2.10. Let N be Poisson-distributed with parameter $\lambda > 0$. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. Bernoulli(p)-distributed random variables, independent of N, and let $X = \sum_{j=1}^{N} X_j$. Then, X and $N - X$ are independent and both Poisson distributed with parameters $p\lambda$ and $(1-p)\lambda$, respectively.

Exercise 6. Proof Lemma 2.10.

Let us now consider a thinning of the Poisson process $\mathcal{N}_t$. Let $T_1, T_2, \ldots$ be its jump times. At each jump time, we now flip an independent coin with probability p to decide whether we retain the jump or discard it. More precisely, let $X_1, X_2, \ldots$ be i.i.d. Bernoullis with parameter p, independent of $\mathcal{N}$. Define two stochastic processes $\mathcal{N}^{(0)}$ and $\mathcal{N}^{(1)}$ via

$$\mathcal{N}_t^{(1)} = \sum_{j=1}^{\mathcal{N}_t} X_j$$

and $\mathcal{N}_t^{(0)} = \mathcal{N}_t - \mathcal{N}_t^{(1)}$. Then $\mathcal{N}^{(1)}$ counts the retained jumps of $\mathcal{N}$ while $\mathcal{N}^{(0)}$ counts the discarded ones.

Proposition 2.11 (Thinning property)**.** The two processes $\mathcal{N}^{(0)}$ and $\mathcal{N}^{(1)}$ are independent Poisson processes with intensity $(1-p)\lambda$ and $p\lambda$, respectively.

Proof. By Exercise 3 we have to show that both processes fulfil the properties of Proposition 2.5 and their independence. Clearly, $\mathcal{N}_0^{(1)} = 0$ and $t \mapsto \mathcal{N}_t^{(1)}$ is almost surely càdlàg and the same holds for $\mathcal{N}^{(0)}$. Further, by Lemma 2.10, the random variables $\mathcal{N}_{t+s}^{(0)} - \mathcal{N}_s^{(0)}$ and $\mathcal{N}_{t+s}^{(1)} - \mathcal{N}_s^{(1)}$ are independently Poisson distributed with parameters $(1-p)\lambda t$ and $p\lambda t$, respectively. For the independent increments, consider

$$\mathcal{N}_t^{(1)} = \sum_{j=1}^{\mathcal{N}_t} X_j \quad \text{and} \quad \mathcal{N}_{t+s}^{(1)} - \mathcal{N}_t^{(1)} = \sum_{j=\mathcal{N}_t+1}^{\mathcal{N}_{t+s}} X_j$$

Since $\mathcal{N}_t$ and $\mathcal{N}_{t+s} - \mathcal{N}_t$ are independent of each other and both sums use independent Bernoullis, the independent increments of $\mathcal{N}^{(1)}$ follow. Next, consider $\mathcal{N}_t^{(0)} = \mathcal{N}_t - \mathcal{N}_t^{(1)}$

and

$$\mathcal{N}^{(0)}_{t+s} - \mathcal{N}^{(0)}_t = (\mathcal{N}_{t+s} - \mathcal{N}_t) - (\mathcal{N}^{(1)}_{t+s} - \mathcal{N}^{(1)}_t).$$

With the argument used above, we have that $\mathcal{N}_t$ and $\mathcal{N}^{(1)}_{t+s} - \mathcal{N}^{(1)}_t$ are independent and $\mathcal{N}^{(1)}_t$ is independent of $\mathcal{N}_{t+s} - \mathcal{N}_t$. The independent increments of $\mathcal{N}^{(0)}$ follow from this and the independent increments of $\mathcal{N}$ and $\mathcal{N}^{(1)}$.

It remains to formally argue that not only the one-dimensional marginals $\mathcal{N}^{(0)}_t$ and $\mathcal{N}^{(1)}_t$ are independent but the whole processes are. To this end, we have to show that the increment vectors $(\mathcal{N}^{(0)}_{t_1}, \dots, \mathcal{N}^{(0)}_{t_n} - \mathcal{N}^{(0)}_{t_{n-1}})$ and $(\mathcal{N}^{(1)}_{t_1}, \dots, \mathcal{N}^{(1)}_{t_n} - \mathcal{N}^{(1)}_{t_{n-1}})$ are independent for all $0 \leq t_1 \leq \dots \leq t_n$. We do this inductively. For a single time t_1, we have already seen the result. For general n, we have by induction hypothesis that the increment vectors $(\mathcal{N}^{(0)}_{t_1}, \dots, \mathcal{N}^{(0)}_{t_{n-1}} - \mathcal{N}^{(0)}_{t_{n-2}})$ and $(\mathcal{N}^{(1)}_{t_1}, \dots, \mathcal{N}^{(1)}_{t_{n-1}} - \mathcal{N}^{(1)}_{t_{n-2}})$ are independent. Moreover, repeating the above arguments, $\mathcal{N}^{(1)}_{t_n} - \mathcal{N}^{(1)}_{t_{n-1}}$ is independent of $\mathcal{N}^{(0)}_{t_n} - \mathcal{N}^{(0)}_{t_{n-1}}$ and further independent of the vector $(\mathcal{N}^{(0)}_{t_1}, \dots, \mathcal{N}^{(0)}_{t_{n-1}} - \mathcal{N}^{(0)}_{t_{n-2}})$ and the same is true for $\mathcal{N}^{(0)}_{t_n} - \mathcal{N}^{(0)}_{t_{n-1}}$ and $\mathcal{N}^{(1)}_{t_n} - \mathcal{N}^{(1)}_{t_{n-1}}$ and the vector $(\mathcal{N}^{(1)}_{t_1}, \dots, \mathcal{N}^{(1)}_{t_{n-1}} - \mathcal{N}^{(1)}_{t_{n-2}})$. This concludes the proof. □

Exercise 7. Give an alternative proof for Proposition 2.11 by constructing the two processes via appropriate independent exponential waiting times.

Proposition 2.12 (Superposition of Poisson processes.)**.** Let $\mathcal{N}$ be a Poisson process of intensity λ and $\mathcal{M}$ be an independent Poisson process of intensity μ. Then, $\mathcal{N} + \mathcal{M}$ is a Poisson process of intensity $\lambda + \mu$.

Exercise 8. Prove Proposition 2.12.

2.3 Oriented percolation

CHAPTER 3

THE CONTACT PROCESS ON $\mathbb{Z}^d$

3.1 The graphical representation

3.2 Monotonicity, duality, and attractiveness

3.3 Survival-extinction phase transition

CHAPTER 4

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