Large Deviations of the Interference in a Wireless Communication Model

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Abstract—**Interference from other users limits the capacity, and possibly the connectivity, of wireless networks. A simple model of a wireless ad hoc network, in which node locations are described by a homogeneous Poisson point process, and node transmission powers are random, is considered in this paper. A large deviation principle for the interference is presented under different assumptions on the distribution of transmission powers.**

Index Terms—**Large deviations, Poisson shot noise, subexponential distributions, fading channels, code division multiple access (CDMA).**

I. Introduction

WIRELESS ad hoc and sensor networks have been the topic of much recent research. Questions of interest include the connectivity of the network, namely, the ability of any two nodes to communicate, possibly via intermediate nodes, and the information transport capacity of the network [14], [6], [7]. The factor limiting the communication between any two nodes is the ratio of signal power to the sum of noise and interference. As such, the signal to interference plus noise ratio (SINR) is an object of interest in its own right. In this paper, we study the large deviations asymptotic of this quantity in the context of a simple model of a wireless network [2], which is described in the next section.

The SINR determines whether a given pair of nodes can talk to each other at a given time. The interference is determined by which other nodes are transmitting simultaneously, as well as the degree of orthogonality between the codes they are using. If the codes are perfectly orthogonal, then there will be no interference. Most cellular systems employ channel assignment schemes to ensure that codes used in nearby cells are indeed orthogonal. Wireless local area networks (LANs) use scheduling to ensure that nearby nodes do not transmit simultaneously. Motivated by ad hoc networks, we are interested in a scenario where there is no centralized infrastructure and where nodes may belong to multiple administrative domains. In such a scenario, neither channel assignment nor sophisticated forms of scheduling may be feasible. Our aim is to determine how the outage probability depends on factors such as node density and spreading (or processing) gain.

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II. Model Description

Let $\{(X_k, P_k)\}_{k\geq 1}$ be a marked point process on the plane, where $\{X_k\}_{k\geq 1}$ denotes the locations of antennas, and the marks $P_k \in (0,\infty)$ denote their transmission powers. Without loss of generality, we will consider a receiver located at the origin. Let w and β be positive constants which denote, respectively, the noise power at the receiver, and the threshold SINR needed for successful reception of a signal. The physical signal propagation is described by a measurable positive function $L : \mathbb{R}^2 \to (0,\infty)$, which gives the attenuation or path loss of the signal. In addition, the signal undergoes random fading (due to occluding objects, reflections, multipath interference, etc.). We denote by H_n the random fading between node n and the origin, and define $Y_n = P_n H_n$. Thus, $Y_n L(X_n)$ is the received power at the origin due to the transmitter at node n.

Within this framework, we say that the receiver at the origin can decode the signal emitted by station n if

$$\frac{Y_n L(X_n)}{w + \sum_{k\neq n} Y_k L(X_k)} \geq \beta. \tag{1}$$

The sum in the denominator is restricted to those stations, which are active during the period of interest. The marked point process can be taken as referring to this subset.

The attenuation function is often taken to be isotropic (i.e., rotation invariant) and of the form $L(x) = \|x\|^{-\alpha}$ or $(1+\|x\|)^{-\alpha}$ or $\max(R, \|x\|)^{-\alpha}$, where the symbol $\|\cdot\|$ denotes the Euclidean norm, and $\alpha, R > 0$ are positive constants. The first function exhibits infinite interference in the immediate vicinity of an antenna, which is not physical. The last choice of attenuation function corresponds to the case of isotropic antennas with ideal Hertzian propagation, and is the one we will work with. (We make the assumption of isotropy only for notational convenience. It should be clear from our derivations that we sum up the contributions to interference at a point from successive annuli around it, and the contribution from an annulus can be readily calculated for a given anisotropic attenuation function. In particular, our framework can deal with directional antennas.) We will assume that $\alpha > 2$, which is an integrability condition needed to ensure that the total interference is finite almost surely, and which is observed to hold in practice.

The basic model described above is quite general and encompasses the case where the signal emitted by station n interferes with the cumulative signals emitted from stations $k \neq n$ in such a way that only some proportion ϕ of these cumulative signals should be considered as noise. Indeed, this situation is recovered replacing $L(\cdot)$ by $\phi\tilde{L}(\cdot)$, where $\tilde{L} : \mathbb{R}^2 \to (0,\infty)$ is again a positive and measurable function. The coefficient ϕ weights the effect of interferences, depending on the orthogonality between codes used during simultaneous transmissions. It is equal to 1 in a narrowband system, and is smaller than 1 in a broadband system that uses code division multiple access (CDMA); see, for instance, [8] and [16]. The physical model of [14] assumes $\phi = 1$; the models of [13], [6], and [7] allow ϕ to be smaller than 1.

From the modeling perspective, the effect of ϕ in (1) can be absorbed into the threshold β (with the noise power suitably modified), so we will assume without loss of generality that $\phi = 1$. From the application perspective, we are interested in $\phi \ll 1$, as is the case, for example, in a CDMA system with large spreading gain. In this case, we can hope that the SINR might exceed the threshold even if all stations transmit simultaneously, in which case no complicated scheduling scheme is needed. With that interpretation, the results of this paper can be seen as providing a guide to the tradeoff achievable between node density and spreading gain of the code. Rewriting (1) allowing $\phi < 1$, we see that the SINR is too small for decoding if and only if

$$\sum_{k\neq n} Y_k L(X_k) > \frac{1}{\phi}\left(\frac{Y_n L(X_n)}{\beta} - w\right).$$

Thus, the probability of decoding failure is given by

$$P\left(\sum_{k\neq n} Y_k L(X_k) > \frac{c}{\phi}\right), \qquad \text{where} \quad c = \frac{Y_n L(X_n)}{\beta} - w.$$

The scaling regime we will consider is the logarithmic asymptotics of this probability as ϕ tends to zero, i.e., a large deviations scaling regime. In other words, we are interested in systems with large bandwidth, and concomitant coding gain, in a regime in which the probability of the SINR falling below the required threshold is small.

As remarked above, we restrict ourselves to the case of ideal Hertzian propagation, so that $L(x) = \ell(\|x\|) = \max(R, \|x\|)^{-\alpha}$. Finally, we assume that the point process $\{X_k\}_{k\geq 1}$ is a homogeneous Poisson process of intensity λ and that both the marks and the shadow fading to the origin are independent identically distributed (i.i.d.) and independent of the locations. In particular, $Y_k = P_k H_k$ is an i.i.d. sequence independent of the point process $\{X_k\}$. From the modeling perspective, it suffices to consider Y_k rather than P_k and H_k separately. With some abuse of terminology, we will henceforth refer to Y_k as the transmission power and use $\{(X_k, Y_k)\}_{k\geq 1}$ as our basic marked point process. With these assumptions, the denominator of the left-hand side of (1) reduces to $w + \sum_{k\neq n} Y_k \ell(\|X_k\|)$.

Look at the SINR between the receiver at the origin and a point located at $x \in \mathbb{R}^2$ of the Poisson process, denote by Y the transmission power of the antenna located at x, and assume Y to be independent of the marked Poisson process $\{(X_k, Y_k)\}_{k\geq 1}$. Let P_x denote the Palm probability of the Poisson process at $x \in \mathbb{R}^2$ (i.e., the conditional law of the Poisson process, given that it has a point at x), and define the random variable

$$V = \sum_{k\geq 1} Y_k \ell(\|X_k\|), \qquad \text{where } \ell(x) = \max(R, x)^{-\alpha}. \quad (2)$$

Due to Slivnyak's theorem (see, e.g., [4]) and the independence between Y and $\{(X_k, Y_k)\}_{k\geq 1}$, we have

$$\begin{aligned} &P_x\left(\frac{Y\ell(\|x\|)}{w + V - Y\ell(\|x\|)} < \beta \,\middle|\, Y = y\right) \\ &= P\left(\frac{Y\ell(\|x\|)}{w + V} < \beta \,\middle|\, Y = y\right) \\ &= P(V > (y\ell(\|x\|)/\beta) - w). \end{aligned}$$

The main aim of this paper is to provide large deviation principles for the total interference V at the origin. Since the noise power w is a positive constant, this will yield the large deviations for the SINR ratio.

If $\alpha > 2$, it can be readily verified from (2) that $EV < \infty$ if $EY_1 < \infty$, and so V is finite almost surely (a.s.) if Y_1 has finite mean. We will consider several different models for the law of Y_1, namely, distributions with bounded support, those with a tail which is asymptotically exponentially equivalent to the tail of a Weibull or exponential distribution (see Section III for the definition of asymptotic exponential equivalence), and those with regularly varying tails. The Weibull assumption on the distribution of the transmission powers is particularly appealing in the context of wireless networks as a recent work by Sagias and Karagiannis [15] states that the fading in wireless channels can be modeled by a Weibull distribution (typically with Gaussian tail).

In order to describe the structure of this paper, we introduce some more notations. Define $R_0 = 0$, $R_k = \sqrt{k}R$ for $k \geq 1$, and let

$$V^k = \sum_{i=1}^{\infty} Y_i L(X_i) 1(R_k \leq \|X_i\| < R_{k+1}), \qquad k \geq 0$$

be the total interference at the origin due to sources at distance between R_k and R_{k+1}; here $1(A)$ denotes the indicator of the event A. In particular, note that $V = \sum_{k\geq 0} V^k$. This paper is organized as follows. In Section III, we give some preliminaries on large deviations, heavy-tailed distributions, and extreme value theory. In Section IV, we establish large deviation principles for the random variables $V_\varepsilon^0 = \varepsilon V^0$ under different assumptions on the transmission powers. Specifically, we consider the cases where the distribution of the transmission power has the following: 1) bounded support, 2) superexponential tails, 3) exponential tails, 4) subexponential tails and it belongs to the domain of attraction of the Gumbel distribution, and 5) regularly varying tails. In Section V, we extend these results to large deviation principles for the total interference $V_\varepsilon = \varepsilon V$. The different cases give rise to different speeds for the large deviation principles, with the speed decreasing as the tail grows heavier.

III. Preliminaries and Notation

We recall here some basic definitions in large deviations theory. A family of probability measures $\{\mu_\varepsilon\}_{\varepsilon>0}$ on $(\mathbb{R}, \mathcal{B})$ obeys a large deviation principle (LDP) with rate function $I(\cdot)$ and speed $v(\cdot)$ if $I : \mathbb{R} \to [0, \infty]$ is a lower semicontinuous function, $v : (0, \infty) \to (0, \infty)$ is a measurable function which diverges to infinity at the origin, and the following inequalities hold for every Borel set $B \in \mathcal{B}$:

$$-\inf_{x \in B^\circ} I(x) \le \liminf_{\varepsilon \to 0} \frac{1}{v(\varepsilon)} \log \mu_\varepsilon(B) \le \limsup_{\varepsilon \to 0} \frac{1}{v(\varepsilon)} \log \mu_\varepsilon(B) \le -\inf_{x \in \overline{B}} I(x)$$

where B° denotes the interior of B and $\overline{B}$ denotes the closure of B. Similarly, we say that a family of $\mathbb{R}$-valued random variables $\{V_\varepsilon\}_{\varepsilon>0}$ obeys an LDP if $\{\mu_\varepsilon\}_{\varepsilon>0}$ obeys an LDP and $\mu_\varepsilon(\cdot) = P(V_\varepsilon \in \cdot)$. We point out that the lower semicontinuity of $I(\cdot)$ means that its level sets

$$\{x \in \mathbb{R} : I(x) \le a\}, \qquad a \ge 0$$

are closed; when the level sets are compact, the rate function $I(\cdot)$ is said to be good.

Throughout this paper, we write $f(x) \sim g(x)$ if $f(x)$ and $g(x)$ are asymptotically equivalent, i.e., $\lim_{x\to\infty} f(x)/g(x) = 1$; moreover, we say that $f(x)$ and $g(x)$ are asymptotically exponentially equivalent if $\log f(x) \sim \log g(x)$.

We conclude this section with some preliminaries on heavy-tailed distributions. Recall that a random variable is said to be subexponential if its distribution function $G(\cdot)$ has support $(0,\infty)$ and $\overline{G^{*2}}(x) \sim 2\bar{G}(x)$ (see, e.g., [1, p. 251] and [9, pp. 39–40]). Here $\bar{G} = 1 - G$ denotes the tail of the distribution function $G(\cdot)$ and $G^{*2}(\cdot)$ denotes the twofold convolution of $G(\cdot)$.

The family of subexponential distribution functions will be denoted by $\mathcal{S}$. It can be classified using extreme value theory, as follows. A positive function $g(\cdot)$ on $(0,\infty)$ is said to be regularly varying at infinity of index $c \in \mathbb{R}$, written $g \in \mathcal{R}(c)$, if $g(x) \sim x^c S(x)$ as $x \to \infty$, where $S(\cdot)$ is a slowly varying function, i.e., $\lim_{x\to\infty} S(tx)/S(x) = 1$ for each $t > 0$. Goldie and Resnick [12] showed that if $G \in \mathcal{S}$ and satisfies some smoothness conditions, then $G(\cdot)$ belongs to the maximum domain of attraction of either the Frechét distribution $\Phi_c(x) = e^{-x^{-c}}$, $c > 0$, or the Gumbel distribution $\Lambda(x) = e^{-e^{-x}}$. In the former case, it has regularly varying tail of index $-c$. We write $G \in \mathrm{MDA}(\Lambda)$ if $G(\cdot)$ belongs to the maximum domain of attraction of the Gumbel distribution.

IV. Large Deviations of the Truncated Interference

In this section, we show the following large deviation principles, which correspond, respectively, to the cases where the transmission powers Y_k are bounded, have Weibullian tails which are superexponential, exponential or subexponential, or have regularly varying tails.

Theorem 1: Suppose that Y_1 has bounded support with supremum b which is strictly positive. Then, the family of random variables $\{V_\varepsilon^0\}$ obeys an LDP on $[0,\infty)$ with speed $\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}$ and good rate function

$$I_1(x) = \frac{R^\alpha x}{b}.$$

Theorem 2: Suppose that there exist constants $c > 0$ and $\gamma > 1$ such that $-\log P(Y_1 > y) \sim cy^\gamma$. Define $\eta = 1-(1/\gamma)$. Then, the family of random variables $\{V_\varepsilon^0\}$ obeys an LDP on $[0,\infty)$ with speed $\frac{1}{\varepsilon}\log^\eta(\frac{1}{\varepsilon})$ and good rate function

$$I_2(x) = \gamma(\gamma-1)^{-\eta} c^{1/\gamma} R^\alpha x.$$

Theorem 3: Suppose that there exists a constant $c > 0$ such that $-\log P(Y_1 > y) \sim cy$. Then, the family of random variables $\{V_\varepsilon^0\}$ obeys an LDP on $[0,\infty)$ with speed $\frac{1}{\varepsilon}$ and good rate function

$$I_3(x) = cR^\alpha x.$$

Theorem 4: Suppose that Y_1 is subexponential and that there exist constants $c > 0$ and $0 < \gamma < 1$ such that $-\log P(Y_1 > y) \sim cy^\gamma$. Then, the family of random variables $\{V_\varepsilon^0\}$ obeys an LDP on $[0,\infty)$ with speed $(\frac{1}{\varepsilon})^\gamma$ and good rate function

$$I_4(x) = cR^{\alpha\gamma} x^\gamma.$$

Theorem 5: Suppose that $P(Y_1 > y) \sim y^{-c} S(y)$, for some constant $c > 1$ and slowly varying function $S(\cdot)$. Then, the family of random variables $\{V_\varepsilon^0\}$ obeys an LDP on $[0,\infty)$ with speed $\log(\frac{1}{\varepsilon})$ and rate function

$$I_5(x) = \begin{cases} 0, & \text{if } x = 0 \\ c, & \text{if } x > 0. \end{cases}$$

Observe that as γ tends to infinity the speed and the rate function of the LDP in Theorem 2 tend to those for the case of bounded transmission powers (with $b = 1$). Similarly, as γ tends to 1, the speed and the rate function of Theorems 2 and 4 tend to those for the case of transmission powers with tails asymptotically exponentially equivalent to the tail of an exponential distribution.

Before going into the details of the proofs, we remark briefly on the intuition behind these results. The theorems above provide LDPs for Poisson shot noise under different conditions on the shot shape (the distribution of Y_1). Theorem 1 basically gives the large deviations for a Poisson random variable, since the shot in this case is effectively a constant. The speed $\frac{1}{\varepsilon}\log(\frac{1}{\varepsilon})$ comes from the fact that the tail of a Poisson distribution behaves like $1/n!$. When Y_1 has superexponential Weibull tails, there is an interaction between the tail asymptotics of the Poisson distribution and that of the shot, both of which combine to contribute to the tail of the Poisson shot noise. Finally, when the shot has exponential or subexponential tails, it dominates and the Poisson distribution plays no role.

This intuition also explains why the intensity λ of the Poisson point process of transmitters plays no role in the large deviation rate function, in any of the theorems above. In the exponential and subexponential cases, a large value of the interference is caused by a single interfering transmitter, and hence, it is the asymptotic of the distribution of transmission powers which governs the rate function. In the superexponential case, a large value of the interference is caused by a combination of 1) there being a large number of interferers in the vicinity of the origin, and 2) each of these having a large transmit power. Of these, only 1) involves the underlying Poisson point process. Now, the number of transmitters within a region of area A is a Poisson random variable with mean λA, whose tail behavior is predominantly described by a $1/n!$ term, which does not depend on λA.

While the absence of the point process intensity in the rate function may appear counterintuitive, it really tells us that the LDPs only capture the shape and scaling behavior of the tail distribution of the interference, and that more refined estimates of the actual probabilities are needed.

We will prove the theorems by providing matching large deviation upper and lower bounds for half intervals, i.e., for $P(V_\varepsilon^0 \geq x)$ and $P(V_\varepsilon^0 > x)$, and showing that these imply a large deviation principle. The proof proceeds through a sequence of lemmas, whose proofs are relegated to the Appendix so as not to interrupt the flow of the arguments.

The following two lemmas provide large deviation upper bounds in the superexponential and exponential cases, respectively.

Lemma 6: Suppose that there exist positive constants $\tilde{c}$ and β such that $\log \phi(\theta) \sim \tilde{c}\theta^\beta$ as $\theta \to \infty$. Then

$$\limsup_{\varepsilon\to 0} \frac{\varepsilon}{\log^{1/\beta}(1/\varepsilon)} \log P(V_\varepsilon^0 \geq x) \leq -\tilde{c}^{-1/\beta} R^\alpha x, \qquad x \geq 0.$$

Lemma 7: Suppose that there is a positive constant c such that $-\log P(Y_1 > y) \sim cy$. Then

$$\limsup_{\varepsilon\to 0} \varepsilon \log P(V_\varepsilon^0 \geq x) \leq -cR^\alpha x, \qquad x \geq 0.$$

The proof of both lemmas uses Chernoff's bound. We present a brief outline here, leaving the details to the Appendix. Clearly

$$V_\varepsilon^0 = \varepsilon R^{-\alpha} \sum_{i=1}^{N_0} Y_i \tag{3}$$

where N_0 is the number of points of the Poisson process falling within the ball of radius R centered at the origin. Using the Chernoff bound, we have

$$P(V_\varepsilon^0 \geq x) \leq \exp(-\theta x + \Lambda_0(\varepsilon R^{-\alpha}\theta)) \qquad \forall\, \theta \geq 0 \tag{4}$$

where $\Lambda_0(\theta) := \log E[\exp(\theta \sum_{i=1}^{N_0} Y_i)]$. However, N_0 is a Poisson distributed random variable with mean $\lambda_0 = \lambda\pi R^2$ and the Y_i are i.i.d. and independent of N_0. Hence, defining $\phi(\theta) := E[\exp(\theta Y_1)]$, we have

$$\Lambda_0(\theta) = \log E[(E[e^{\theta Y_1}])^{N_0}] = \lambda_0(\phi(\theta) - 1). \tag{5}$$

The proof now proceeds by substituting (5) in (4) and optimizing over θ.

The upper bound in Lemma 6 is in terms of the logarithmic moment generating function of Y_1 whereas the assumptions in Theorem 2 are in terms of the tail of its distribution. The next lemma relates a tail condition on the law of Y_1 to the tail behavior of its logarithmic moment generating function. Its proof requires an extension of Laplace's method and is set out in the Appendix.

Lemma 8: Suppose that there exist constants $c > 0$ and $\gamma > 1$ such that $-\log P(Y_1 > y) \sim cy^\gamma$. Define $\eta = 1 - (1/\gamma)$. Then

$$\log \phi(\theta) \sim (\gamma - 1)\gamma^{-1/\eta} c^{-1/(\gamma-1)} \theta^{1/\eta}.$$

Next, we consider the subexponential Weibull case. A large deviation upper bound is given by the following.

Lemma 9: Suppose that Y_1 is subexponential and that there exist constants $c > 0$ and $0 < \gamma < 1$ such that $-\log P(Y_1 > y) \sim cy^\gamma$, then

$$\limsup_{\varepsilon\to 0} \varepsilon^\gamma \log P(V_\varepsilon^0 \geq x) \leq -cR^{\alpha\gamma} x^\gamma, \qquad x \geq 0.$$

The proof uses a key fact about subexponential distributions, namely, that the tail distribution of a sum of i.i.d. subexponential random variables is asymptotically equivalent to that of their maximum, and this is still true if the number of terms in the sum is a random variable provided that this random variable has exponentially decaying tail. Details are in the Appendix.

The next lemma gives the large deviation lower bound needed to prove Theorem 1. It is a straightforward consequence of the tail behavior of the Poisson distribution for N_0.

Lemma 10: If Y_1 has compact support whose supremum, denoted b, is strictly positive (i.e., Y_1 is not identically zero), then

$$\liminf_{\varepsilon\to 0} \frac{\varepsilon}{\log(1/\varepsilon)} \log P(V_\varepsilon^0 > x) \geq -\frac{R^\alpha x}{b}, \qquad x \geq 0.$$

The large deviation lower bound in the superexponential Weibull case is given by the following lemma.

Lemma 11: Suppose that there exist constants $c > 0$ and $\gamma > 1$ such that $-\log P(Y_1 > y) \sim cy^\gamma$. Define $\eta = 1 - (1/\gamma)$. Then, for all $x \geq 0$, we have

$$\liminf_{\varepsilon\to 0} \frac{\varepsilon}{\log^\eta(1/\varepsilon)} \log P(V_\varepsilon^0 > x) \geq -\gamma(\gamma-1)^{-\eta} c^{1/\gamma} R^\alpha x.$$

The proof involves identifying the *most likely way* that a large value of V^0 arises. Specifically, in (3), it involves identifying the typical value of N_0, the number of interferers within distance R of the receiver, as well as the typical value of their transmission powers, conditional on $V^0 > x/\varepsilon$. See the Appendix for details.

Proof of Theorem 1: The function $I_1(x) = R^\alpha x/b$ is continuous on $[0, \infty)$ and has compact level sets. Hence, it is a good rate function.

If Y_1 has compact support with supremum b, then it is easy to see that $\log \phi(\theta) \sim b\theta$, where $\phi(\cdot)$ is the moment generating function of Y_1. Hence, we have by Lemma 6 that

$$\limsup_{\varepsilon\to 0} \frac{\varepsilon}{\log(1/\varepsilon)} \log P(V_\varepsilon^0 \geq x) \leq -\frac{R^\alpha x}{b}, \qquad x \geq 0. \tag{6}$$

This upper bound matches the corresponding lower bound in Lemma 10.

The upper and lower bounds can be extended from half intervals $[x, \infty)$ and (x, ∞) to arbitrary closed and open sets in a standard way, which is repeated in the proofs of Theorems 2–5 [actually the rate function of Theorem 5 is not continuous in 0; however, it is readily checked that the argument we consider below holds for rate functions that are equal to 0 at the origin and continuous on $(0, \infty)$]. We, therefore, sketch it for completeness.

Let F be a closed subset of $[0,\infty)$ and let x denote the infimum of F. Since $I_1(\cdot)$ is increasing, $I_1(x)=\inf_{y\in F} I_1(y)$. Now F is contained in $[x,\infty)$, and so we obtain using (6) that

$$\begin{aligned}\limsup_{\varepsilon\to 0}&\frac{\varepsilon}{\log(1/\varepsilon)}\log P(V_\varepsilon^0\in F)\\ &\le \limsup_{\varepsilon\to 0}\frac{\varepsilon}{\log(1/\varepsilon)}\log P\left(V_\varepsilon^0\ge x\right)\\ &\le -I_1(x)=-\inf_{y\in F} I_1(y).\end{aligned}$$

This establishes the large deviation upper bound for arbitrary closed sets.

Next, let G be an open subset of $[0,\infty)$. Suppose first that $0\notin G$ and set $\nu=\inf_{y\in G} I_1(y)$. Then, ν is finite and, for arbitrary $\delta>0$, we can find $x\in G$ such that $I_1(x)\le \nu+\delta$. Since G is open, we can also find $\eta>0$ such that $(x-\eta,x+\eta)\subseteq G$. Now

$$\begin{aligned}P(V_\varepsilon^0\in G)&\ge P(V_\varepsilon^0\in(x-\eta,x+\eta))\\ &=P(V_\varepsilon^0>x-\eta)-P(V_\varepsilon^0\ge x+\eta).\end{aligned}\quad (7)$$

Moreover

$$\liminf_{\varepsilon\to 0}\frac{\varepsilon}{\log(1/\varepsilon)}\log P(V_\varepsilon^0>x-\eta)\ge -I_1(x-\eta)$$

by Lemma 10, whereas

$$\limsup_{\varepsilon\to 0}\frac{\varepsilon}{\log(1/\varepsilon)}\log P(V_\varepsilon^0\ge x+\eta)\le -I_1(x+\eta)$$

by (6). Since $I_1(x-\eta)<I_1(x+\eta)$, we obtain using (7) and Lemma 19 in the Appendix that

$$\liminf_{\varepsilon\to 0}\frac{\varepsilon}{\log(1/\varepsilon)}\log P\left(V_\varepsilon^0\in G\right)\ge -I_1(x-\eta).$$

Since $I_1(\cdot)$ is continuous, by letting η decrease to zero, we get

$$\liminf_{\varepsilon\to 0}\frac{\varepsilon}{\log(1/\varepsilon)}\log P\left(V_\varepsilon^0\in G\right)\ge -I_1(x)\ge -\inf_{y\in G} I_1(y)-\delta$$

where the last inequality follows from the choice of x. The large deviation lower bound now follows upon letting δ decrease to zero.

If $0\in G$, then, since G is open, there is an $\eta>0$ such that $[0,\eta)\subseteq G$. Hence

$$P(V_\varepsilon^0\in G)\ge 1-P(V_\varepsilon^0\ge\eta).$$

By similar arguments to the above, we can show that

$$\liminf_{\varepsilon\to 0}\frac{\varepsilon}{\log(1/\varepsilon)}\log P\left(V_\varepsilon^0\in G\right)\ge 0.$$

Since $\inf_{y\in G} I_1(y)=I_1(0)=0$ as $I_1(\cdot)$ is increasing, this establishes the large deviation lower bound if $0\in G$, and completes the proof of the theorem. □

Proof of Theorem 2: We obtain from Lemmas 6 and 8 the upper bound

$$\begin{aligned}\limsup_{\varepsilon\to 0}&\frac{\varepsilon}{\log^\eta(1/\varepsilon)}\log P\left(V_\varepsilon^0\ge x\right)\\ &\le -\left((\gamma-1)\gamma^{-1/\eta}c^{-1/(\gamma-1)}\right)^{-\eta}R^\alpha x\\ &=-\gamma(\gamma-1)^{-\eta}c^{1/\gamma}R^\alpha x.\end{aligned}$$

This matches the lower bound from Lemma 11. The extension from the half intervals (x,∞) and $[x,\infty)$ to arbitrary open and closed sets follows along the lines of the proof of Theorem 1. □

Proof of Theorems 3 and 4: The large deviation upper bound for half intervals $[x,\infty)$ is provided by Lemma 7 in the exponential case and by Lemma 9 in the subexponential Weibull case. For the lower bound, observe that for all $x\ge 0$

$$\begin{aligned}P(V_\varepsilon^0>x)&\ge P(V_\varepsilon^0>x,N_0\ge 1)\\ &\ge P(Y_1>R^\alpha x/\varepsilon)P(N_0\ge 1)\end{aligned}$$

and so, by the assumption that $-\log P(Y_1>y)\sim cy^\gamma$ for some $\gamma\in(0,1]$, we obtain

$$\begin{aligned}\liminf_{\varepsilon\to 0}\varepsilon^\gamma\log P\left(V_\varepsilon^0>x\right)&\ge\liminf_{\varepsilon\to 0}\varepsilon^\gamma\log P(Y_1>R^\alpha x/\varepsilon)\\ &=-cR^{\alpha\gamma}x^\gamma.\end{aligned}$$

Finally, the upper and lower bounds can be extended to arbitrary closed and open sets using standard techniques, as in the proof of Theorem 1. □

Proof of Theorem 5: The claim follows if we give upper and lower bounds on half intervals $[x,\infty)$ and (x,∞), $x\ge 0$. Since $P(V_\varepsilon^0>0)=1$,the upper and lower bounds for half intervals $[0,\infty)$ and $(0,\infty)$ are obvious. Thus,we consider$x>0$.

Recall that regularly varying distributions are subexponential (see, e.g., [9, Corollary 1.3.2]). Therefore, we have by [1, Lemma 2.2] that

$$\begin{aligned}P\left(V_\varepsilon^0>x\right)&=P\left(\sum_{i=1}^{N_0}Y_i>R^\alpha x/\varepsilon\right)\\ &\sim E[N^0]P(Y_1>R^\alpha x/\varepsilon)\\ &=\lambda_0(R^\alpha x/\varepsilon)^{-c}S(R^\alpha x/\varepsilon).\end{aligned}$$

Likewise, we have for all $\delta>0$ small enough, that

$$\begin{aligned}P\left(V_\varepsilon^0\ge x\right)&\le P\left(\sum_{i=1}^{N_0}Y_i>R^\alpha(x-\delta)/\varepsilon\right)\\ &\sim\lambda_0\left(R^\alpha(x-\delta)/\varepsilon\right)^{-c}S(R^\alpha(x-\delta)/\varepsilon).\end{aligned}\quad (9)$$

The large deviation lower and upper bounds for half intervals readily follow from (8) and (9) upon taking logarithms and letting ε, and then δ, tend to zero. □

V. Large Deviations of the Total Interference

So far we have restricted attention to V^0, the contribution to interference due to transmitters within range R of the location of interest. We now extend our results to the total interference V. Here $G(\cdot)$ denotes the distribution function of the transmission powers, i.e., $G(x)=P(Y_1\le x)$.

Theorem 12: We have the following.

i) If Y_1 has bounded support with supremum $b > 0$, then the family of random variables $\{V_\varepsilon\}$ obeys an LDP on $[0,\infty)$ with speed $\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}$ and good rate function $I_1(\cdot)$ given by Theorem 1.

ii) If $-\log P(Y_1 > y) \sim cy^\gamma$ for some $c > 0$ and $\gamma > 1$, then the family of random variables $\{V_\varepsilon\}$ obeys an LDP on $[0,\infty)$ with speed $\frac{1}{\varepsilon}\log^\eta(\frac{1}{\varepsilon})$, where $\eta = 1 - (1/\gamma)$, and good rate function $I_2(\cdot)$ given by Theorem 2.

iii) If $-\log P(Y_1 > y) \sim cy$ for some $c > 0$, then the family of random variables $\{V_\varepsilon\}$ obeys an LDP on $[0,\infty)$ with speed $\frac{1}{\varepsilon}$, and good rate function $I_3(\cdot)$ given by Theorem 3.

iv) If $G \in \mathrm{MDA}(\Lambda) \cap \mathcal{S}$ and $-\log \bar{G}(y) \sim cy^\gamma$ for some $c > 0$ and $0 < \gamma < 1$, then the family of random variables $\{V_\varepsilon\}$ obeys an LDP on $[0,\infty)$ with speed $(\frac{1}{\varepsilon})^\gamma$ and good rate function $I_4(\cdot)$ given by Theorem 4.

v) If $\bar{G} \in \mathcal{R}(-c), c > 1$, then the family of random variables $\{V_\varepsilon\}$ obeys an LDP on $[0,\infty)$ with speed $\log(\frac{1}{\varepsilon})$ and rate function $I_5(\cdot)$ given by Theorem 5.

The proof of the theorem requires different techniques in the cases where the tails of Y_1 decay exponentially or faster, and where they decay subexponentially. In the former case, we will make use of the following lemma and Chernoff's bound.

Lemma 13: Suppose that the family of random variables $\{X_\varepsilon\}$ obeys an LDP on $[0,\infty)$ with speed $v(\cdot)$ and rate function $I(x) = \gamma x$ for some $\gamma > 0$. Let $\{Y_\varepsilon\}$ be a family of nonnegative random variables independent of $\{X_\varepsilon\}$, satisfying

$$P(Y_\varepsilon \geq x) \leq \exp(-v(\varepsilon)\gamma' x)$$

for all $\varepsilon > 0$ and $x \geq 0$, and for some $\gamma' \geq \gamma$. Define $Z_\varepsilon = X_\varepsilon + Y_\varepsilon$. Then, $\{Z_\varepsilon\}$ obeys an LDP on $[0,\infty)$ with speed $v(\cdot)$ and rate function $I(\cdot)$.

The proof is in the Appendix. Loosely speaking, the lemma says that making a small perturbation to the random variables X_ε by adding a noise term Y_ε does not change the rate function in the LDP if the tails of the random variables Y_ε decay sufficiently rapidly. Observe that it is not the case that $\{X_\varepsilon\}$ and $\{Z_\varepsilon\}$ are exponentially equivalent (see [5] for a definition). Nevertheless, they do have the same large deviations rate function.

Next, define

$$U^j = \sum_{i=1}^{\infty} Y_i 1(R_j \leq \|X_i\| < R_{j+1}), \qquad j \geq 1$$

$$W^k = V^0 + \sum_{j=1}^{k} R_j^{-\alpha} U^j, \qquad k \geq 1$$

and

$$W = \lim_{k\to\infty} W^k$$

where $R_j = \sqrt{j}R$. The limit above exists since the random variables U^j are positive, and so the sequence $\{W^k\}$ is increasing. The reason for the choice $R_j = \sqrt{j}R$ is that this makes the areas of the successive annuli $\{x : R_{j-1} \leq \|x\| < R_j\}$ equal.

Observe that $R_j^{-\alpha}U^j$ is an upper bound for the interference due to nodes in the annulus $\{x : R_j \leq \|x\| < R_{j+1}\}$. Moreover, the random variables U^j are i.i.d. because they are the sum of the marks of a homogeneous marked Poisson process over disjoint intervals of equal area. In addition, W is a.s. finite. Indeed

$$EW = \lambda_0 R^{-\alpha}\left(1 + \sum_{j\geq 1} j^{-\alpha/2}\right) EY_1$$

and this quantity is finite since $\alpha > 2$ and the assumptions of Theorem 12 guarantee $EY_1 < \infty$.

Define $W_\varepsilon = \varepsilon W$ and note that $V_\varepsilon^0 \leq V_\varepsilon \leq W_\varepsilon$. The following lemma holds.

Lemma 14: Suppose that the assumptions of either part i), ii), or iii) of Theorem 12 are satisfied. Then, the family $\{W_\varepsilon\}$ obeys an LDP on $[0,\infty)$ with the same speed and rate function as stated for $\{V_\varepsilon\}$ in the corresponding part of Theorem 12.

The proof is in the Appendix. The idea is to show that the total contribution to the interference from nodes in all annuli sufficiently far from the origin is negligible and hence, by Lemma 13, that they do not change the rate function in the LDP. In addition, the contraction principle [5, Th. 4.2.1] is used to show that the contribution from any finite number of nearby annuli also does not change the rate function.

We will use the above lemma to prove parts i)–iii) of Theorem 12. For the proof of parts iv) and v), we need the two lemmas stated below following some definitions; we refer the reader to [9, Lemma A3.27] and [9, Lemma A3.26] for their proofs.

Consider the random variable

$$X = \sum_{k\geq 0} \psi_k Z_k$$

where the Z's are i.i.d. positive random variables with distribution function $F(\cdot)$ and the ψ's are positive constants. We assume without loss of generality that $\max_{k\geq 0}\psi_k = 1$.

Lemma 15: Suppose $F \in \mathrm{MDA}(\Lambda) \cap \mathcal{S}$ and $\sum_{k\geq 0}\psi_k^\delta < \infty$, for some $0 < \delta < 1$. Then,

$$P(X \geq x) \sim k^+ \bar{F}(x)$$

where k^+ is the cardinality of $\{k \geq 0 : \psi_k = 1\}$.

Lemma 16: Assume $\overline{F} \in \mathcal{R}(-c)$, for some positive constant c, say $\overline{F}(x) \sim x^{-c}S(x)$. If moreover $\sum_{k\geq 0}\psi_k^\delta < \infty$, for some $0 < \delta < \min(c, 1)$, then

$$P(X \geq x) \sim x^{-c}\tilde{S}(x), \qquad \text{where } \tilde{S}(x) = \left(\sum_{k\geq 0}\psi_k^c\right) S(x).$$

Proof of Theorem 12: We first give the proofs of parts i)–iii). Under these assumptions, we established the LDP for $\{V_\varepsilon^0\}$ in Theorems 1–3, and the LDP for $\{W_\varepsilon\}$ with the same speed and rate function in Lemma 14. Since $V_\varepsilon^0 \leq V_\varepsilon \leq W_\varepsilon$ for all $\varepsilon > 0$, the large deviation upper and lower bounds on half intervals also hold for V_ε. These bounds can be extended to a full LDP as in the proof of Theorem 1.

We now prove part iv). As usual, it suffices to prove large deviation upper and lower bounds for half intervals $[x,\infty)$ and (x,∞), respectively. Set $U^0 = R^\alpha V^0$ and recall that

$$W = R^{-\alpha}\sum_{k=0}^{\infty}\psi_k U^k, \qquad \text{where } \psi_0 = 1$$

and

$$\psi_k = k^{-\alpha/2}, \qquad k \geq 1. \tag{10}$$

Since $\alpha > 2$, it is clear that there is a $\delta \in (0,1)$ such that $\alpha\delta > 2$, and so $\sum_{k\geq 0}\psi_k^\delta < \infty$. We also note that, by [1, Lemma 2.2]

$$P(U^0 > x) = P\left(\sum_{i=1}^{N_0} Y_i > x\right) \sim E[N_0]P(Y_1 > x) \\ = \lambda_0 P(Y_1 > x).$$

Therefore, by the closure property of MDA(Λ) and $\mathcal{S}$ under tail equivalence (see, e.g., [9, Proposition 3.3.28 and Lemma A3.15]), the law of U^0 also belongs to MDA$(\Lambda)\cap\mathcal{S}$, so, by Lemma 15, we get

$$P(W_\varepsilon \geq x) = P\left(R^\alpha W \geq \frac{R^\alpha x}{\varepsilon}\right) \sim 2P\left(U^0 \geq \frac{R^\alpha x}{\varepsilon}\right) \\ = 2P(V_\varepsilon^0 \geq x)$$

where the last equality follows from the equality $U^0 = R^\alpha V^0$. Since $V_\varepsilon^0 \leq V_\varepsilon \leq W_\varepsilon$, we have for all $x \geq 0$ that

$$\limsup_{\varepsilon\to 0}\varepsilon^\gamma \log P(V_\varepsilon \geq x) \leq \limsup_{\varepsilon\to 0}\varepsilon^\gamma \log[2P(V_\varepsilon^0 \geq x)] \\ \leq -cR^{\alpha\gamma}x^\gamma.$$

We have used Lemma 9 to obtain the last inequality. A matching lower bound for open half intervals (x,∞) follows from the LDP for V_ε^0 stated in Theorem 4.

The proof of (v) is very similar. Since $P(V_\varepsilon > 0) = 1$ for all $\varepsilon > 0$, the large deviation upper and lower bounds for $[0,\infty)$ and $(0,\infty)$ are obvious, so we consider $x > 0$. To obtain the upper bound, we first note that $V \leq W$, where W is defined by (10). We also have, by [1, Lemma 2.2], that

$$P(U^0 > x) = P\left(\sum_{i=1}^{N_0} Y_i > x\right) \sim E[N_0]P(Y_1 > x) \\ = \lambda_0 x^{-c} S(x)$$

for some slowly varying function $S(\cdot)$. This implies that the law of U^0 is in $\mathcal{R}(-c)$. Hence, by Lemma 16

$$P(W_\varepsilon \geq x) = P\left(R^\alpha W \geq \frac{R^\alpha x}{\varepsilon}\right) \sim \left(\frac{x}{\varepsilon}\right)^{-c}\tilde{S}\left(\frac{x}{\varepsilon}\right)$$

for a suitable slowly varying function $\tilde{S}(\cdot)$. It readily follows, using the definition of a slowly varying function, that

$$\frac{1}{\log(1/\varepsilon)}\log P(W_\varepsilon \geq x) \to -c \quad \text{as } \varepsilon \to 0.$$

Since $V_\varepsilon \leq W_\varepsilon$, we have the desired large deviation upper bound for V_ε on half intervals $[x,\infty)$. A matching lower bound for open half intervals (x,∞) follows from the LDP for V_ε^0 stated in Theorem 5. This concludes the proof of the theorem. □

VI. Discussion

In this section, we discuss some variants of the model we have studied as well as some implications for communication networks.

We assumed an attenuation function of the form $L(x) = \max(R,\|x\|)^{-\alpha}$ for convenience. In fact, our analysis easily carries over to quite general attenuation functions, as we now argue. Suppose that $L(x) = \ell(\|x\|)$ for some continuous, non-increasing function $\ell : [0,\infty) \to (0,\infty)$. Suppose also that the following tail condition holds for all r sufficiently large:

$$\exists c > 0, \alpha > 2 : \ell(r) \leq cr^{-\alpha}. \tag{11}$$

The attenuation function $L(x) = \ell(\|x\|) = (1+\|x\|)^{-\alpha}$, for instance, satisfies this condition if $\alpha > 2$.

We now claim that all the conclusions of Theorem 12 continue to hold if we replace R^α by $1/\ell(0)$ in the corresponding rate functions.

Here is a brief outline of the proof. Fix $\rho > 0$ arbitrarily small. Observe that from (2) the total interference V at the origin is bounded from below by

$$\hat{V}^0 = \ell(\rho)\sum_{k\geq 1} Y_k 1(\|X_k\| < \rho) = \ell(\rho)\sum_{k=1}^{\hat{N}_\rho} Y_k$$

where $\hat{N}_\rho$ denotes the number of nodes within the disc of radius ρ centered at the origin, and has a Poisson distribution with mean $\pi\rho^2\lambda$. Next, define

$$\hat{U}^j = \sum_{i=1}^{\infty} Y_i 1(\hat{R}_j \leq \|X_i\| < \hat{R}_{j+1}), \qquad j \geq 0$$

$$\hat{W}_k = \ell(0)\hat{U}^0 + \sum_{j=1}^{k}\ell(\hat{R}_j)\hat{U}^j, \qquad k \geq 1$$

$$\hat{W} = \lim_{k\to\infty}\hat{W}_k$$

where $\hat{R}_0 = 0$ and $\hat{R}_k = \sqrt{k}\rho$, $k \geq 1$. Clearly, the total interference V is bounded from above by $\hat{W}$.

We can now derive LDPs for the families $\{\varepsilon\hat{V}^0\}$ and $\{\varepsilon\hat{W}\}$ exactly as in the previous sections. The one technical condition that needs to be checked is that $\sum_{k\geq 0}\psi_k^\delta < \infty$ for some $\delta \in (0,1)$, where $\psi_0 = 1$ and $\psi_k = \ell(\hat{R}_k)/\ell(0)$, $k \geq 1$. This condition is satisfied because of the assumption in (11) about the tail of the attenuation function.

Next, we note that the rate functions of the LDPs obtained in the previous sections did not depend on the intensity λ of the Poisson process, and that the parameter R only entered via the attenuation function for nodes in the disc closest to the origin. Thus, by analogy, the rate functions for $\{\varepsilon\hat{V}^0\}$ and $\{\varepsilon\hat{W}\}$ will simply have $R^{-\alpha}$ replaced by $\ell(\rho)$ and $\ell(0)$, respectively. So, letting $v_i(\cdot)$ denote the speed corresponding to the rate function $I_i(\cdot)(i=1,\ldots,5)$ in Theorem 12 and by $I_i^{(r)}(\cdot)$, $r \geq 0$, the rate function obtained by $I_i(\cdot)$ replacing $R^{-\alpha}$ by $\ell(r)$, we have, for all $\rho > 0$ and $x \geq 0$

$$-I_i^{(\rho)}(x) \leq \liminf_{\varepsilon\to 0} \frac{1}{v_i(\varepsilon)} \log P(\varepsilon \hat{W} > x) \leq \limsup_{\varepsilon\to 0} \frac{1}{v_i(\varepsilon)} \log P(\varepsilon \hat{W} \geq x) \leq I_i^{(0)}(x).$$

Letting ρ tend to zero and using the continuity of $I_i(\cdot)$ and $\ell(\cdot)$, we obtain the desired LDPs for the total interference V (more precisely, we obtain upper and lower bounds on closed and open half intervals, which can be extended to any closed and open set by standard techniques; see the proof of Theorem 1).

Let $r_e > 0$ be a positive constant. Using similar arguments, one can show that all the conclusions of Theorem 12 continue to hold with R^α replaced by $1/\ell(r_e)$ if we assume that the attenuation function is of the form

$$\hat{\ell}(x) = \begin{cases} 0, & \text{if } x < r_e \\ \ell(x), & \text{if } x \geq r_e \end{cases}$$

where $\ell : (0,\infty) \to (0,\infty)$ is a continuous and nonincreasing function which satisfies (11). (In Section II, we followed convention and defined the attenuation function to be strictly positive, but there is no harm in allowing it to take the value zero.) To be more specific, we provide here a possible choice for the bounds on V; the LDPs are then obtained arguing as above (i.e., the case $r_e = 0$). For a fixed $\rho > 0$, consider the lower bound for V defined by

$$\tilde{V}^0 = \ell((\rho+1)r_e) \sum_{k\geq 1} Y_k 1(r_e \leq \|X_k\| < (\rho+1)r_e) = \ell((\rho+1)r_e) \sum_{k=1}^{\tilde{N}_\rho} Y_k$$

where $\tilde{N}_\rho$ denotes the number of nodes within the annulus $\{x : r_e \leq \|x\| < (\rho+1)r_e\}$, and has a Poisson distribution with mean $\pi(\rho^2 + 2\rho) r_e^2 \lambda$. Next, define

$$\tilde{U}^j = \sum_{i=1}^{\infty} Y_i 1(\tilde{R}_j \leq \|X_i\| < \tilde{R}_{j+1}), \qquad j \geq 1$$
$$\tilde{W}_k = \ell(r_e)\tilde{U}^1 + \sum_{j=2}^{k} \ell(\tilde{R}_j)\tilde{U}^j, \qquad k \geq 1$$
$$\tilde{W} = \lim_{k\to\infty} \tilde{W}_k$$

where $\tilde{R}_k = \sqrt{k} r_e$, $k \geq 1$. Clearly, the total interference at the origin V is bounded from above by $\tilde{W}$.

We can use this generalization to analyze the effect of a scheduling strategy which ensures that all transmitters within some vicinity of the receiver must remain silent. (This can be thought of as a simplistic model of the 802.11 protocol with request-to-send/clear-to-send (RTS/CTS), with the exclusion zone corresponding to the region within which the CTS can be heard. It is simplistic because this will actually be a random region and, moreover, the ability of a node to hear the CTS will be correlated with the fading of its own signal to the receiver. By assuming that nodes within a fixed radius are silenced, we are ignoring this correlation.) Say this exclusion zone is a circle of radius r_e centered on the receiver. Assuming that no other transmitters are silenced (again, a simplifying assumption, as other transmissions going on in parallel will create their own exclusion zones), this can be modeled by simply considering the attenuation function $\hat{\ell}(\cdot)$ defined above. We can also incorporate the effect of spreading gain in a CDMA system, as described in Section II. Suppose the spreading gain is $1/\phi$, i.e., only a fraction ϕ of the transmitted power interferes with the receiver. If this is combined with the above scheduling strategy, then the attenuation function is effectively $\tilde{\ell} = \phi\hat{\ell}$, and so $\tilde{\ell}_{\max} := \phi\ell(r_e)$ is the quantity that enters into the rate function in place of $R^{-\alpha}$ in Theorem 12.

The above expression gives us some insight into the relative benefits of spreading versus scheduling. Suppose $\ell(x)$ is roughly of the form $x^{-\alpha}$. Then, doubling r_e decreases $\tilde{\ell}_{\max}$ by 2^α, at the cost of silencing four times as many nodes during each transmission period. On the other hand, increasing the spreading gain by 2^α would require a proportionate increase in bandwidth. Since $\alpha > 2$, this suggests that scheduling is more efficient than coding. (In fact, this holds *a fortiori* if our simplifying assumptions are removed. Without those assumptions, more nodes would be silenced, and it would also be nodes with higher channel gain to the receiver that would be more likely to be silenced.) Of course, this is only one aspect of the design; in an ad hoc network, coding may be simpler to implement than scheduling.

The LDPs we have obtained are crude estimates of the probability of the interference exceeding a threshold. Indeed, since LDPs provide the asymptotics of probabilities on the logarithmic scale, the rate functions do not even depend on the intensity of the Poisson process of node locations, whereas the actual probability certainly does. A natural question, therefore, is whether we can get more refined estimates of this exceedance probability. We leave this as a problem for future research. Here we limit ourselves to noting that one approach to estimating the exceedance probability is via fast simulation. Our LDPs can be of help in developing such a scheme as they provide some insight into the required change of measure.

While we have presented LDPs for the interference in the case of general signal power distributions, more precise results are available in the special case of Rayleigh fading; see [11]. The throughput achievable by a network of sensors transmitting to a cluster head is considered in [3]. The authors model the system at packet level using a loss network, and model interference using a Poisson point process of node locations and Rayleigh fading. Our results here could form the basis for studying throughput and other performance measures in sensor networks with more general attenuation functions.

VII. Concluding Remarks

We established a large deviation principle for the total interference in a model of an ad hoc wireless network. We also identified the most likely way in which such large deviations arise.

We modeled node locations using a Poisson point process and considered a number of different models for the signal power distribution. While we considered a Hertzian model for attenuation, the techniques used can be extended to other models as well, and we outlined some such extensions. Our main findings were as follows: if signal powers have superexponentially decaying tails, large values of the interference are due to a combination of a large number of interfering nodes and higher signal power at these nodes, whereas for signal powers with exponential or subexponential tails, large values of the total interference are due to a single interferer with high power.

It remains an open problem to extend the results to node location models other than the Poissonian one. Such models could be motivated, for example, by algorithms for channel allocation that ensure that nearby nodes do not transmit on the same channel. We considered one very simple example of such a model, but it would be of interest to study more realistic examples.

Appendix

Proof of Lemma 6: Note that $\beta \geq 1$ by the convexity of $\log \phi(\,\cdot\,)$ and that β and $\tilde{c}$ are unique. It is implicit in the assumption of the lemma that $\phi(\,\cdot\,)$ is finite everywhere, i.e., that Y_1 has a superexponentially decaying tail. Now consider

$$\theta = \frac{R^\alpha}{\varepsilon}\left(\gamma \log \frac{x}{\varepsilon}\right)^{1/\beta}, \qquad x > 0$$

where $\gamma > 0$ is a constant we will specify later. Note that for any $x > 0$, θ is strictly positive for all ε sufficiently small. For ε and θ as above, we have by (4) and (5) that

$$\begin{aligned}\frac{\varepsilon}{\log^{1/\beta}(1/\varepsilon)} \log P\left(V_\varepsilon^0 \geq x\right) &\leq -\frac{R^\alpha x}{\log^{1/\beta}(1/\varepsilon)}\left(\gamma \log \frac{x}{\varepsilon}\right)^{1/\beta} \\ &\quad + \frac{\varepsilon \lambda_0}{\log^{1/\beta}(1/\varepsilon)}(\phi(\varepsilon R^{-\alpha}\theta) - 1). \end{aligned} \tag{12}$$

Now $\varepsilon R^{-\alpha}\theta = (\gamma \log(x/\varepsilon))^{1/\beta}$ tends to infinity as ε tends to zero. Hence, by the assumption of the lemma, we have for arbitrary $\delta > 0$ that

$$\begin{aligned}\limsup_{\varepsilon\to 0} \frac{\varepsilon}{\log^{1/\beta}(1/\varepsilon)}\phi(\varepsilon R^{-\alpha}\theta) &\leq \limsup_{\varepsilon\to 0} \frac{\varepsilon}{\log^{1/\beta}(1/\varepsilon)} \exp((1+\delta)\tilde{c}(\varepsilon R^{-\alpha}\theta)^\beta) \\ &= \limsup_{\varepsilon\to 0} \frac{\varepsilon}{\log^{1/\beta}(1/\varepsilon)} \exp\left((1+\delta)\tilde{c}\gamma \log \frac{x}{\varepsilon}\right) \\ &= \limsup_{\varepsilon\to 0} \frac{\varepsilon}{\log^{1/\beta}(1/\varepsilon)} \left(\frac{x}{\varepsilon}\right)^{(1+\delta)\tilde{c}\gamma}.\end{aligned}$$

Now take $\gamma = 1/((1+\delta)\tilde{c})$. Then, it follows from the above that

$$\begin{aligned}\limsup_{\varepsilon\to 0} \frac{\varepsilon}{\log^{1/\beta}(1/\varepsilon)}\phi(\varepsilon R^{-\alpha}\theta) &\leq \limsup_{\varepsilon\to 0} \frac{x}{\log^{1/\beta}(1/\varepsilon)} \\ &= 0.\end{aligned} \tag{13}$$

By (12) and (13), we get

$$\limsup_{\varepsilon\to 0} \frac{\varepsilon}{\log^{1/\beta}(1/\varepsilon)} \log P\left(V_\varepsilon^0 \geq x\right) \leq -\gamma^{1/\beta} R^\alpha x$$

where $\gamma = \frac{1}{(1+\delta)\tilde{c}}$. As $\delta > 0$ can be chosen arbitrarily small, this establishes the claim of the lemma. □

Proof of Lemma 7: First, note that for each $\theta < c$ we have $Ee^{\theta Y_1} < \infty$. Indeed, for all $\delta > 0$ small enough, we have $(1-\delta)c > \theta$, and by the assumption $-\log P(Y_1 > y) \sim cy$, it follows that there exists $M > 0$ such that

$$P(Y_1 > y) \leq e^{-(1-\delta)cy}, \qquad \text{for all } y > M.$$

Since $Y_1 \geq 0$, we have

$$Ee^{\theta Y_1} = 1 + \theta \int_0^\infty e^{\theta y} P(Y_1 > y)dy, \qquad \text{for each } \theta \in \mathbb{R}. \tag{14}$$

Then, the finiteness of the Laplace transform for each $\theta < c$ follows by (14) noticing that by the choice of δ it holds $\int_M^\infty e^{-((1-\delta)c-\theta)y}dy < \infty$. By (4), for each $x > 0$ and $\varepsilon, \theta > 0$ such that $0 < \theta < cR^\alpha/\varepsilon$, we have

$$\varepsilon \log P(V_\varepsilon^0 \geq x) \leq -\theta\varepsilon x + \varepsilon \Lambda_0(\varepsilon R^{-\alpha}\theta). \tag{15}$$

Choose $\theta = (c-\delta)R^\alpha/\varepsilon$ and take the $\limsup$ as $\varepsilon \to 0$ in (15). By the finiteness of the Laplace transform and (5), we have

$$\limsup_{\varepsilon\to 0} \varepsilon \log P(V_\varepsilon^0 \geq x) \leq -(c-\delta)R^\alpha x, \qquad x \geq 0.$$

The conclusion follows letting δ tend to 0. □

Proof of Lemma 8: Fix $\delta > 0$. By assumption, there exists $M \geq 0$ such that

$$\exp(-(1-\delta)cy^\gamma) \geq P(Y_1 > y) \geq \exp(-(1+\delta)cy^\gamma)$$

for all $y > M$. Since $Y_1 \geq 0$, we have

$$E[e^{\theta Y_1}] = 1 + \theta \int_0^\infty e^{\theta y} P(Y_1 > y)dy, \qquad \theta \in \mathbb{R}.$$

Hence, for all $\theta \geq 1$

$$\phi(\theta) \geq \psi_-(\theta) := \int_M^\infty e^{\theta y - (1+\delta)cy^\gamma} dy. \tag{16}$$

In order to obtain the logarithmic asymptotics of the above integral, we need a version of Laplace's method that is detailed in Lemma 17. By this lemma, we have

$$\log \psi_-(\theta) \sim (\gamma-1)\gamma^{-1/\eta}((1+\delta)c)^{-1/(\gamma-1)}\theta^{1/\eta}. \tag{17}$$

Similarly, for all $\theta > 0$

$$\phi(\theta) \leq \psi_+(\theta) := e^{\theta M} + \theta \int_M^\infty e^{\theta y - (1-\delta)cy^\gamma} dy. \tag{18}$$

By Lemma 17 in the Appendix, together with the principle of the largest term (see, e.g., [10, Lemma 2.1]), we deduce that

$$\log \psi_+(\theta) \sim (\gamma-1)\gamma^{-1/\eta}((1-\delta)c)^{-1/(\gamma-1)}\theta^{1/\eta}. \quad (19)$$

Since $\delta > 0$ can be chosen arbitrarily small, it follows from (16)–(19) that

$$\log \phi(\theta) \sim (\gamma-1)\gamma^{-1/\eta}c^{-1/(\gamma-1)}\theta^{1/\eta}$$

which is the claim of the lemma. □

We need the following variant of Laplace's method to prove Lemma 8.

Lemma 17: For fixed constants $M \geq 0$, $\alpha > 0$, and $\gamma > 1$, define the function $\psi(\theta) = \int_M^\infty e^{\theta y - \alpha y^\gamma}dy$, $\theta > 0$, and $\eta = 1-(1/\gamma)$. Then

$$\log \psi(\theta) \sim (\gamma-1)\gamma^{-1/\eta}\alpha^{-1/(\gamma-1)}\theta^{1/\eta}.$$

Proof: It is implicit from the assumption that $\psi(\theta)$ is finite everywhere. Consider the function

$$F_\theta(y) = \theta y - \alpha y^\gamma, \qquad y > 0.$$

Note that it is differentiable with a unique maximum attained at $y_0(\theta) = (\frac{\theta}{\alpha\gamma})^{1/(\gamma-1)}$ and

$$F_\theta(y_0(\theta)) = (\gamma-1)\gamma^{-1/\eta}\alpha^{-1/(\gamma-1)}\theta^{1/\eta}.$$

For each $\theta > 0$ large enough and $\varepsilon > 0$, we have

$$\begin{aligned}\int_M^{y_0(\theta)-\varepsilon} & e^{F_\theta(y)}dy \\ &= \int_M^{y_0(\theta)-\varepsilon} \frac{1}{F'_\theta(y)}\, de^{F_\theta(y)} \\ &\leq \left(\sup_{y\in[M,y_0(\theta)-\varepsilon]} \frac{1}{F'_\theta(y)}\right)\left[e^{F_\theta(y_0(\theta)-\varepsilon)} - e^{F_\theta(M)}\right] \\ &\leq \frac{1}{F'_\theta(y_0(\theta)-\varepsilon)}e^{F_\theta(y_0(\theta)-\varepsilon)} \\ &= \frac{1}{\theta - \alpha\gamma(y_0(\theta)-\varepsilon)^{\gamma-1}}e^{\theta(y_0(\theta)-\varepsilon)-\alpha(y_0(\theta)-\varepsilon)^\gamma}.\end{aligned}$$

Choose $\varepsilon = \theta^{-\beta}$ for some $\beta > 1$, and take logarithms in the above inequality. Then

$$\begin{aligned}\limsup_{\theta\to\infty} & \frac{\log \int_M^{y_0(\theta)-\theta^{-\beta}} e^{F_\theta(y)}dy}{(\gamma-1)\gamma^{-1/\eta}\alpha^{-1/(\gamma-1)}\theta^{1/\eta}} \\ &\leq \limsup_{\theta\to\infty} \frac{\theta\left(\left(\frac{\theta}{\alpha\gamma}\right)^{\frac{1}{1-\gamma}} - \theta^{-\beta}\right) - \alpha\left(\left(\frac{\theta}{\alpha\gamma}\right)^{\frac{1}{1-\gamma}} - \theta^{-\beta}\right)^\gamma}{(\gamma-1)\gamma^{-1/\eta}\alpha^{-1/(\gamma-1)}\theta^{1/\eta}} \\ &= 1.\end{aligned} \quad (20)$$

Similarly, for each $\theta > 0$ large enough and $\varepsilon > 0$, we have

$$\begin{aligned}\int_M^{y_0(\theta)-\varepsilon} & e^{F_\theta(y)}dy \\ &= \int_M^{y_0(\theta)-\varepsilon} \frac{1}{F'_\theta(y)}\, de^{F_\theta(y)} \\ &\geq \left(\inf_{y\in[M,y_0(\theta)-\varepsilon]} \frac{1}{F'_\theta(y)}\right)\left[e^{F_\theta(y_0(\theta)-\varepsilon)} - e^{F_\theta(M)}\right] \\ &\geq \frac{1}{\theta - \alpha\gamma M^{\gamma-1}}\left[e^{\theta y_0(\theta)-\alpha y_0(\theta)^\gamma - \varepsilon\theta} - e^{\theta M - \alpha M^\gamma}\right].\end{aligned}$$

Again, choose $\varepsilon = \theta^{-\beta}$, for some $\beta > 1$, and take logarithms in the above inequality. Then

$$\begin{aligned}\liminf_{\theta\to\infty} & \frac{\log \int_M^{y_0(\theta)-\theta^{-\beta}} e^{F_\theta(y)}dy}{(\gamma-1)\gamma^{-1/\eta}\alpha^{-1/(\gamma-1)}\theta^{1/\eta}} \\ &\geq \liminf_{\theta\to\infty} \frac{\log(e^{\theta y_0(\theta)-\alpha y_0(\theta)^\gamma-\theta^{1-\beta}} - e^{\theta M-\alpha M^\gamma})}{(\gamma-1)\gamma^{-1/\eta}\alpha^{-1/(\gamma-1)}\theta^{1/\eta}} \\ &= 1\end{aligned} \quad (21)$$

where the last equality follows from the fact that

$$e^{\theta y_0(\theta)-\alpha y_0(\theta)^\gamma-\theta^{1-\beta}} - e^{\theta M-\alpha M^\gamma} \sim e^{\theta y_0(\theta)-\alpha y_0(\theta)^\gamma-\theta^{1-\beta}}$$

as $\theta \to \infty$. By (20) and (21), we get

$$\log \int_M^{y_0(\theta)-\theta^{-\beta}} e^{F_\theta(y)}dy \sim (\gamma-1)\gamma^{-1/\eta}\alpha^{-1/(\gamma-1)}\theta^{1/\eta}. \quad (22)$$

Similarly, one can prove that

$$\log \int_{y_0(\theta)+\theta^{-\beta}}^\infty e^{F_\theta(y)}dy \sim (\gamma-1)\gamma^{-1/\eta}\alpha^{-1/(\gamma-1)}\theta^{1/\eta}. \quad (23)$$

Finally, for any $\theta, \varepsilon > 0$, we have

$$2\varepsilon e^{\min\{F_\theta(y_0(\theta)-\varepsilon), F_\theta(y_0(\theta)+\varepsilon)\}} \leq \int_{y_0(\theta)-\varepsilon}^{y_0(\theta)+\varepsilon} e^{F_\theta(y)}dy \leq 2\varepsilon e^{F_\theta(y_0(\theta))}.$$

Choose again $\varepsilon = \theta^{-\beta}$, $\beta > 1$, and take the logarithm in the above inequality. Then, it is readily seen that

$$\begin{aligned}\log \int_{y_0(\theta)-\theta^\beta}^{y_0(\theta)+\theta^\beta} e^{F_\theta(y)}dy &\sim F_\theta(y_0(\theta)) \\ &= (\gamma-1)\gamma^{-1/\eta}\alpha^{-1/(\gamma-1)}\theta^{1/\eta}\end{aligned} \quad (24)$$

as $\theta \to \infty$. The conclusion follows by (22)–(24) and the principle of the largest term (see, e.g., [10, Lemma 2.1]). □

Proof of Lemma 9: The proof uses a key result about subexponential distributions [1, Lemma 2.2 p. 259], which states that

$$P\left(\sum_{i=1}^{N_0} Y_i > x\right) \sim E[N_0]P(Y_1 > x)$$

as $x \to \infty$, under the assumption that the Y_i are i.i.d. subexponential and independent of N_0, which has exponential tail. In our case, N_0 is Poisson with mean λ_0, so the assumptions hold. Thus, for all $\delta > 0$ small enough, it holds

$$P\left(V_\varepsilon^0 \geq x\right) \leq P\left(\sum_{i=1}^{N_0} Y_i > R^\alpha (x-\delta)/\varepsilon\right) \sim \lambda_0 P(Y_1 > R^\alpha(x-\delta)/\varepsilon), \qquad \text{as } \varepsilon \to 0.$$

The claim of the lemma follows from the assumption $-\log P(Y_1 > y) \sim cy^\gamma$, by letting δ decrease to zero. □

Proof of Lemma 10: Since Y_1 has compact support with supremum $b > 0$, for arbitrarily small $\delta > 0$, there is a $p > 0$ (depending on δ) such that $P(Y_1 > (1-\delta)b) = p$. Recall that the independent thinning with retention probability p of a Poisson point process with intensity μ is a Poisson point process with intensity $p\mu$ (see, e.g., [4]). Therefore, if we define $\tilde{N}_0 = \sum_{i=1}^{N_0} 1(Y_i > (1-\delta)b)$, then $\tilde{N}_0$ is a Poisson random variable with mean $p\lambda_0$. We now have

$$V_\varepsilon^0 \geq \varepsilon R^{-\alpha}(1-\delta)b \sum_{i=1}^{N_0} 1(Y_i > (1-\delta)b) = \varepsilon R^{-\alpha}(1-\delta)b\tilde{N}_0.$$

Thus

$$P\left(V_\varepsilon^0 > x\right) \geq P\left(\tilde{N}_0 > \frac{R^\alpha x}{\varepsilon(1-\delta)b}\right)$$

from which we deduce that, for $x \geq 0$

$$\liminf_{\varepsilon\to 0} \frac{\varepsilon}{\log(1/\varepsilon)} \log P\left(V_\varepsilon^0 > x\right) \geq \liminf_{\varepsilon\to 0} \frac{\varepsilon}{\log(1/\varepsilon)} \log P\left(\tilde{N}_0 > \frac{R^\alpha x}{\varepsilon(1-\delta)b}\right) = -\frac{R^\alpha x}{(1-\delta)b}. \tag{25}$$

The equality in (25) follows by Lemma 18 (which guarantees that the family of random variables $\{\varepsilon\tilde{N}_0\}$ obeys an LDP on $[0,\infty)$ with speed $\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}$ and good rate function $J(x) = x$). Letting δ decrease to zero, we obtain the claim of the lemma. □

We now provide a large deviation principle for the Poisson distribution, which was used in the proof of Lemma 10 (see also the proof of Lemma 11).

Lemma 18: Let X be a Poisson random variable with mean $\mu > 0$. Then, the family of random variables $\{\varepsilon X\}$ obeys an LDP on $[0,\infty)$ with speed $\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}$ and good rate function $J(x) = x$.

Proof: We will show that

$$-x \leq \liminf_{\varepsilon\to 0} \frac{\varepsilon}{\log(1/\varepsilon)} \log P(\varepsilon X > x) \leq \limsup_{\varepsilon\to 0} \frac{\varepsilon}{\log(1/\varepsilon)} \log P(\varepsilon X \geq x) \leq -x \tag{26}$$

for all $x \geq 0$. Then, the claim follows by extending these bounds to arbitrary open and closed sets by standard techniques (see the proof of Theorem 1). The bounds in (26) are obvious for $x = 0$. For $x > 0$ and $\varepsilon > 0$ small enough, we have

$$P(X \geq [x/\varepsilon] - 1) \leq P(\varepsilon X > x) \leq P(\varepsilon X \geq x) \leq P(X \geq [x/\varepsilon])$$

where $[x]$ denotes the integer part of x, so using the usual upper bound for the tail of the Poisson distribution, we get for any $x > 0$ and $\varepsilon > 0$ small enough

$$\frac{\mu^{[x/\varepsilon]-1}}{([x/\varepsilon]-1)!} e^{-\mu} \leq P(\varepsilon X > x) \leq P(\varepsilon X \geq x) \leq \exp\{-\mu + [x/\varepsilon] - [x/\varepsilon]\log(\mu^{-1}[x/\varepsilon])\}. \tag{27}$$

By Stirling's formula, we have

$$([x/\varepsilon]-1)! \sim \sqrt{2\pi}([x/\varepsilon]-1)^{([x/\varepsilon]-1)+(1/2)} \exp\{-([x/\varepsilon]-1)\}$$

as $\varepsilon \to 0$. Finally, we get (26) by taking the logarithm, multiplying by $\varepsilon/\log(1/\varepsilon)$ and passing to the limit as $\varepsilon \to 0$ in (27). □

Proof of Lemma 11: For arbitrary $n > 0$, we have

$$P\left(V_\varepsilon^0 > x\right) \geq P\left(N_0 \geq n, \min\{Y_1,\ldots,Y_n\} > \frac{R^\alpha x}{\varepsilon n}\right) = P(N_0 \geq n)\left(P\left(Y_1 > \frac{R^\alpha x}{\varepsilon n}\right)\right)^n. \tag{28}$$

Denote by $[z]$ the integer part of $z \in \mathbb{R}$, and take $n = [\kappa/(\varepsilon\log^{1/\gamma}(1/\varepsilon))]$, where the constant $\kappa > 0$ will be specified later. Along similar lines as in the proof of Lemma 18 one can show that the family of random variables $\{\varepsilon\log^{1/\gamma}(1/\varepsilon)N_0\}$ obeys an LDP on $[0,\infty)$ with speed $\frac{1}{\varepsilon\log^{1/\gamma}(1/\varepsilon)}\log(\frac{1}{\varepsilon\log^{1/\gamma}(1/\varepsilon)})$ and good rate function $J(x) = x$. Therefore

$$\log P\left(N_0 \geq \frac{\kappa}{\varepsilon\log^{1/\gamma}(1/\varepsilon)}\right) \sim -\frac{\kappa}{\varepsilon\log^{1/\gamma}(1/\varepsilon)}\log\left(\frac{1}{\varepsilon\log^{1/\gamma}(1/\varepsilon)}\right) \sim -\frac{\kappa}{\varepsilon}\log^\eta(1/\varepsilon)$$

as $\varepsilon \to 0$, and so

$$\lim_{\varepsilon\to 0} \frac{\varepsilon}{\log^\eta(1/\varepsilon)} \log P(N_0 \geq n) = -\kappa. \tag{29}$$

We have suppressed the dependence of n on ε for notational convenience. Next, by the assumption that $-\log P(Y_1 > y) \sim cy^\gamma$, we obtain

$$n\log P\left(Y_1 > \frac{R^\alpha x}{\varepsilon n}\right) \sim -\frac{\kappa c}{\varepsilon\log^{1/\gamma}(1/\varepsilon)}\left(\frac{R^\alpha x\log^{1/\gamma}(1/\varepsilon)}{\kappa}\right)^\gamma = -\frac{c(R^\alpha x)^\gamma}{\kappa^{\gamma-1}}\frac{\log^\eta(1/\varepsilon)}{\varepsilon}$$

and so

$$\lim_{\varepsilon\to 0} \frac{\varepsilon}{\log^\eta(1/\varepsilon)} n\log P\left(Y_1 > \frac{R^\alpha x}{\varepsilon n}\right) = -\frac{c(R^\alpha x)^\gamma}{\kappa^{\gamma-1}}. \tag{30}$$

Substituting (29) and (30) in (28), we get

$$\liminf_{\varepsilon\to 0} \frac{\varepsilon}{\log^\eta(1/\varepsilon)} \log P\left(V_\varepsilon^0 > x\right) \geq -\kappa - \frac{c(R^\alpha x)^\gamma}{\kappa^{\gamma-1}}.$$

The maximum value of the lower bound is attained at $\kappa = ((\gamma-1)c)^{1/\gamma} R^\alpha x$. Substituting this into the right-hand side of the latter inequality, we get

$$\liminf_{\varepsilon\to 0} \frac{\varepsilon}{\log^\eta(1/\varepsilon)} \log P\left(V_\varepsilon^0 > x\right) \geq -\gamma(\gamma-1)^{-\eta} c^{1/\gamma} R^\alpha x$$

as claimed. This completes the proof of the lemma. □

The following variant of the principle of the largest term is used in the proof of Theorem 1.

Lemma 19: Let $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ be two sequences of positive numbers such that $a_n > b_n$ for all $n \geq 1$. Assume that

$$\liminf_{n\to\infty} c_n \log a_n \geq -a \quad \text{and} \quad \limsup_{n\to\infty} c_n \log b_n \leq -b \tag{31}$$

where $\{c_n\}_{n\geq 1}$ is a sequence of positive numbers converging to 0, and $0 < a < b$. Then

$$\liminf_{n\to\infty} c_n \log(a_n - b_n) \geq -a.$$

Proof: Using assumption (31) and the fact that $0 < a < b$, it follows that for all $\varepsilon > 0$ small enough, $b > a + 2\varepsilon$ and there exists $\bar{n} = \bar{n}(\varepsilon)$ such that for all $n \geq \bar{n}$, it holds

$$a_n \geq e^{-(a+\varepsilon)/c_n} \quad \text{and} \quad b_n \leq e^{-(b-\varepsilon)/c_n}.$$

Thus

$$\begin{aligned} a_n - b_n &\geq e^{-(a+\varepsilon)/c_n} - e^{-(b-\varepsilon)/c_n} \\ &= e^{-(a+\varepsilon)/c_n}\left(1 - e^{-(b-a-2\varepsilon)/c_n}\right). \end{aligned}$$

Taking the logarithm, multiplying by c_n, and passing to the limit as $n \to \infty$ in the above inequality, we have

$$\liminf_{n\to\infty} c_n \log(a_n - b_n) \geq -(a+\varepsilon).$$

The claim follows since ε is arbitrary. □

Proof of Lemma 13: For each $\varepsilon > 0$, let $\tilde{Y}_\varepsilon$ be independent of X_ε and have distribution given by

$$P(\tilde{Y}_\varepsilon \geq x) = \exp(-v(\varepsilon)\gamma' x) \qquad \forall\, x \geq 0.$$

Then, $\tilde{Y}_\varepsilon$ stochastically dominates Y_ε and so they can be coupled in such a way that $\tilde{Y}_\varepsilon \geq Y_\varepsilon$ almost surely. Moreover, it is easy to see that $\{\tilde{Y}_\varepsilon\}$ obeys an LDP on $[0,\infty)$ with speed $v(\cdot)$ and rate function $\gamma' x$. Since $\tilde{Y}_\varepsilon$ is independent of X_ε, we have by [10, Th. 4.14] that the family $\{(X_\varepsilon, \tilde{Y}_\varepsilon)\}$ satisfies an LDP on $[0,\infty)^2$ with speed $v(\cdot)$ and rate function $J(x,y) = \gamma x + \gamma' y$. Therefore, by the contraction principle (see, e.g., [5, Th. 4.2.1]) the family $\tilde{Z}_\varepsilon = X_\varepsilon + \tilde{Y}_\varepsilon$ obeys an LDP on $[0,\infty)$ with speed $v(\cdot)$ and good rate function

$$I(z) = \inf_{x+y=z} \gamma x + \gamma' y.$$

Since $\gamma' \geq \gamma$, it holds $I(z) = \gamma z$.

We have thus shown that $\{\tilde{Z}_\varepsilon\}$ obeys an LDP with the same speed and rate function as $\{X_\varepsilon\}$. Moreover, since Y_ε is nonnegative, we have $X_\varepsilon \leq Z_\varepsilon \leq \tilde{Z}_\varepsilon$ for all $\varepsilon > 0$. Hence, we can obtain lower bounds on $P(Z_\varepsilon > x)$ and upper bounds on $P(Z_\varepsilon \geq x)$ from the corresponding bounds on X_ε and $\tilde{Z}_\varepsilon$, respectively. These can be extended to a full LDP as in the proof of Theorem 1. □

Proof of Lemma 14: We will prove the lemma in the case of bounded Y_1. The other two cases are similar. Define $U_\varepsilon^j = \varepsilon U^j$ and $W_\varepsilon^k = \varepsilon W^k$. Since the random variables $U^1, U^2, \ldots$ are i.i.d., and for each j the family $\{U_\varepsilon^j\}$ obeys an LDP on $[0,\infty)$ with speed $\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}$ and good rate function $J(x) = \frac{x}{b}$ (the proof is identical to Theorem 1), then the family $\{(U_\varepsilon^1, \ldots, U_\varepsilon^k)\}$ satisfies an LDP on $[0,\infty)^k$ with speed $\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}$ and good rate function $J(x_1,\ldots,x_k) = (1/b)\sum_{j=1}^k x_j$ (see, e.g., [10, Th. 4.14]). Similarly, by Theorem 1, the family $\{(V_\varepsilon^0, U_\varepsilon^1, \ldots, U_\varepsilon^k)\}$ satisfies an LDP on $[0,\infty)^{k+1}$ with speed $\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}$ and good rate function $I(x_0, x_1, \ldots, x_k) = (R^\alpha x_0 + \sum_{j=1}^k x_j)/b$. Therefore, by the contraction principle (see, e.g., [5, Th. 4.2.1]), it follows that the family $\{W_\varepsilon^k\}$ obeys an LDP on $[0,\infty)$ with speed $\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}$ and good rate function given by

$$\begin{aligned} &\inf\left\{\frac{R^\alpha x_0 + \sum_{j=1}^k x_j}{b} : x_0 + \sum_{j=1}^k R_j^{-\alpha} x_j = x\right\} \\ &= \frac{R^\alpha x}{b} \\ &= I_1(x). \end{aligned}$$

Now define $\tilde{W}^k = \sum_{j=k+1}^\infty R_j^{-\alpha} U^j$. We have

$$\begin{aligned} \log E[e^{\theta \tilde{W}^k}] &= \sum_{j=k+1}^\infty \log E\left[e^{\theta R_j^{-\alpha} U^j}\right] \\ &= \sum_{j=k+1}^\infty \lambda_0\left[\phi(\theta R_j^{-\alpha}) - 1\right]. \end{aligned} \tag{32}$$

Indeed, the $R^{-\alpha}U^j$ are i.i.d. and have the same distribution as V^0. Now, recall that $R_j = \sqrt{j}R$, thus $R_j^{-\alpha} = (\frac{k}{j})^{\alpha/2} R_k^{-\alpha}$. Using the convexity of $\phi(\cdot)$, we obtain that, for all $j \geq k+1$

$$\begin{aligned} \phi\left(\theta R_j^{-\alpha}\right) &\leq \left(\frac{k}{j}\right)^{\alpha/2} \phi(\theta R_k^{-\alpha}) + \left(1 - \left(\frac{k}{j}\right)^{\alpha/2}\right)\phi(0) \\ &= \left(\frac{k}{j}\right)^{\alpha/2} [\phi(\theta R_k^{-\alpha}) - 1] + 1. \end{aligned}$$

Substituting this in (32), we get

$$\log E[e^{\theta \tilde{W}^k}] \leq \lambda_0 \left[\phi(\theta R_k^{-\alpha}) - 1\right] \sum_{j=k+1}^{\infty} \left(\frac{k}{j}\right)^{\alpha/2}$$
$$= c_k \Lambda_0(\theta R_k^{-\alpha})$$

where $c_k = \sum_{j=k+1}^{\infty} (\frac{k}{j})^{\alpha/2}$. Note that this infinite sum is finite by the assumption that $\alpha > 2$. Now, using the Chernoff's bound, we have for all $\theta \geq 0$

$$P(\tilde{W}_\varepsilon^k \geq x) \leq \exp\left(-\theta x + c_k \Lambda_0(\theta R_k^{-\alpha})\right).$$

By similar computations as in the proof of Lemma 6, it follows

$$\limsup_{\varepsilon \to 0} \frac{\varepsilon}{\log(1/\varepsilon)} \log P(\tilde{W}_\varepsilon^k \geq x) \leq -\frac{R_k^\alpha x}{b}.$$

Moreover, $W_\varepsilon = W_\varepsilon^k + \tilde{W}_\varepsilon^k$, and we showed above that $\{W_\varepsilon^k\}$ obeys an LDP on $[0, \infty)$ with speed $\frac{1}{\varepsilon} \log \frac{1}{\varepsilon}$ and rate function $I_1(x) = R^\alpha x / b$. Hence, by Lemma 13, we see that $\{W_\varepsilon\}$ also obeys an LDP with the same speed and rate function, as claimed. □

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