THE UNIVERSALITY CLASSES IN THE PARABOLIC ANDERSON MODEL

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Abstract. We discuss the long time behaviour of the parabolic Anderson model, the Cauchy problem for the heat equation with random potential on $\mathbb{Z}^d$. We consider general i.i.d. potentials and show that exactly *four* qualitatively different types of intermittent behaviour can occur. These four universality classes depend on the upper tail of the potential distribution: (1) tails at ∞ that are thicker than the double-exponential tails, (2) double-exponential tails at ∞ studied by Gärtner and Molchanov, (3) a new class called *almost bounded potentials*, and (4) potentials bounded from above studied by Biskup and König. The new class (3), which contains both unbounded and bounded potentials, is studied in both the annealed and the quenched setting. We show that intermittency occurs on unboundedly increasing islands whose diameter is slowly varying in time. The characteristic variational formulas describing the optimal profiles of the potential and of the solution are solved explicitly by parabolas, respectively, Gaussian densities.

1. Introduction and main results

1.1 The parabolic Anderson model

We consider the continuous solution $v\colon [0,\infty) \times \mathbb{Z}^d \to [0,\infty)$ to the Cauchy problem for the heat equation with random coefficients and localised initial datum,

$$\frac{\partial}{\partial t} v(t,z) = \Delta^{\mathrm{d}} v(t,z) + \xi(z) v(t,z), \qquad \text{for } (t,z) \in (0,\infty) \times \mathbb{Z}^d, \tag{1.1}$$

$$v(0,z) = 1\!\!1_0(z), \qquad \text{for } z \in \mathbb{Z}^d. \tag{1.2}$$

Here $\xi = (\xi(z)\colon z \in \mathbb{Z}^d)$ is an i.i.d. random potential with values in $[-\infty,\infty)$, and Δ^{d} is the discrete Laplacian,

$$\Delta^{\mathrm{d}} f(z) = \sum_{y\sim z} \big[f(y) - f(z)\big], \qquad \text{for } z \in \mathbb{Z}^d,\ f\colon \mathbb{Z}^d \to \mathbb{R}.$$

The parabolic problem (1.1) is called the *parabolic Anderson model*. The operator $\Delta^{\mathrm{d}} + \xi$ appearing on the right is called the *Anderson Hamiltonian*; its spectral properties are well-studied in mathematical physics. Equation (1.1) describes a random mass transport through a random field of sinks and sources, corresponding to lattice points z with $\xi(z) < 0$, respectively, > 0. It is a linearised model for chemical kinetics [GM90], is equivalent to Burger's equation in hydrodynamics [CM94], and describes magnetic phenomena [MR94]. We refer the reader to [GM90], [M94] and [CM94] for more background and to [GK05] for a survey on mathematical results.

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The long-time behaviour of the parabolic Anderson problem is well-studied in the mathematics and mathematical physics literature because it is the prime example of a model exhibiting an *intermittency effect*. This means, loosely speaking, that most of the total mass of the solution,

$$U(t) = \sum_{z \in \mathbb{Z}^d} v(t,z), \qquad \text{for } t > 0, \tag{1.3}$$

is concentrated on a small number of remote islands, called the *intermittent islands*. A manifestation of intermittency in terms of the moments of $U(t)$ is as follows. For $0 < p < q$, the main contribution to the q^{th} moment of $U(t)$ comes from islands that contribute only negligibly to the p^{th} moments. Therefore, intermittency can be defined by the requirement,

$$\limsup_{t\to\infty} \frac{\langle U(t)^p \rangle^{1/p}}{\langle U(t)^q \rangle^{1/q}} = 0, \qquad \text{for } 0 < p < q, \tag{1.4}$$

where $\langle \cdot \rangle$ denotes expectation with respect to ξ. Whenever ξ is truly random, the parabolic Anderson model is intermittent in this sense, see [GM90, Theorem 3.2].

However, one wishes to understand the intermittent behaviour in much greater detail. The following has been heuristically argued in the literature and has been verified, at least partially, for important special examples of potentials: the intermittent islands are characterized by a particularly high exceedance of the potential and an optimal shape, which is determined by a deterministic variational formula. A universal picture is present: the location and number of the intermittent islands are random, their size and the absolute height of the potential in the islands is t-dependent, but the (rescaled) shape depends neither on randomness nor on t. Examples studied include the double-exponential distribution [GM98], potentials bounded from above [BK01] and continuous analogues on $\mathbb{R}^d$ instead of $\mathbb{Z}^d$ like Poisson obstacle fields [S98] and Gaussian and other Poisson fields [GK00, GKM00]. A finer analysis of the geometry of the intermittent islands has been carried out for Poisson obstacle fields [S98] and the double-exponential distribution [GKM05].

In the present paper we initiate the study of the parabolic Anderson model for *arbitrary* potentials, with the aim of identifying *all universality classes* of intermittent behaviour that can arise for different potential distributions. Our standing assumption is that the potentials $(\xi(z) \colon z \in \mathbb{Z}^d)$ are independent and identically distributed and that all positive exponential moments of $\xi(0)$ are finite, which is necessary and sufficient for the finiteness of the p^{th} moments of $U(t)$ at all times. The long-term behaviour of the solutions depends strongly and exclusively on the upper tail behaviour of the random variable $\xi(0)$. It is fully described by the top of the spectrum of the Anderson Hamiltonian $\Delta^{\mathrm{d}} + \xi$ in large t-dependent boxes.

The outline of the remainder of this section is as follows. In Section 1.2, we formulate and discuss a mild regularity condition on the potential. In Section 1.3, we show that under this condition the potentials can be split into exactly *four* classes, which exhibit four different types of intermittent behaviour. Three of these classes have been studied in the literature up to now. A fourth class, the class of *almost bounded potentials*, is studied in the present paper for the first time. We present our results on the moment and almost-sure large-time asymptotics for $U(t)$ in Section 1.4. In Section 1.5, we give a heuristic derivation of the moment asymptotics, and in Section 1.6, we explain the variational problems involved.

1.2 Regularity assumptions

We first state and discuss our regularity assumptions on the potential. Roughly speaking, the purpose of these assumptions is to ensure that the potential has the same qualitative behaviour at different scales, and therefore the system does not belong to different universality classes at different times. Our assumptions refer to the upper tail of $\xi(0)$, and are conveniently formulated in terms of the regularity of its logarithmic moment generating function,

$$H(t) = \log \big\langle e^{t\xi(0)} \big\rangle, \qquad \text{as } t \uparrow \infty. \tag{1.5}$$

Note that H is convex and $t \mapsto H(t)/t$ is increasing with $\lim_{t\to\infty} H(t)/t = \operatorname{esssup} \xi(0)$. To simplify the presentation, we make the assumption that if ξ is bounded from above, then $\operatorname{esssup} \xi(0) = 0$, so that $\lim_{t\to\infty} H(t)/t \in \{0, \infty\}$. This is no loss of generality, as additive constants in the potential appear as additive constants both in $\frac{1}{pt} \log\langle U(t)^p\rangle$ and $\frac{1}{t} \log U(t)$. The first central assumption on H is the following:

Assumption (H). *$t \mapsto \frac{H(t)}{t}$ is in the de Haan class.*

We recall that a measurable function $\widetilde{H}$ is said to be in the *de Haan class* if, for some regularly varying function $g\colon (0,\infty) \to \mathbb{R}$, the term $g(t)^{-1}\big(\widetilde{H}(\lambda t) - \widetilde{H}(t)\big)$ converges to a nonzero limit as $t \uparrow \infty$, for any $\lambda > 1$. Recall that a measurable function g is called *regularly varying* if $g(\lambda t)/g(t)$ converges to a positive limit for every $\lambda > 0$. If this is the case, then the limit takes the form λ^{ϱ}, and ϱ is called the *index of regular variation.* If $\varrho = 0$, then the function is called *slowly varying.*

When $H(t)/t$ is in the de Haan class, then H is regularly varying with some index $\gamma \in \mathbb{R}$. By convexity of H, we have $\gamma \geq 0$. If H is regularly varying with index $\gamma \neq 1$, then $H(t)/t$ is in the de Haan class, so that the statements are equivalent for $\gamma \neq 1$. However, if $\gamma = 1$, then this does not necessarily hold, see [BGT87, Theorem 3.7.4].

From the theory of regular functions we derive the existence of a function $\widehat{H}$ which can be characterized by two parameters, $\gamma \in [0,\infty)$ and $\rho \in (0,\infty)$, and plays an important role in the sequel.

Proposition 1.1. *Assumption (H) is equivalent to the existence of a function $\widehat{H}\colon (0,\infty) \to \mathbb{R}$ and a continuous auxiliary function $\kappa\colon (0,\infty) \to (0,\infty)$ such that*

$$\lim_{t\uparrow\infty} \frac{H(ty) - yH(t)}{\kappa(t)} = \widehat{H}(y) \neq 0, \qquad \textit{for } y \in (0,1) \cup (1,\infty). \tag{1.6}$$

The convergence holds uniformly on every interval $[0, M]$, with $M > 0$. Moreover, with γ the index of variation of H, the following statements hold:

(i) *κ is regularly varying of index $\gamma \geq 0$. In particular, $\kappa(t) = t^{\gamma+o(1)}$ as $t \uparrow \infty$.*
(ii) *There exists a parameter $\rho > 0$ such that, for every $y > 0$,*

  (a) *if $\gamma \neq 1$, then $\widehat{H}(y) = \rho \dfrac{y - y^{\gamma}}{1-\gamma}$, and $\displaystyle\lim_{t\uparrow\infty} \frac{H(t)}{\kappa(t)} = \frac{\rho}{\gamma - 1}$,*

  (b) *if $\gamma = 1$, then $\widehat{H}(y) = \rho y \log y$, and $\displaystyle\lim_{t\uparrow\infty} \frac{|H(t)|}{\kappa(t)} = \infty$.*

Proof. See Chapter 3 in [BGT87]. More accurately, using the notation $f(t) = H(t)/t$ and $g(t) = \kappa(t)/t$, (i) is shown in [BGT87, Section 3.0], see also [BGT87, Theorem 1.4.1]. The uniformity of the convergence follows since the left hand side of (1.6) is convex in y, negative on the interval $(0, 1)$, and continuous in zero.

(ii) follows from [BGT87, Lemma 3.2.1]. The implication stated in (ii)(a) follows from [BGT87, Theorems 3.2.6, 3.2.7], and the implication stated in (ii)(b) is shown in [BGT87, Theorem 3.7.4]. □

Note that κ is an asymptotic scale function, and $\widehat{H}$ an asymptotic shape function for H. While $\gamma \in [0,\infty)$ is unambiguously determined by the potential distribution, the parameter ρ could be absorbed in either κ or $\widehat{H}$. The latter option makes it possible to keep track of ρ in the sequel. If ξ is unbounded from above, then ξ and $\xi + C$ have the same pair of $\widehat{H}$ and κ for any $C \in \mathbb{R}$. If ξ is replaced by $C\xi$ for some $C > 0$, then the pair $(\widehat{H}, \kappa)$ may be replaced by $(C^\gamma \widehat{H}, \kappa)$. In the case $\gamma \neq 1$ one may choose $\kappa(t) = H(t)$ in (1.6), if $\gamma = 1$ one may take $\kappa(t) = H(t) - \int_1^t H(s)/s\, ds$, see [BGT87, Theorem 3.7.3].

The three regimes $0 \le \gamma < 1$, $\gamma = 1$ and $\gamma > 1$ obviously distinguish three qualitatively different classes of (upper tail behaviour of) potentials. However, in order to appropriately describe the asymptotics of the parabolic Anderson model in the case $\gamma = 1$, a finer distinction is necessary. For this we need an additional mild assumption on the auxiliary function κ:

Assumption (K). *The limit* $\kappa^* = \lim_{t\to\infty} \dfrac{\kappa(t)}{t}$ *exists as an element of* $[0, \infty]$.

Assumption (K) is obviously satisfied in the cases $\gamma \neq 1$ and for potentials bounded from above in the case $\gamma = 1$. Indeed, when $\gamma < 1$, then $\kappa^* = 0$, while when $\gamma > 1$, then $\kappa^* = \infty$ by Proposition 1.1(ii)(a). When $\gamma = 1$ and $H(t)/t \to 0$, then, by Proposition 1.1(ii)(b), $H(t)/\kappa(t) \to \infty$, so that $\kappa(t)/t \to 0$. Hence, Assumption (K) can be a restriction only for potentials unbounded from above in the case $\gamma = 1$.

1.3 The universality classes

In this section, we define and discuss the four universality classes of the parabolic Anderson model under the Assumptions (H) and (K). In particular, we explain the relation between the asymptotics of the parabolic Anderson model and the parameters γ and κ^* introduced in Assumptions (H) and (K).

For the moment, we focus on the large time behaviour of the p^{th} moment $\langle U(t)^p\rangle$ for any $p > 0$. We show that there is a scale function $\alpha\colon (0,\infty) \to (0,\infty)$ and a number $\chi \in \mathbb{R}$ such that

$$\frac{1}{pt} \log\langle U(t)^p\rangle = \frac{H\big(pt\,\alpha(pt)^{-d}\big)}{pt\,\alpha(pt)^{-d}} - \frac{1}{\alpha(pt)^2}\big(\chi + o(1)\big), \qquad \text{as } t \uparrow \infty, \tag{1.7}$$

The scale function α describes how fast the expected total mass, which at time $t = 0$ is localised at the origin, spreads, in the sense that

$$\lim_{R\uparrow\infty} \liminf_{t\uparrow\infty} \frac{\alpha(t)^2}{t} \log \frac{\big\langle \sum_{z\in\mathbb{Z}^d} v(t,z)\, \mathbb{1}\{|z| \le R\,\alpha(t)\}\big\rangle}{\big\langle \sum_{z\in\mathbb{Z}^d} v(t,z)\big\rangle} = 0. \tag{1.8}$$

Heuristically, $\alpha(t)$ also determines the size of the intermittent islands for the almost sure behaviour of $U(t)$. The order of their diameter is given as $(\alpha \circ \beta)(t)$, where $\beta(t)$ is the asymptotic inverse of $t \mapsto t/\alpha(t)^2$ evaluated at $d \log t$, cf. Section 1.4.2 below. The numbers χ are naturally given in terms of minimisation problems, where the minimisers correspond to the typical shape of the solution on an intermittent island. A rigorous proof of these heuristic statements, however, is beyond the means of this paper.

One expects that $\alpha(t)$ is asymptotically the larger, the thinner the upper tails of $\xi(0)$ are. It will turn out that when $\kappa^* = \infty$, then (1.8) is satisfied with $\alpha(t) = 1$ independently of R. Therefore, we only need to analyse $\alpha(t)$ in the case when $\kappa^* < \infty$. Analytically, if $\kappa^* < \infty$, then $\alpha(t)$ may be defined by a fixed point equation as follows:

Proposition 1.2 (The scale function α). *Suppose that Assumptions (H) and (K) are satisfied and $\kappa^* < \infty$. There exists a regularly varying scale function $\alpha\colon (0,\infty) \to (0,\infty)$, which is unique up to asymptotic equivalence, such that for all sufficiently large $t > 0$*

$$\frac{\kappa\big(t\alpha(t)^{-d}\big)}{t\alpha(t)^{-d}} = \frac{1}{\alpha(t)^2}. \tag{1.9}$$

The index of regular variation is $\frac{1-\gamma}{d+2-d\gamma}$ *and hence* $\lim_{t\uparrow\infty} \frac{t}{\alpha(t)^d} = \infty$. *Moreover,*

(i) *If* $\gamma = 1$ *and* $0 < \kappa^* < \infty$, *then* $\lim_{t\uparrow\infty} \alpha(t) = 1/\sqrt{\kappa^*} \in (0, \infty)$.
(ii) *If* $\gamma = 1$ *and* $\kappa^* = 0$, *or if* $\gamma < 1$, *then* $\lim_{t\uparrow\infty} \alpha(t) = \infty$.

Proof. To see that α is regularly varying and unique up to asymptotic equivalence we note that $f(t) = t(\kappa(t)/t)^{-d/2}$ is regularly varying with index at least one. By [BGT87, Theorem 1.5.12], there exists an asymptotically unique inverse g such that $f(g(t)) \sim t$ for $t \uparrow \infty$. This inverse is regularly varying. By definition, $t \mapsto t\alpha(t)^{-d}$ satisfies $f(t\alpha(t)^{-d}) = t$ and hence $\alpha(t) \sim (t/g(t))^{1/d}$ is regularly varying. The index of regular variation of α is immediate from the defining equation and the fact that $\kappa(t)$ is regularly varying with index γ.

Under the assumptions of (i), for large t, the mapping $x \mapsto \kappa(tx^{d/2})/tx^{d/2}$ maps a compact interval centred in κ^* to itself, and hence the existence of a solution to (1.9) follows from a fixed-point argument. The stated properties of $\alpha(\,\cdot\,)$ follow immediately from the definition.

Under the assumptions of (ii), we look at the problem of finding $s > 0$ such that $\kappa(s)/s = (s/t)^{2/d}$. For any fixed t, as we increase s the left hand side goes to zero and the right hand side to infinity. Hence for sufficiently large t, there exists a solution $s = s(t)$, which is going to infinity as $t \uparrow \infty$. Then $\alpha(t) = (t/s(t))^{1/d}$ solves (1.9) and converges to infinity. □

Now we introduce the four universality classes, ordered from thick to thin upper tails of $\xi(0)$. Recall the general formula for the asymptotics of the moments $\langle U(t)^p \rangle$ from (1.7).

(1) $\boxed{\gamma > 1, \text{ or } \gamma = 1 \text{ and } \kappa^* = \infty.}$

This case is included in [GM98] as the upper boundary case $\rho = \infty$ in their notation. Here $\chi = 2d$, the scale function $\alpha(t) = 1$ is constant, and the first term on the right hand side in (1.7) dominates the sum, which diverges to infinity. The asymptotics in (1.8) can be strengthened to

$$\lim_{t\uparrow\infty} \frac{1}{t} \log \frac{\langle v(t,0) \rangle}{\left\langle \sum_{z\in\mathbb{Z}^d} v(t,z) \right\rangle} = 0,$$

i.e. the expected total mass remains essentially in the origin and the intermittent islands are single sites, a phenomenon of *complete localisation*. We call this the *single-peak case*. ◇

(2) $\boxed{\gamma = 1 \text{ and } \kappa^* \in (0, \infty).}$

This case, the *double-exponential case*, is the main objective of [GM98]. The prime example is the double exponential distribution with parameter $\rho \in (0, \infty)$,

$$\mathrm{Prob}\big\{\xi(0) > r\big\} = \exp\{-e^{r/\rho}\},$$

which implies $H(t) = \rho t \log(\rho t) - \rho t + o(t)$. Here $\alpha(t) \to 1/\sqrt{\kappa^*} \in (0, \infty)$, so that the size of the intermittent islands is constant in time. The first term on the right hand side in (1.7) dominates the sum, which goes to infinity. Moreover,

$$\chi = \min_{\substack{g\colon \mathbb{Z}^d \to \mathbb{R} \\ \sum g^2 = 1}} \Big\{ \frac{1}{2} \sum_{\substack{x,y\in\mathbb{Z}^d \\ x\sim y}} \big(g(x) - g(y)\big)^2 - \rho \sum_{x\in\mathbb{Z}^d} g^2(x) \log g^2(x) \Big\}, \tag{1.10}$$

where we write $x \sim y$ if x and y are neighbours. This variational problem is difficult to analyse. It has a solution, which is unique for sufficiently large values of ρ, and heuristically this minimizer represents the shape of the solution. As noted in [GH99], for any family of minimizers g_ρ, as $\rho \uparrow \infty$, g_ρ converges to δ_0, which links to the single-peak case. Furthermore, as $\rho \downarrow 0$, the minimisers g_ρ are asymptotically given by

$$g_\rho^2(\lfloor x/\sqrt{\rho}\rfloor) = (1+o(1))\, e^{-|x|^2}\pi^{-d/2},$$

uniformly on compacts and in $L^1(\mathbb{R}^d)$. Consequently,

$$\chi = \rho\, d\Big(1 - \frac{1}{2}\log\frac{\rho}{\pi} + o(1)\Big) \qquad \text{as } \rho \downarrow 0. \qquad \diamond$$

(3) $\boxed{\gamma = 1 \text{ and } \kappa^* = 0.}$

Potentials in this class are called *almost bounded* in [GM98] and may be seen as the degenerate case for $\rho = 0$ in their notation. This class contains both bounded and unbounded potentials, and is analysed for the first time in the present paper. The scale function $\alpha(t)$ and hence the diameter of the intermittent islands goes to infinity and is slowly varying, in particular it is slower than any power of t. The first term on the right hand side in (1.7) dominates the sum, which may go to infinity or zero. Moreover,

$$\chi = \min_{\substack{g\in H^1(\mathbb{R}^d)\\ \|g\|_2=1}} \Big\{ \int_{\mathbb{R}^d} |\nabla g(x)|^2\,dx - \rho\int g^2(x)\log g^2(x)\,dx\Big\}, \tag{1.11}$$

see Theorem 1.4. This variational formula is obviously the continuous variant of (1.10), and it is much easier to solve. There is a unique minimiser, given by

$$g_\rho(x) = \Big(\frac{\rho}{\pi}\Big)^{d/4}\exp\Big(-\frac{\rho}{2}|x|^2\Big),$$

representing the rescaled shape of the solution on an intermittent island. In particular, $\chi = \rho d\big(1-\frac{1}{2}\log\frac{\rho}{\pi}\big)$, which is the asymptotics of (1.10) as $\rho \downarrow 0$. Hence, on the level of variational problems, (3) is the boundary case of (2) for $\rho \downarrow 0$. $\diamond$

(4) $\boxed{\gamma < 1.}$

This is the case of *potentials bounded from above*, which is treated in [BK01]. Indeed, in [BK01], it is assumed that there exists a non-decreasing function $\alpha(t)$ and a nonpositive function $\widetilde{H}\colon (0,\infty) \to (-\infty,0]$ such that

$$\lim_{t\uparrow\infty} \tfrac{\alpha(t)^{d+2}}{t} H\big(\tfrac{t}{\alpha(t)^d}\, y\big) = \widetilde{H}(y),$$

uniformly on compact sets in $(0,\infty)$. It is easy to infer from the results of Section 1.2 above that this assumption holds if Assumption (H) holds for the index $\gamma < 1$, for α defined by (1.9) and

$$\widetilde{H}(y) = \frac{\rho}{\gamma-1}\, y^\gamma.$$

Here $\alpha(t) \to \infty$ as $t \mapsto \alpha(t)$ is regularly varying with index $\frac{1-\gamma}{d+2-d\gamma}$. The potential ξ is necessarily bounded from above. In this case, the two terms on the right hand side in (1.7) are of the *same order*, and (1.7) converges to zero. Moreover,

$$\chi = \inf_{\substack{g\in H^1(\mathbb{R}^d)\\ \|g\|_2=1}} \Big\{ \int_{\mathbb{R}^d} |\nabla g(x)|^2\,dx - \rho\int_{\mathbb{R}^d} \frac{g^{2\gamma}(x)-g^2(x)}{\gamma-1}\,dx\Big\}. \tag{1.12}$$

In the lower boundary case where $\gamma = 0$, the functional $\int g^{2\gamma}$ must be replaced by the Lebesgue measure of $\mathrm{supp}\,(g)$. In this case the formula is well-known and well-understood. In particular, the minimizer exists, is unique up to spatial shifts, and has compact support. To the best of our knowledge, for $\gamma \in (0,1)$, the formula in (1.12) has not been analysed explicitly, unless in $d = 1$. In Proposition 1.15 below, we show that (1.12) converges to (1.11), as would follow from interchanging the limit $\gamma \uparrow 1$ with the infimum on g. This means that, on the level of variational formulas, (3) is the boundary case of (4) for $\gamma \uparrow 1$.

Remark 1.3. The variational problems in (1.10), (1.11), and (1.12) encode the asymptotic shape of the rescaled and normalised solution $v(t,\cdot)$ in the centred ball with radius of order $\alpha(t)$. Informally, the main contribution to $\langle U(t)\rangle$ comes from the events that

$$\frac{v\big(t, \lceil \cdot\, \alpha(t)\rceil\big)}{\big\|v\big(t, \lceil \cdot\, \alpha(t)\rceil\big)\big\|_2} \approx g,$$

where g is a minimiser in the definition of χ. To the best of our knowledge this heuristics has not been made rigorous in any nontrivial case so far. Note that in case (1), formally, (1.10) holds with $\rho = \infty$ and hence the optimal g is $\mathbb{1}_0$. $\diamond$

Since the cases (1), (2) and and (4) have been studied in the literature [BK01, GM98], the possible scaling picture of the parabolic Anderson model under the Assumptions (H) and (K) is complete once the case (3) is resolved. This is the content of the remainder of this paper.

1.4 Long time tails in the almost bounded case

In this section we present our results on the almost bounded case (3). In other words, we assume that $\kappa(t)/t$ is slowly varying and converges to zero.

1.4.1. *Moment asymptotics.* Our main result on the annealed asymptotics of $U(t)$ gives the first two terms in the asymptotics of $\langle U(t)^p\rangle$ for any $p > 0$, as $t \uparrow \infty$.

Theorem 1.4 (Moment asymptotics). *Suppose Assumptions (H) and (K) hold, and assume that we are in case* (3)*, i.e.,* $\gamma = 1$ *and* $\kappa^* = 0$*. Let* $\rho > 0$ *be as in Proposition 1.1(ii)(b). Then, for any* $p \in (0,\infty)$,

$$\frac{1}{pt}\log\langle U(t)^p\rangle = \frac{H\big(pt\,\alpha(pt)^{-d}\big)}{pt\,\alpha(pt)^{-d}} - \frac{1}{\alpha(pt)^2}\big(\rho d(1-\tfrac12\log\tfrac{\rho}{\pi}) + o(1)\big), \qquad \text{as } t\uparrow\infty. \tag{1.13}$$

Remark 1.5 (The constant). Recall from (1.7) and (1.11) that the constant $\rho d(1-\frac12\log\frac{\rho}{\pi})$ arises as a variational problem; see Section 1.6. The variational problem plays an essential role in the proof. $\diamond$

Remark 1.6 (Intermittency). Note from (1.9) that the first term in (1.13) is of higher order than the second term. Formula (1.13), together with the results of Proposition 1.1 and the fact that $\alpha(\cdot)$ is slowly varying, imply that

$$\begin{aligned}\log\frac{\langle U(t)^p\rangle^{1/p}}{\langle U(t)^q\rangle^{1/q}} &= \frac{H\big(pt\,\alpha(pt)^{-d}\big)}{p\,\alpha(pt)^{-d}} - \frac{H\big(qt\,\alpha(qt)^{-d}\big)}{q\,\alpha(qt)^{-d}} + o\big(t/\alpha(t)^2\big)\\ &= \frac{t}{\alpha(t)^2}\Big(\tfrac{q}{p}\hat H\big(\tfrac{p}{q}\big) + o(1)\Big) \qquad \text{for } p,q\in(0,\infty).\end{aligned} \tag{1.14}$$

In particular, we have intermittency in the sense of (1.4), and the convergence is exponential on the scale $t/\alpha(t)^2$. $\diamond$

In spite of the simplicity of the variational formula (1.11), the derivation of (1.13) is technically rather involved and requires a number of demanding tools. We use both representations of $U(t)$ available to us: an approximative representation in terms of an eigenfunction expansion, and the Feynman-Kac formula involving simple random walk. The heart of the proof is an application of a large deviation principle for the rescaled local times of simple random walk. However, there are three major obstacles to be removed, which require a variety of novel techniques. The first one is a compactification argument for the space, which is based on an estimate for Dirichlet eigenvalues in large boxes against maximal Dirichlet eigenvalues in small subboxes. This is an adaptation of a method from [BK01]. The second technique is a cutting argument for the large potential values, which we trace back to a large deviations estimate for the self-intersection number of the simple random walk. This is of independent interest and is carried out in Section 2. Finally, the third obstacle, which appears in the proof of the *upper* bound, is the lack of upper semi-continuity of the map $f \mapsto \int f(x) \log f(x)\,dx$ in the topology of the large deviation principle, even after compactification and removal of large values. Therefore, in the proof of the upper bound we replace the classical large deviation principle by a new approach, taken from [BHK05], which identifies and estimates the joint density of the family of the random walk local times. See Proposition 3.3 below.

An alternative heuristic derivation of formula (1.16) is given in Section 1.5. The proof of Theorem 1.4 is given in Sections 2 and 3.

1.4.2. *Almost-sure asymptotics.* We define another scale function β such that

$$\frac{\beta(t)}{\alpha\big(\beta(t)\big)^2} \sim d\,\log t\,. \tag{1.15}$$

In other words, $\beta(t)$ is the asymptotic inverse of $t \mapsto t/\alpha(t)^2$ evaluated at $d\,\log t$, which by [BGT87, Theorem 1.15.12] exists and is slowly varying. In order to avoid technical inconveniences, we assume that the field ξ is bounded from *below*. See Remark 1.10 for comments on this issue.

Theorem 1.7 (Almost sure asymptotics)**.** *Suppose Assumptions (H) and (K) hold, and assume that we are in case (3), i.e.,* $\gamma = 1$ *and* $\kappa^* = 0$. *Furthermore, suppose that* β *is defined by* (1.15) *and that* $\operatorname{essinf}\xi(0) > -\infty$. *Let* $\rho > 0$ *be as in Proposition 1.1. Then, almost surely,*

$$\frac{1}{t}\log U(t) = \frac{H\big(\beta(t)\alpha(\beta(t))^{-d}\big)}{\beta(t)\,\alpha(\beta(t))^{-d}} - \frac{1}{\alpha(\beta(t))^2}\,\big(\rho(d - \tfrac{d}{2}\log\tfrac{\rho}{\pi} + \log\tfrac{\rho}{e}) + o(1)\big), \qquad \textit{as } t \uparrow \infty. \tag{1.16}$$

Remark 1.8 (The constant)**.** In Section 1.6, we will see that also the constant $\rho(d - \frac{d}{2}\log\frac{\rho}{\pi} + \log\frac{\rho}{e})$ arises as a variational problem. A remarkable fact is that the first two leading contributions to $U(t)$ are *deterministic.* $\diamond$

Remark 1.9 (Interpretation)**.** Heuristically, $\alpha(\beta(t))$ is the order of the diameter of the intermittent islands, which almost surely carry most of the mass of $U(t)$. Note that $\beta(t) = (\log t)^{1+o(1)}$ and $\alpha(\beta(t)) = (\log t)^{o(1)}$, i.e., the size of the intermittent islands increases extremely slowly. The crucial point in the proof of Theorem 1.7 is to show the existence of an island with radius of order $\alpha(\beta(t))$ within the box $[-t,t]^d$ on which the shape of the vertically shifted and rescaled potential is optimal, i.e., resembles a certain parabola. To prove this, we use the first moment asymptotics at time $\beta(t)$ locally on that island. The exponential rate, which is $\beta(t)/\alpha(\beta(t))^2$ has to be balanced against the *number* of possible islands, which has exponential rate $d\log t$, cf. (1.15). $\diamond$

Remark 1.10 (Lower tails of the potential)**.** The assertion of Theorem 1.7 remains true *mutatis mutandis* if the assumption $\operatorname{essinf}\xi(0) > -\infty$ is replaced, in $d \geq 2$, by the assumption that $\mathrm{Prob}\{\xi(0) > -\infty\}$ exceeds the critical nearest-neighbour site percolation threshold. This ensures the existence of an infinite component in the set $\mathcal{C} = \{z \in \mathbb{Z}^d \colon \xi(z) > -\infty\}$, and thus (1.16) holds conditional on the event that the origin belongs to the infinite cluster in $\mathcal{C}$. In $d = 1$, an infinite cluster exists if and only if $\mathrm{Prob}\{\xi(0) > -\infty\} = 1$. If we assume that $\xi(0) > -\infty$ almost surely and $\langle \log(-\xi(0) \vee 1)\rangle < \infty$, (1.16) is true verbatim, while otherwise the rate of the almost sure asymptotics depends on the lower tails of $\xi(0)$; see [BK01a] for details. The effect of the assumption is to ensure sufficient connectivity in the sense that the mass flow from the origin to regions where the random potential assumes high values and an approximately optimal shape is not hampered by deep valleys on the way.

We decided to detail the proof of the almost sure asymptotics under the stronger assertion that $\operatorname{essinf} \xi(0) > -\infty$. See [BK01, Section 5.2] for the proof of the analogous assertion in the bounded-potential case under the weaker assumptions. The arguments given there can be extended with some effort to the situation of the present paper. ◇

The proof of Theorem 1.7 is given in Section 4. It essentially follows the strategy of [BK01].

1.4.3. *Examples.* We now explain what kind of upper tail behaviour is covered by the almost bounded case, arguing separately for the bounded and unbounded case, denoted by (B) and (U), respectively. Suppose the distribution of the field $\xi(0)$ satisfies

$$\log \mathrm{Prob}\big\{\xi(0) > r\big\} \sim -e^{f(r)}, \qquad \text{as } \begin{cases} r \uparrow \infty & \text{in case (U)}, \\ r \uparrow 0 = \operatorname{esssup} \xi(0), & \text{in case (B)}. \end{cases} \tag{1.17}$$

Here f is a positive, strictly increasing smooth function satisfying $f'(r) \uparrow \infty$ as $r \uparrow \infty$ in case (U) and $f'(r)r \uparrow \infty$ as $r \uparrow 0$ in case (B). Note that typical representatives of case (2) of the four universality classes are $f(r) \approx cr$ as $r \uparrow \infty$, violating the condition in case (U); and typical representatives of case (4) of the four universality classes are $f(r) \approx -\frac{\gamma}{1-\gamma} \log |r|$ as $r \uparrow 0$, violating the condition in case (B). The cumulant generating function behaves like

$$H(t) \approx \log \int e^{tr} \exp\big\{-e^{f(r)}\big\}\, dr \approx \sup_r \big[tr - e^{f(r)}\big] = tr(t) - e^{f(r(t))}, \tag{1.18}$$

where $r(t)$ is asymptotically, as $t \uparrow \infty$, defined via $t = f'(r(t))e^{f(r(t))}$. Note that $r(t) \uparrow \infty$ in case (U), while $r(t) \uparrow 0$ in case (B), as $t \uparrow \infty$. Hence, $f'(r(t)) \uparrow \infty$ in case (U), while $f'(r(t))r(t) \uparrow \infty$ in case (B). Rewriting the definition of $r(t)$ as

$$e^{f(r(t))} = \frac{tr(t)}{f'(r(t))r(t)} = o(tr(t)),$$

we thus obtain that the first term on the right hand side of (1.18) dominates the second term. Therefore, we can approximate $H(t)/t \approx r(t)$, as $t \uparrow \infty$. We next assume that $f'(r(\cdot))$ is slowly varying at infinity. We then see that, using the fact that $r(t) = f^{-1}\big(\log \frac{t}{f'(r(t))}\big)$ in the last equality,

$$\begin{aligned} H(ty) - yH(t) &\approx ty\Big(f^{-1}\big(\log \tfrac{ty}{f'(r(ty))}\big) - f^{-1}\big(\log \tfrac{t}{f'(r(t))}\big)\Big) \\ &\approx ty\Big(f^{-1}\big(\log \tfrac{t}{f'(r(t))} + \log y\big) - f^{-1}\big(\log \tfrac{t}{f'(r(t))}\big)\Big) \\ &\approx t\,(y \log y)\,(f^{-1})'\big(\log \tfrac{t}{f'(r(t))}\big) = (y \log y)\,\frac{t}{f'(r(t))}. \end{aligned}$$

Using Proposition 1.1, this means that the scaling relation in (1.6) is satisfied with $\kappa(t) = t/f'(r(t))$ and $\rho = 1$. As $f'(r(t)) \uparrow \infty$ is slowly varying, we see that we are in case (3) of the four universality classes.

1.5 Heuristic derivation of Theorem 1.4

In this section, we give a heuristic explanation of Theorem 1.4 in terms of large deviations for the scaled potential ξ. Our proof of Theorem 1.4 follows a different strategy.

We use the setup and notation of Section 1.4.3 and handle the cases (B) respectively (U) simultaneously. Consequently, the definition (1.9) of $\alpha(t)$ reads

$$\alpha(t)^2 = \frac{t\alpha(t)^{-d}}{\kappa(t\alpha(t)^{-d})} = f'\big(r(t\alpha(t)^{-d})\big). \tag{1.19}$$

We introduce the shifted, scaled potential

$$\begin{aligned}\overline{\xi}_t(x) &:= \alpha(t)^2\Big[\xi\big(\lfloor x\alpha(t)\rfloor\big) - \frac{H(t\alpha(t)^{-d})}{t\alpha(t)^{-d}}\Big]\\ &\approx \alpha(t)^2\Big[\xi\big(\lfloor x\alpha(t)\rfloor\big) - r(t\alpha(t)^{-d}) + \tfrac{\alpha(t)^d}{t}e^{f(r(t\alpha(t)^{-d}))}\Big],\end{aligned} \tag{1.20}$$

for $x\in Q_R=[-R,R]^d$. The process $\overline{\xi}_t$ satisfies a large deviation principle, for every $R>0$, on the cube Q_R with rate $t\alpha(t)^{-2}$ and rate function $\varphi\mapsto\int_{Q_R}e^{\varphi(x)-1}\,dx$. Indeed, with $B_R=[-R,R]^d\cap\mathbb{Z}^d$,

$$\begin{aligned}\mathrm{Prob}\big\{\overline{\xi}_t\approx\varphi\text{ on }Q_R\big\} &\approx \prod_{z\in B_{R\alpha(t)}}\mathrm{Prob}\Big\{\xi(0)\approx\tfrac{\varphi(z\alpha(t)^{-1})}{\alpha(t)^2}+r(t\alpha(t)^{-d})-\tfrac{\alpha(t)^d}{t}e^{f(r(t\alpha(t)^{-d}))}\Big\}\\ &\approx \prod_{z\in B_{R\alpha(t)}}\exp\Big\{-\exp\Big[f\Big(r(t\alpha(t)^{-d})+\tfrac{\varphi(z\alpha(t)^{-1})}{\alpha(t)^2}-\tfrac{\alpha(t)^d}{t}e^{f(r(t\alpha(t)^{-d}))}\Big)\Big]\Big\}\end{aligned}$$

By a Taylor expansion around $r(t\alpha(t)^{-d})$, using that $s=f'(r(s))e^{f(r(s))}$ for $s=t\alpha(t)^{-d}$ as well as (1.19), we can continue with

$$\begin{aligned}\mathrm{Prob}\big\{\overline{\xi}_t\approx\varphi\text{ on }Q_R\big\} &\approx \exp\Big\{-\alpha(t)^d\int_{Q_R}\exp\big[f(r(t\alpha(t)^{-d}))+\frac{\varphi(x)}{\alpha(t)^2}f'(r(t\alpha(t)^{-d}))-1\big]\,dx\Big\}\\ &=\exp\Big\{-\frac{t}{f'(r(t\alpha(t)^{-d}))}\int_{Q_R}e^{\varphi(x)-1}\,dx\Big\}\\ &\approx\exp\Big\{-\frac{t}{\alpha(t)^2}\int_{Q_R}e^{\varphi(x)-1}\,dx\Big\}.\end{aligned}$$

The asymptotics of $\langle U(t)^p\rangle$ can now be explained as follows. Note that $U(t)=u(t,0)$, where $u(t,\cdot)$ is the solution of the parabolic Anderson model (1.1) with initial condition $u(0,\cdot)=1$. We can approximate $u(t,0)$ by the $w_t(t,0)$ where $(s,z)\mapsto w_t(s,z)$ is the solution to the initial boundary value problem (1.1) with zero boundary condition outside the box B_t and initial condition $w_t(0,\cdot)=\mathbb{1}_{B_t}$. Let $\lambda_t^{\mathrm{d}}(\xi)$ denote the principal eigenvalue of $\Delta^{\mathrm{d}}+\xi$ in $\ell^2(B_t)$ with zero boundary condition. Then an eigenfunction expansion shows that

$$U(t)^p = u(t,0)^p\approx w_t(t,0)^p\approx e^{pt\lambda_t^{\mathrm{d}}(\xi)}.$$

This already explains why the asymptotics of the p^{th} moments of $U(t)$ are the same as the asymptotics of the moments of $U(pt)$. We proceed by taking $p=1$. Now the shift invariance and the asymptotic scaling properties of the discrete Laplace operator yield that

$$\lambda_t^{\mathrm{d}}(\xi)=\tfrac{H(t\alpha(t)^{-d})}{t\alpha(t)^{-d}}+\lambda_t^{\mathrm{d}}\big(\alpha(t)^{-2}\overline{\xi}_t(\lfloor\cdot\,\alpha(t)^{-1}\rfloor)\big)\approx\tfrac{H(t\alpha(t)^{-d})}{t\alpha(t)^{-d}}+\alpha(t)^{-2}\lambda(\overline{\xi}_t),$$

where $\lambda(\psi)$ denotes the principal eigenvalue of $\Delta+\psi$ in $L^2(Q_{t\alpha(t)^{-d}})$, with zero boundary condition. Hence,

$$\langle U(t)\rangle\approx e^{H(t\alpha(t)^{-d})\alpha(t)^d}\Big\langle\exp\Big\{\frac{t}{\alpha(t)^2}\lambda(\overline{\xi}_t)\Big\}\Big\rangle. \tag{1.21}$$

Using the large deviation principle for $\overline{\xi}_t$ with $R = t\alpha(t)^{-d}$, and anticipating that $\psi \mapsto \lambda(\psi)$ has the appropriate continuity and boundedness properties, we may use Varadhan's lemma to deduce that

$$\frac{1}{t}\log\langle U(t)\rangle \approx \tfrac{H(t\alpha(t)^{-d})}{t\alpha(t)^{-d}} - \frac{1}{\alpha(t)^2}\chi,$$

where χ is given by

$$\chi = \inf_{\psi}\Big\{\int_{\mathbb{R}^d} e^{\psi(x)-1}\,dx - \lambda(\psi)\Big\}. \tag{1.22}$$

We show in Section 1.6 that χ is equal to $\rho d(1-\frac{1}{2}\log\frac{\rho}{\pi})$. This completes the heuristic derivation of Theorem 1.4. The interpretation of the above heuristics is that the moments of the total mass $U(t)$ are mainly governed by potentials ξ whose shape is approximately given as

$$\xi(\cdot) \approx \tfrac{H(t\alpha(t)^{-d})}{t\alpha(t)^{-d}} + \alpha(t)^{-2}\psi\big(\cdot\,\alpha(t)^{-1}\big)$$

where ψ is a minimiser of the formula in (1.22).

1.6 Variational representations of the constants in Theorem 1.4 and 1.7

1.6.1. *The constant in Theorem 1.4.* Fix $\rho > 0$ and define $\chi(\rho) \in \mathbb{R}$ by

$$\chi(\rho) = \inf_{\substack{g\in H^1(\mathbb{R}^d)\\ \|g\|_2=1}}\Big\{\|\nabla g\|_2^2 - \mathcal{H}(g^2)\Big\}, \tag{1.23}$$

where $H^1(\mathbb{R}^d)$ is the usual Sobolev space, ∇ the usual (distributional) gradient, and

$$\mathcal{H}(g^2) = \rho\int_{\mathbb{R}^d} g^2(x)\log g^2(x)\,dx. \tag{1.24}$$

By the logarithmic Sobolev inequality in (1.29) below, $\mathcal{H}(g^2) \in [-\infty,\infty)$ is well-defined for $g \in H^1(\mathbb{R}^d)$. Furthermore, we introduce the Legendre transform of $\mathcal{H}$ on $L^2(\mathbb{R}^d)$ and the top of the spectrum of the operator $\Delta + \psi$ in $H^1(\mathbb{R}^d)$,

$$\mathcal{L}(\psi) = \sup_{g\in L^2(\mathbb{R}^d)}\big\{\langle g^2,\psi\rangle - \mathcal{H}(g^2)\big\} \qquad\text{and}\qquad \lambda(\psi) = \sup_{\substack{g\in H^1(\mathbb{R}^d)\\ \|g\|_2=1}}\big\{\langle \psi, g^2\rangle - \|\nabla g\|_2^2\big\}. \tag{1.25}$$

Introduce the functions

$$g_\rho(x) = \Big(\frac{\rho}{\pi}\Big)^{\frac{d}{4}} e^{-\frac{\rho}{2}|x|^2} \qquad\text{and}\qquad \psi_\rho(x) = \rho + \rho\frac{d}{2}\log\frac{\rho}{\pi} - \rho^2|x|^2, \qquad \text{for } x \in \mathbb{R}^d. \tag{1.26}$$

Note that the Gaussian density g_ρ is the unique L^2-normalized positive eigenfunction of the operator $\Delta + \psi_\rho$ in $H^1(\mathbb{R}^d)$ with eigenvalue $\lambda(\psi_\rho) = \rho - \rho d + \rho\frac{d}{2}\log\frac{\rho}{\pi}$. It satisfies $\mathcal{L}(\psi_\rho) = \rho$.

Proposition 1.11 (Solution of the variational formula in (1.23))**.** *For any $\rho \in (0,\infty)$, the infimum in* (1.23) *is, up to horizontal shift, uniquely attained at g_ρ. In particular, $\chi(\rho) = \rho d\big(1-\frac{1}{2}\log\frac{\rho}{\pi}\big)$ is the constant appearing in Theorem 1.4. Moreover, $\mathcal{L}$ is identified as*

$$\mathcal{L}(\psi) = \frac{\rho}{e}\int_{\mathbb{R}^d} e^{\frac{1}{\rho}\psi(x)}\,dx, \tag{1.27}$$

and the 'dual' representation is

$$\chi(\rho) = \inf_{\psi\in\mathcal{C}(\mathbb{R}^d)}\big\{\mathcal{L}(\psi) - \lambda(\psi)\big\}, \tag{1.28}$$

where $\mathcal{C}(\mathbb{R}^d)$ is the set of continuous functions $\mathbb{R}^d \to \mathbb{R}$. Up to horizontal shift, the infimum in (1.28) *is uniquely attained at the parabola ψ_ρ in* (1.26).

Proof. By the *logarithmic Sobolev inequality* in the form of [LL01, Th. 8.14] with $a = \sqrt{\pi/\rho}$, we have

$$\|\nabla g\|_2^2 \geq \rho \int_{\mathbb{R}^d} g^2(x) \log g^2(x)\, dx + \rho d\big(1 - \tfrac{1}{2}\log \tfrac{\rho}{\pi}\big), \tag{1.29}$$

with equality exactly for the Gaussian density g_ρ and its horizontal shifts. This proves the first statement. In order to see that (1.27) holds, use Jensen's inequality for any $g \in L^2(\mathbb{R}^d)$ to obtain

$$\langle g^2, \psi\rangle - \mathcal{H}(g^2) = \rho\|g\|_2^2 \int \frac{g^2}{\|g\|_2^2} \log \frac{e^{\frac{1}{\rho}\psi}}{g^2} \leq \rho \|g\|_2^2 \log \frac{\int e^{\frac{1}{\rho}\psi}}{\|g\|_2^2}. \tag{1.30}$$

Equality holds if and only if $g^2 = Ce^{\frac{1}{\rho}\psi}$ for some $C > 0$. The right side of (1.30) is maximal precisely for $\|g\|_2^2 = \frac{1}{e}\int e^{\frac{1}{\rho}\psi}$. Substituting this value, we arrive at (1.27).

To see the last two statements, we use (1.27) and the formula in (1.25) for $\lambda(\psi)$ to obtain, for any $\psi \in \mathcal{C}(\mathbb{R}^d)$,

$$\mathcal{L}(\psi) - \lambda(\psi) = \inf_{\substack{g\in H^1(\mathbb{R}^d)\\ \|g\|_2=1}} \Big(\|\nabla g\|_2^2 - \mathcal{H}(g^2) - \rho \int g^2 \Big[\frac{\psi}{\rho} - \log g^2 - e^{\frac{1}{\rho}\psi - \log g^2 - 1}\Big]\Big). \tag{1.31}$$

The term in square brackets is equal to $\theta - e^{\theta-1}$ for $\theta = \frac{\psi}{\rho} - \log g^2$. Since this is nonpositive and is zero only for $\theta = 1$, we have that '$\leq$' holds in (1.28). Furthermore, by restricting the infimum over g to strictly positive continuous functions and interchanging the order of the infima, we see that

$$\begin{aligned}
\inf_{\psi\in\mathcal{C}(\mathbb{R}^d)} \big\{\mathcal{L}(\psi) - \lambda(\psi)\big\} &\leq \inf_{\substack{g\in H^1(\mathbb{R}^d)\\ \|g\|_2=1, g>0}} \inf_{\psi\in\mathcal{C}(\mathbb{R}^d)} \Big(\|\nabla g\|_2^2 - \mathcal{H}(g^2) - \rho\int g^2\Big[\frac{\psi}{\rho} - \log g^2 - e^{\frac{1}{\rho}\psi - \log g^2 - 1}\Big]\Big)\\
&\leq \inf_{\substack{g\in H^1(\mathbb{R}^d)\\ \|g\|_2=1, g>0}} \|\nabla g\|_2^2 - \mathcal{H}(g^2) = \chi(\rho),
\end{aligned}$$

by substituting $\psi = \rho + \rho \log g^2$, and we use that the maximizer g of the right hand side is strictly positive. Therefore, equality holds in (1.28). We also know that, by uniqueness of the solution in (1.23), the unique minimizer in (1.28) is $\psi = \rho + \rho\log g_\rho^2 = \psi_\rho$. □

Remark 1.12 (Interpretation)**.** Both representations (1.23) and (1.28) may be interpreted in terms of optimal rescaled profiles for the moment asymptotics of the total mass $U(t)$. While the minimizer ψ_ρ in (1.28) describes the shape of the potential ξ (see Section 1.5), the minimizer g_ρ in (1.23) describes the solution $u(t,\cdot)$, cf. Remark 1.3. ◇

1.6.2. *The constant in Theorem 1.7.* We now turn to the variational representation of the constant appearing in Theorem 1.7. We define $\widetilde{\chi}(\rho)$ by

$$\widetilde{\chi}(\rho) = \inf\{-\lambda(\psi) \colon \psi \in \mathcal{C}(\mathbb{R}^d), \mathcal{L}(\psi) \leq 1\}. \tag{1.32}$$

where we recall that $\mathcal{C}(\mathbb{R}^d)$ is the set of continuous functions $\mathbb{R}^d \to \mathbb{R}$.

Proposition 1.13 (Solution of the variational formula in (1.32))**.** *For any $\rho \in (0,\infty)$, the function $\psi_\rho - \rho\log\frac{\rho}{e}$, with ψ_ρ as defined in* (1.26)*, is the unique minimizer in* (1.32)*, and $\widetilde{\chi}(\rho) = \chi(\rho) + \rho\log\frac{\rho}{e}$.*

Proof. Obviously, the condition $\mathcal{L}(\psi) \leq 1$ in (1.32) may be replaced by $\mathcal{L}(\psi) = 1$. In the representation

$$\widetilde{\chi}(\rho) = \inf\Big\{\rho\log\mathcal{L}(\psi) - \lambda(\psi) \colon \psi \in \mathcal{C}(\mathbb{R}^d), \mathcal{L}(\psi) = 1\Big\}$$

we may omit the condition $\mathcal{L}(\psi) = 1$ completely since $\rho \log \mathcal{L}(\psi) - \lambda(\psi)$ is invariant under adding constants to ψ. We use the definition of $\lambda(\psi)$ in (1.25) and obtain, after interchanging the infima,

$$\widetilde{\chi}(\rho) = \inf_{\substack{g\in H^1(\mathbb{R}^d)\\ \|g\|_2=1}} \Big\{ \|\nabla g\|_2^2 - \sup_{\psi\in\mathcal{C}(\mathbb{R}^d)} \Big(\langle \psi, g^2\rangle - \rho \log \int e^{\frac{1}{\rho}\psi(x)}\, dx \Big) \Big\} + \rho \log \frac{\rho}{e}. \tag{1.33}$$

The supremum over ψ is uniquely (up to additive constants) attained at $\psi = \rho \log g^2$ with value $\mathcal{H}(g^2)$, as an application of Jensen's inequality shows:

$$\begin{aligned}\rho \log \int e^{\frac{1}{\rho}\psi(x)}\, dx &= \rho \log \int dx\, g^2(x)\, e^{\frac{1}{\rho}\psi(x) - \log g^2(x)} \geq \rho \int dx\, g^2(x) \Big(\frac{1}{\rho}\psi(x) - \log g^2(x) \Big) \\ &= \langle \psi, g^2\rangle - \mathcal{H}(g^2).\end{aligned}$$

Hence, $\widetilde{\chi}(\rho) = \chi(\rho) + \rho \log \frac{\rho}{e}$. Since g_ρ is, up to horizontal shifts, the unique minimiser in (1.23), $\widetilde{\psi}_\rho = \rho \log g_\rho^2 + C$ is the unique minimizer in (1.33). By the above reasoning, $\widetilde{\psi}_\rho$ is the unique minimizer of (1.32), where $C = -\rho \log \frac{\rho}{e}$ is determined by requiring that $\mathcal{L}(\widetilde{\psi}_\rho) = 1$. □

Remark 1.14 (Interpretation)**.** There is an interpretation of the minimiser of (1.32) in terms of the optimal rescaled profile of the potential ξ for the almost-sure asymptotics of the total mass $U(t)$. Indeed, the condition $\mathcal{L}(\psi) \leq 1$ guarantees that, almost surely for all large t, the profile ψ appears in some 'microbox' in the rescaled landscape ξ within the 'macrobox' $B_t = [-t,t]^d \cap \mathbb{Z}^d$, which is one of the intermittent islands. The logarithmic rate of the total mass, $\frac{1}{t}\log U(t) \approx \lambda_{B_t}(\xi)$, can be bounded from below against the eigenvalue of ξ in the microbox, which is described by $\lambda(\psi)$. Optimising over all admissible ψ explains the lower bound in (1.16). Our proof of the lower bound in Section 4 makes this heuristics precise.

The Gaussian density g_ρ in (1.26) is the unique positive L^2-normalized eigenfunction of $\Delta + \psi_\rho - \rho \log \frac{\rho}{e}$ corresponding to the eigenvalue $-\widetilde{\chi}(\rho) = \lambda(\psi_\rho - \rho \log \frac{\rho}{e})$. It describes the rescaled shape of the solution $u(t,\cdot)$ in the intermittent island. An interesting consequence is that the appropriately rescaled potential and solution shapes are identical for the moment asymptotics and for the almost sure asymptotics. This phenomenon also occurs in the cases of the double-exponential distribution and the potentials bounded from above. ◇

1.6.3. *Convergence of the variational problem in* (1.12)*.* We close this section by showing that the variational problem in (1.12) converges to the variational problem in (1.11) as $\gamma \uparrow 1$. We define

$$\chi(\rho,\gamma) = \inf_{\substack{g\in H^1(\mathbb{R}^d)\\ \|g\|_2=1}} \Big\{ \int_{\mathbb{R}^d} |\nabla g(x)|^2\, dx + \rho \int_{\mathbb{R}^d} \frac{g^{2\gamma}(x) - g^2(x)}{1-\gamma}\, dx \Big\}, \tag{1.34}$$

which is equal to the variational problem in (1.12).

Proposition 1.15 (Convergence of the variational problem in (1.34))**.** *For any* $\rho \in (0,\infty)$,

$$\lim_{\gamma\uparrow 1} \chi(\rho,\gamma) = \chi(\rho). \tag{1.35}$$

Proof. The upper bound in (1.35) follows by substituting the Gaussian density $g = g_\rho$ in (1.26) into the infimum in (1.34), and by noting that

$$\lim_{\gamma\uparrow 1} \int_{\mathbb{R}^d} \frac{g_\rho^{2\gamma}(x) - g_\rho^2(x)}{\gamma - 1}\, dx = \int_{\mathbb{R}^d} g_\rho^2(x) \log g_\rho^2(x)\, dx,$$

by an explicit computation of the integrals involved.

For the lower bound in (1.35), we bound, for any $\gamma \in [0,1)$ and $g \in H^1(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \frac{g^{2\gamma}(x) - g^2(x)}{1-\gamma}\,dx = \int_{\mathbb{R}^d} g^2(x) \frac{e^{(\gamma-1)\log g^2(x)} - 1}{1-\gamma}\,dx \geq -\int_{\mathbb{R}^d} g^2(x)\log g^2(x)\,dx,$$

since $e^\theta - 1 \geq \theta$ for every $\theta \in \mathbb{R}$. Therefore, $\chi(\rho,\gamma) \leq \chi(\rho)$ for every $\gamma \in [0,1)$. □

The remainder of the paper is as follows. In Section 2 we present an important auxiliary result on self-intersections of random walks, which will be used in the proof of Theorem 1.4 in Section 3. The proof of Theorem 1.7 is given in Section 4. Finally, in Section 5 we use the opportunity to correct an error in the proof of the moment asymptotics in case (4) from [BK01].

2. An auxiliary result on self-intersections of random walks

In this section we provide a result on q-fold self-intersections of random walks, for small $q > 1$, which is an important tool in the proof of the upper bound in Theorem 1.4.

It is also of independent interest. Let $\ell_t(z) = \int_0^t \delta_z(X(s))\,ds$ denote the local time at z of the simple random walk $(X(s)\colon s \in [0,t])$ on $\mathbb{Z}^d$ with generator Δ^{d}, starting at the origin.

Proposition 2.1. *Fix $q > 1$ such that $q(d-2) < d$ and $R > 0$. Let $\alpha(t) \to \infty$ such that $\alpha(t) = \mathcal{O}(t^{2/(2d+2)-\varepsilon})$ for some $\varepsilon > 0$. Then*

$$\limsup_{\theta\downarrow 0}\limsup_{t\uparrow\infty} \frac{\alpha(t)^2}{t} \log \mathbb{E}\Big[\exp\Big\{\theta\alpha(t)^{-\frac{1}{q}[d+(2-d)q]}\Big(\sum_{x\in\mathbb{Z}^d} \ell_t(x)^q\Big)^{\frac{1}{q}}\Big\}1\!\!1\{\mathrm{supp}\,(\ell_t) \subseteq B_{R\alpha(t)}\}\Big] = 0. \quad (2.1)$$

Remark 2.2. The result is better understood when rephrasing it in terms of the normalised and rescaled local times, $L_t(\cdot) = \frac{1}{t}\alpha(t)^d \ell_t(\lfloor \cdot\, \alpha(t)\rfloor)$. Then the exponent may be rewritten as

$$\alpha(t)^{-\frac{1}{q}[d+(2-d)q]}\Big(\sum_{x\in\mathbb{Z}^d} \ell_t(x)^q\Big)^{\frac{1}{q}} = \frac{t}{\alpha(t)^2}\|L_t\|_q,$$

where $\|\cdot\|_q$ is the norm on $L^q(\mathbb{R}^d)$. Hence, (2.1) is a large deviations result for the q-norm of L_t on the scale $t/\alpha(t)^2$. It is known that $(L_t\colon t > 0)$ satisfies a large deviation principle on this scale in the weak topology generated by bounded continuous functions, see for example [GKS04]. However, (2.1) does not follow from a routine application of Varadhan's lemma, since the q-norm is neither bounded nor continuous in this topology. See [Ch04] for an analogous result for a smoothed version of L_t. ◇

Remark 2.3. Our proof yields (2.1) also without indicator on $\{\mathrm{supp}\,(\ell_t) \subseteq B_{R\alpha(t)}\}$ if the sum is restricted to a finite subset of $\mathbb{Z}^d$. It can easily be extended to a large class of random walks, also in discrete time. The proof is based on a combinatorial analysis of the high integer moments of the random variable $\sum_x \ell_t(x)^q$. This method is of crucial importance in the analysis of intersections and self-intersections of random paths [KM02], and of random walk in random scenery [GKS04]. ◇

Proof of Proposition 2.1. By B we denote the box $B = B_{R\alpha(t)} = [-R\alpha(t), R\alpha(t)]^d \cap \mathbb{Z}^d$. In the exponent on the left side of (2.1), we restrict the sum to $x \in B$ and forget about the indicator on $\{\mathrm{supp}\,(\ell_t) \subseteq B_{R\alpha(t)}\}$. In the following we write $\|\cdot\|_q$ for the norm in $\ell^q(B)$.

In a first step we reduce the problem to a problem on asymptotics of high integer moments. Suppose first that there are constants $T, C > 0$ such that

$$\mathbb{E}\big[\|\ell_t\|_q^{kq}\big] \leq k^{kq} C^{kq} \alpha(t)^{k[d+(2-d)q]}, \qquad \text{for any } t \geq T, k \geq \frac{t}{\alpha(t)^2}. \quad (2.2)$$

We now show that this assumption implies (2.1). Expanding the exponential series, we rewrite

$$\mathbb{E}\Big[\exp\Big\{\theta\alpha(t)^{-\frac{1}{q}[d+(2-d)q]}\|\ell_t\|_q\Big\}\Big] = \sum_{k=0}^{\infty}\frac{1}{k!}\Big(\theta\alpha(t)^{-\frac{1}{q}[d+(2-d)q]}\Big)^k\mathbb{E}\big[\|\ell_t\|_q^k\big].$$

Abbreviate $k_t = q\lceil t/\alpha(t)^2\rceil$. Under our assumption,

$$\mathbb{E}\big[\|\ell_t\|_q^k\big] \le \Big(\frac{k}{q}\Big)^k C^k\alpha(t)^{\frac{k}{q}[d+(2-d)q]}, \qquad \text{for } t\ge T, k\ge k_t, \tag{2.3}$$

and hence we obtain

$$\mathbb{E}\Big[\exp\Big\{\theta\alpha(t)^{-\frac{1}{q}[d+(2-d)q]}\|\ell_t\|_q\Big\}\Big] \le \sum_{k=0}^{k_t-1}\frac{1}{k!}\theta^k\alpha(t)^{-\frac{k}{q}[d+(2-d)q]}\mathbb{E}\big[\|\ell_t\|_q^k\big] + \sum_{k=k_t}^{\infty}\frac{1}{k!}\Big(\frac{\theta Ck}{q}\Big)^k. \tag{2.4}$$

For all sufficiently small $\theta>0$, the second term is estimated as follows:

$$\sum_{k=k_t}^{\infty}\frac{1}{k!}\Big(\frac{\theta Ck}{q}\Big)^k \le \sum_{k=k_t}^{\infty}\Big(\frac{\theta C}{eq}\Big)^k = \frac{\Big(\frac{\theta C}{eq}\Big)^{k_t}}{1-\frac{\theta C}{eq}},$$

and the exponential rate (in $t\alpha(t)^{-2}$) of the right hand side tends to $-\infty$ as $\theta\downarrow 0$.

For the first term, we bound, using Hölder's inequality and (2.3), for $k\le k_t$,

$$\mathbb{E}\big[\|\ell_t\|_q^k\big] \le \mathbb{E}\big[\|\ell_t\|_q^{k_t}\big]^{\frac{k}{k_t}} \le \Big(\Big(\frac{k_t}{q}\Big)^{k_t}C^{k_t}\alpha(t)^{\frac{k_t}{q}[d+(2-d)q]}\Big)^{\frac{k}{k_t}} = \Big(\frac{k_t}{q}\Big)^k C^k\alpha(t)^{\frac{k}{q}[d+(2-d)q]}.$$

Therefore, the first term in (2.4) is bounded by

$$\sum_{k=0}^{k_t}\frac{1}{k!}\Big(\frac{\theta Ck_t}{q}\Big)^k \le e^{\frac{\theta C}{q}k_t}.$$

This proves that (2.2) implies the statement (2.1). Therefore, it suffices to prove (2.2) with some constants $C,T>0$. We use C to denote a generic constant which depends on R, d and q, but not on k and t, and C may change its value from appearance to appearance.

To prove (2.2), we write A_k for the set of maps $\beta\colon B\to\mathbb{N}_0$ satisfying $\sum_{x\in B}\beta_x = k$. First we write out

$$\begin{aligned}\mathbb{E}\big[\|\ell_t\|_q^{kq}\big] &= \sum_{z_1,\dots,z_k\in B}\mathbb{E}\Big[\prod_{x\in B}\ell_t(x)^{q\#\{i\colon z_i=x\}}\Big]\\ &= \sum_{\beta\in A_k}\mathbb{E}\Big[\prod_{x\in B}\ell_t(x)^{q\beta_x}\Big]\#\{z\in B^k\colon \beta_x=\#\{z_i=x\}\forall x\}\\ &= k!\sum_{\beta\in A_k}\mathbb{E}\Big[\prod_{x\in B}\ell_t(x)^{q\beta_x}\Big]\prod_{x\in B}\frac{1}{\beta_x!}.\end{aligned} \tag{2.5}$$

Note that, for $\beta\in A_k$, the numbers $q\beta_x$ are not necessarily integers. We resolve this problem, in an upper bound, by introducing a further sum over the set $A_k(\beta)$ of all $\widetilde\beta\colon B\to\mathbb{N}_0$ satisfying $|\widetilde\beta_x - q\beta_x|<1$ for every $x\in B$. Then, clearly,

$$\mathbb{E}\Big[\prod_{x\in B}\ell_t(x)^{q\beta_x}\Big] \le \sum_{\widetilde\beta\in A_k(\beta)}\mathbb{E}\Big[\prod_{x\in B}\ell_t(x)^{\widetilde\beta_x}\Big]. \tag{2.6}$$

We fix $\beta \in A_k$ and $\widetilde{\beta} \in A_k(\beta)$ and denote $\widetilde{k} = \sum_{x\in B} \widetilde{\beta}_x$. Writing out the local times, we have

$$\mathbb{E}\Big[\prod_{x\in B} \ell_t(x)^{\widetilde{\beta}_x}\Big] = \Big[\prod_{x\in B}\prod_{i=1}^{\widetilde{\beta}_x}\int_0^t ds_i^x\Big]\,\mathbb{P}\big\{X(s_i^x) = x\,\forall x \in B\,\forall i = 1,\dots,\widetilde{\beta}_x\big\}.$$

The next step is to give new names to the integration variables s_i^x such that we can order the time variables. Fix some function $\varrho\colon \{1,\dots,\widetilde{k}\} \to B$ such that $|\varrho^{-1}(\{x\})| = \widetilde{\beta}_x$ for any $x \in B$. We continue with, denoting the set of permutations of $1,\dots,\widetilde{k}$ by $\mathfrak{S}_{\widetilde{k}}$,

$$\begin{aligned}
\mathbb{E}\Big[\prod_{x\in B} \ell_t(x)^{\widetilde{\beta}_x}\Big] &= \int_{[0,t]^{\widetilde{k}}} dt_1\dots dt_{\widetilde{k}}\,\mathbb{P}\{X(t_i) = \varrho(i)\forall i = 1,\dots,\widetilde{k}\}\\
&= \int_{0<t_1<\dots<t_{\widetilde{k}}\le t} dt_1\dots dt_{\widetilde{k}} \sum_{\sigma\in\mathfrak{S}_{\widetilde{k}}} \mathbb{P}\big\{X(t_{\sigma(i)}) = \varrho(i)\ \ \forall i\big\}\\
&= \sum_{\sigma\in\mathfrak{S}_{\widetilde{k}}} \int_{(0,\infty)^{\widetilde{k}}} ds_1\dots ds_{\widetilde{k}}\mathbb{1}\Big\{\sum_{i=1}^{\widetilde{k}} s_i \le t\Big\} \prod_{i=1}^{\widetilde{k}} p_{s_i}\big(\varrho(\sigma(i-1)),\varrho(\sigma(i))\big),
\end{aligned} \tag{2.7}$$

where we switched from σ to σ^{-1} and substituted $s_i = t_i - t_{i-1}$ (with $t_0 = 0$), and we introduced the transition probabilities of continuous time simple random walk, $p_s(x,y) = \mathbb{P}_x\{X(s) = y\}$. Here we use the convention $\sigma(0) = 0$ and $\varrho(0) = 0$, the starting point of the random walk.

We estimate the indicator on the right hand side of (2.7) against $e^{\lambda t}\prod_{i=1}^{\widetilde{k}} e^{-\lambda s_i}$ for $\lambda = \alpha(t)^{-2}$. Then we integrate out over all the s_i, to obtain

$$\mathbb{E}\Big[\prod_{x\in B} \ell_t(x)^{\widetilde{\beta}_x}\Big] \le e^{\lambda t} \sum_{\sigma\in\mathfrak{S}_{\widetilde{k}}} \prod_{i=1}^{\widetilde{k}} G_\lambda\big(\varrho(\sigma(i-1)),\varrho(\sigma(i))\big), \tag{2.8}$$

where G_λ is the Green's function of the walk given by $G_\lambda(x,y) = \int_0^\infty e^{-\lambda s} p_s(x,y)\,ds$. It will be convenient to use a closed loop of sites, i.e., to change the convention $\sigma(0) = 0$ to the convention $\sigma(0) = \sigma(\widetilde{k})$. This change of conventions leads to a factor

$$\frac{G_\lambda\big(\varrho(0),\varrho(\sigma(1))\big)}{G_\lambda\big(\varrho(\sigma(\widetilde{k})),\varrho(\sigma(1))\big)},$$

which can be bounded by $e^{o(k)}$ since $\sup_{x,y\in B} G_\lambda(0,y)/G_\lambda(x,y) \le e^{o(k)}$, where we recall that $\lambda = \alpha(t)^{-2}$, $k > t\alpha(t)^{-2}$ and $B = B_{R\alpha(t)}$.

We denote by $P(\widetilde{\beta})$ the set of maps $\gamma\colon B\times B \to \mathbb{N}_0$ such that $\sum_{y\in B}\gamma_{x,y} = \sum_{y\in B}\gamma_{y,x} = \widetilde{\beta}_x$ for any $x \in B$. Then we can rewrite

$$\begin{aligned}
\mathbb{E}\Big[\prod_{x\in B} \ell_t(x)^{\widetilde{\beta}_x}\Big] \le e^{\lambda t + o(k)} &\sum_{\gamma\in P(\widetilde{\beta})} \prod_{x,y\in B} G_\lambda(x,y)^{\gamma_{x,y}} \sum_{\substack{w\in B^{\widetilde{k}}\\ (w_0 = w_{\widetilde{k}})}} \sum_{\sigma\in\mathfrak{S}_{\widetilde{k}}}\\
&\times \mathbb{1}\{\varrho\circ\sigma = w\}\mathbb{1}\big\{\gamma_{x,y} = \#\{i\colon w_{i-1} = x, w_i = y\}\ \ \forall x,y \in B\big\}.
\end{aligned}$$

We can evaluate the sums over w and σ using elementary combinatorics. Indeed, note that

$$\#\{\sigma\colon \varrho\circ\sigma = w\} = \prod_{x\in B} \widetilde{\beta}_x!, \tag{2.9}$$

since, given w and ϱ, the left hand side equals the number of orders in which one can put $\widetilde{k}$ objects, of which $\widetilde{\beta}_x$ for each $x \in B$ are indistinguishable, into a row such that the same vector of elements arises. Since one can only permute within those indices which belong to the same class of indistinguishable objects, we obtain (2.9). This performs the sum over σ. To perform the sum over w for fixed γ, we use [dH00, p.17], to obtain

$$\#\big\{w\colon \gamma_{x,y} = \#\{i\colon w_{i-1} = x, w_i = y\}\ \ \forall x, y \in B\big\} \leq \widetilde{k} \prod_{x,y\in B} \frac{\widetilde{\beta}_x!}{\gamma_{x,y}!}.$$

Therefore, we obtain

$$\begin{aligned}\mathbb{E}\Big[\prod_{x\in B} \ell_t(x)^{\widetilde{\beta}_x}\Big] &\leq e^{\lambda t+o(k)} \sum_{\gamma\in P(\widetilde{\beta})} \prod_{x,y\in B} \Big[G_\lambda(x,y)^{\gamma_{x,y}} \frac{\widetilde{\beta}_x!}{\gamma_{x,y}!}\Big] \prod_{x\in B} \widetilde{\beta}_x! \\ &\leq e^{\lambda t+o(k)} \sum_{\gamma\in P(\widetilde{\beta})} e^{\sum_{x,y\in B}\gamma_{x,y}} \prod_{x,y\in B} \Big[\frac{G_\lambda(x,y)^q}{\gamma_{x,y}}\Big]^{\frac{1}{q}\gamma_{x,y}} \prod_{x,y\in B} \gamma_{x,y}^{\frac{1-q}{q}\gamma_{x,y}} \prod_{x\in B} \widetilde{\beta}_x^{2\widetilde{\beta}_x},\end{aligned} \tag{2.10}$$

where we use that $n^n e^{-n} \leq n! \leq n^n$. We next use that, since $|\widetilde{\beta}_x - q\beta_x| < 1$,

$$\widetilde{k} = \sum_{x,y\in B} \gamma_{x,y} = \sum_{x\in B} \widetilde{\beta}_x \leq q \sum_{x\in B} \beta_x + |B| = qk + |B|.$$

By our assumption on the growth of $\alpha(t)$, we have $|B| \leq C\alpha(t)^d \leq o(k)$ and hence $e^{\sum_{x,y\in B}\gamma_{x,y}} \leq C^k$. Fix $\gamma \in P(\widetilde{\beta})$. We use Jensen's inequality for the logarithm to obtain

$$\begin{aligned}\prod_{x,y\in B} \Big[\frac{G_\lambda(x,y)^q}{\gamma_{x,y}}\Big]^{\frac{1}{q}\gamma_{x,y}} &= \exp\Big\{\frac{1}{q}\sum_{x\in B} \widetilde{\beta}_x \sum_{y\in B} \frac{\gamma_{x,y}}{\widetilde{\beta}_x} \log \frac{G_\lambda(x,y)^q}{\gamma_{x,y}}\Big\} \\ &\leq \exp\Big\{\frac{1}{q}\sum_{x\in B} \widetilde{\beta}_x \log\Big(\sum_{y\in B} \frac{G_\lambda(x,y)^q}{\widetilde{\beta}_x}\Big)\Big\}.\end{aligned} \tag{2.11}$$

Recall that $\lambda = \alpha(t)^{-2}$. Since $(d-2)q < d$, there is a constant C (only depending on R, d and q) such that, for any $x \in B$,

$$\sum_{y\in B} G_{\alpha(t)^{-2}}(x,y)^q \leq C\alpha(t)^{d+(2-d)q}. \tag{2.12}$$

This gives that

$$\prod_{x,y\in B} \Big[\frac{G_\lambda(x,y)^q}{\gamma_{x,y}}\Big]^{\frac{1}{q}\gamma_{x,y}} \leq C^k \alpha(t)^{[d+(2-d)q]\widetilde{k}/q} \prod_{x\in B} \beta_x^{-\frac{1}{q}\beta_x}. \tag{2.13}$$

We substitute (2.13) into (2.10) and summarise (2.5), (2.6) and (2.10). Using that $|\widetilde{k} - qk| \leq |B|$, we obtain

$$\mathbb{E}\Big[\|\ell_t\|_q^{qk}\Big] \leq \widetilde{k}^{\widetilde{k}} C^k \alpha(t)^{[d+(2-d)q](k+\frac{1}{q}|B|)} \sum_{\beta\in A_k} \sum_{\widetilde{\beta}\in A_k(\beta)} \sum_{\gamma\in P(\widetilde{\beta})} \prod_{x,y\in B} \Big[\frac{(\frac{1}{k}\gamma_{x,y})^{\frac{1-q}{q}}}{(\frac{1}{k}\widetilde{\beta}_x)^{\frac{1}{q}-2} (\frac{1}{k}\beta_x)^{\beta_x/\widetilde{\beta}_x}}\Big]^{\gamma_{x,y}}. \tag{2.14}$$

Note that, by our growth assumption on $\alpha(t)$ and since $k \geq t/\alpha(t)^2$,

$$\widetilde{k}^{\widetilde{k}} \leq (qk)^{qk} C^{|B|} k^{|B|} \leq (qk)^{qk} e^{o(k)}.$$

The product is estimated with the help of Jensen's inequality for the logarithm, together with the fact that $y \mapsto \frac{\gamma_{x,y}}{\widetilde\beta_x}$ is a probability measure, as follows:

$$\begin{aligned}\prod_{x,y\in B}\Big[\frac{(\frac1k\gamma_{x,y})^{\frac{1-q}{q}}}{(\frac1k\widetilde\beta_x)^{\frac1q-2}(\frac1k\beta_x)^{\beta_x/\widetilde\beta_x}}\Big]^{\gamma_{x,y}} &= \exp\Big\{\frac{q-1}{q}\sum_{x\in B}\widetilde\beta_x\sum_{y\in B}\frac{\gamma_{x,y}}{\widetilde\beta_x}\log\frac{\widetilde\beta_x(\frac1k\widetilde\beta_y)}{\gamma_{x,y}}\Big\}\\ &\quad\times\exp\Big\{-\sum_{x\in B}\beta_x\log\frac{\beta_x}{k}+\frac1q\sum_{x\in B}\widetilde\beta_x\log\frac{\widetilde\beta_x}{\widetilde k}\Big\}\\ &\le \exp\Big\{-\sum_{x\in B}\beta_x\log\frac{\beta_x}{k}+\frac1q\sum_{x\in B}\widetilde\beta_x\log\frac{\widetilde\beta_x}{\widetilde k}\Big\}.\end{aligned}$$

Now recall that $q\beta_x-1\le\widetilde\beta_x\le q\beta_x+1\le 2q\beta_x$ for $\beta_x>0$ to bound

$$\sum_{x\in B}\widetilde\beta_x\log\frac{\widetilde\beta_x}{\widetilde k}\le\sum_{x\in B}(q\beta_x-1)\log\frac{2q\beta_x}{\widetilde k}=q\sum_{x\in B}\beta_x\log\frac{\beta_x}{k}+\sum_{x\in B}\log\frac{\widetilde k}{2q\beta_x}+qk\log\frac{k}{\widetilde k},$$

so that we arrive at

$$\begin{aligned}-\sum_{x\in B}\beta_x\log\frac{\beta_x}{k}+\frac1q\sum_{x\in B}\widetilde\beta_x\log\frac{\widetilde\beta_x}{\widetilde k}&\le k\log\frac{k}{\widetilde k}+\frac1q\sum_{x\in B}\log\frac{\widetilde k}{2q\beta_x}\\ &\le k\log\frac{qk}{\widetilde k}+Ck+C|B|\log k\le Ck,\end{aligned}$$

since $qk/\widetilde k$ converges to one and since $|B|\log k\le C\alpha(t)^d\log k\le o(k)$. Hence, we have estimated the product on the right hand side of (2.14) against C^k uniformly in $\beta\in A_k$, $\widetilde\beta\in A_k(\beta)$ and $\gamma\in P(\widetilde\beta)$. Our growth condition on $\alpha(t)$ implies that each of the sums can be estimated against $e^{o(k)}$. Indeed,

$$|P(\widetilde\beta)|\le k^{|B|^2}\le e^{C\alpha(t)^{2d}\log k}\le e^{o(k)},$$

for any $\widetilde\beta\in A_k(\beta)$ and for any $\beta\in A_k$. Furthermore, $|A_k(\beta)|\le 2^{|B|}\le e^{o(k)}$ for any $\beta\in A_k$, and finally $|A_k|\le k^{|B|}\le e^{o(k)}$. Therefore, we obtain

$$\mathbb{E}\big[\|\ell_t\|_q^{kq}\big]\le C^k k^{qk}\alpha(t)^{[d+(2-d)q]k}\alpha(t)^{C|B|}\le k^{kq}C^{kq}\alpha(t)^{k[d+(2-d)q]},$$

where we again used our growth condition on $\alpha(t)$. This completes the proof. □

3. The moment asymptotics: Proof of Theorem 1.4

Our analysis is based on the link between the random-walk and random-field descriptions provided by the Feynman-Kac formula. Let $(X(s)\colon s\in[0,\infty))$ be the continuous-time simple random walk on $\mathbb{Z}^d$ with generator $\Delta^{\mathrm d}$. By $\mathbb{P}_z$ and $\mathbb{E}_z$ we denote the probability measure, respectively, the expectation with respect to the walk starting at $X(0)=z\in\mathbb{Z}^d$.

For any potential $V\colon\mathbb{Z}^d\to[-\infty,\infty)$, we denote by u^V the unique nonnegative, continuous solution of the initial-value problem

$$\begin{aligned}\partial_t u(t,z) &= \Delta^{\mathrm d}u(t,z)+V(z)u(t,z), &&\text{for } (t,z)\in(0,\infty)\times\mathbb{Z}^d,\\ u(0,z)&=1, &&\text{for } z\in\mathbb{Z}^d.\end{aligned}\tag{3.1}$$

The Feynman-Kac formula allows us to express u^V as

$$u^V(t,z)=\mathbb{E}_z\Big[\exp\Big\{\int_0^t V\big(X(s)\big)\,ds\Big\}\Big],\qquad \text{for } z\in\mathbb{Z}^d,\ t>0.\tag{3.2}$$

Hence, u^ξ is the solution of the parabolic Anderson problem in (1.1) with initial condition $u(0,z)=1$ for all $z\in\mathbb{R}^d$, and the main object of our study is $U(t)=u^\xi(t,0)$. Introduce the vertically shifted random potential

$$\xi_t(z)=\xi(z)-H\Big(\frac{t}{\alpha(t)^d}\Big)\frac{\alpha(t)^d}{t}. \tag{3.3}$$

Note that t is a parameter here, and ξ_t should not be seen as a time-dependent random potential. Fix $p\in(0,\infty)$. Then Theorem 1.4 is equivalent to the statement

$$\lim_{t\uparrow\infty}\frac{\alpha(pt)^2}{pt}\log\big\langle u^{\xi_{pt}}(t,0)^p\big\rangle=-\chi(\rho), \tag{3.4}$$

where $\chi(\rho)$ is defined in (1.23). We approximate $u^{\xi_{pt}}$ by finite-space versions. Let $R>0$ and let $B_R=[-R,R]^d\cap\mathbb{Z}^d$ be the centred box in $\mathbb{Z}^d$ with radius R. Introduce $u_R^V\colon[0,\infty)\times\mathbb{Z}^d\to[0,\infty)$ by

$$u_R^V(t,z)=\mathbb{E}_z\Big[\exp\Big\{\int_0^t V\big(X(s)\big)\,ds\Big\}1\!\!1\{\operatorname{supp}(\ell_t)\subseteq B_R\}\Big], \tag{3.5}$$

where $\ell_t(z)=\int_0^t\delta_z(X(s))\,ds$ are the local times of the random walk. Note that $u_r^V\le u_R^V\le u^V$ for $0<r<R<\infty$. In the finite space setting we can work easily with eigenfunction expansions: We look at the function

$$p_R^V(t,y,z)=\mathbb{E}_y\Big[e^{\langle V,\ell_t\rangle}1\!\!1\{\operatorname{supp}(\ell_t)\subseteq B_R\}1\!\!1\{X(t)=z\}\Big]\qquad\text{for } y,z\in\mathbb{Z}^d, \tag{3.6}$$

and the eigenvalues, $\lambda_1>\lambda_2\ge\lambda_3\ge\cdots\ge\lambda_n$, of the operator $\Delta^{\mathrm{d}}+V$ in $\ell^2(B_R)$ with zero boundary condition, where we abbreviate $n=|B_R|$. We may pick an orthonormal basis of corresponding eigenfunctions e_k. By convention, e_k vanishes outside B_R. Note that $\sum_{z\in B_R}p_R^V(t,y,z)=u_R^V(t,y)$. Furthermore, we have the eigenfunction expansion

$$p_R^V(t,y,z)=\sum_k e^{t\lambda_k}\mathrm{e}_k(y)\mathrm{e}_k(z). \tag{3.7}$$

In particular,

$$u_R^V(t,z)=\sum_k e^{t\lambda_k}\langle\mathrm{e}_k,1\!\!1\rangle\mathrm{e}_k(z). \tag{3.8}$$

The following proposition carries out the necessary large deviations arguments for the case $p=1$, and is the key result for the proof of (3.4).

Proposition 3.1.
(i) *Let* $R>0$. *Then* $\displaystyle\limsup_{t\uparrow\infty}\frac{\alpha(t)^2}{t}\log\Big\langle u^{\xi_t}_{R\alpha(t)}(t,0)\Big\rangle\le-\chi(\rho)$.

(ii) $\displaystyle\liminf_{R\uparrow\infty}\liminf_{t\uparrow\infty}\frac{\alpha(t)^2}{t}\log\Big\langle u^{\xi_t}_{R\alpha(t)}(t,0)\Big\rangle\ge-\chi(\rho)$.

The proofs of Proposition 3.1(i) and (ii) are deferred to Sections 3.2 and 3.3, respectively.

3.1 Proof of (3.4) subject to Proposition 3.1

Proof of the lower bound in (3.4). All we have to do is to show that, as $t\uparrow\infty$,

$$\big\langle u^{\xi_{pt}}(t,0)^p\big\rangle\ge e^{o(t\alpha(pt)^{-2})}\Big\langle u^{\xi_{pt}}_{R\alpha(pt)}(pt,0)\Big\rangle. \tag{3.9}$$

To prove this, we repeat the proof of [BK01, Lemmas 4.1 and 4.3] for the reader's convenience. We abbreviate $r=R\alpha(pt)$, $V=\xi_{pt}$, $u=u^V$, $u_r=u_r^V$ and $p_r=p_r^V$. Note that $|B_r|=e^{o(t\alpha(pt)^{-2})}$.

Now we prove (3.9). First we assume that $p \in (0,1)$. Use the shift invariance of the distribution of the field V and the inequality $\sum_i x_i^p \geq (\sum_i x_i)^p$ for nonnegative x_i to estimate

$$\big\langle u(t,0)^p \big\rangle = \Big\langle \frac{1}{|B_r|} \sum_{z \in B_r} u(t,z)^p \Big\rangle \geq e^{o(t\alpha(pt)^{-2})} \Big\langle \sum_{z \in B_r} u(t,z)^p \Big\rangle \geq e^{o(t\alpha(pt)^{-2})} \Big\langle \Big(\sum_{z \in B_r} u(t,z) \Big)^p \Big\rangle. \tag{3.10}$$

By $\|\cdot\|$ we denote the norm on $\ell^2(B_r)$. According to Parseval's identity, the numbers $\langle \mathrm{e}_k, \mathbb{1} \rangle^2 / \|\mathbb{1}\|^2$ sum up to one. Using the Fourier expansion in (3.8) and Jensen's inequality, we obtain

$$\begin{aligned} \Big\langle \Big(\sum_{z \in B_r} u(t,z) \Big)^p \Big\rangle &\geq \|\mathbb{1}\|^{2p} \Big\langle \Big(\sum_k e^{t\lambda_k} \frac{\langle \mathrm{e}_k, \mathbb{1} \rangle^2}{\|\mathbb{1}\|^2} \Big)^p \Big\rangle \geq e^{o(t\alpha(pt)^{-2})} \Big\langle \sum_k e^{pt\lambda_k} \frac{\langle \mathrm{e}_k, \mathbb{1} \rangle^2}{\|\mathbb{1}\|^2} \Big\rangle \\ &\geq e^{o(t\alpha(pt)^{-2})} \Big\langle \sum_{z \in B_r} u_r(pt,z) \Big\rangle \geq e^{o(t\alpha(pt)^{-2})} \big\langle u_r(pt,0) \big\rangle. \end{aligned} \tag{3.11}$$

Substituting (3.11) in (3.10) completes the proof of (3.9) in the case $p \in (0,1)$.

Now we turn to the case $p \in [1,\infty)$. We use the first equation in (3.10), Jensen's inequality, the eigenfunction expansion in (3.8) and the inequality $(\sum_i x_i)^p \geq \sum_i x_i^p$ for nonnegative x_i to obtain

$$\begin{aligned} \big\langle u(t,0)^p \big\rangle &\geq \Big\langle \Big(\frac{1}{|B_r|} \sum_{z \in B_r} u(t,z) \Big)^p \Big\rangle \geq e^{o(t\alpha(pt)^{-2})} \Big\langle \Big(\sum_k e^{t\lambda_k} \langle \mathrm{e}_k, \mathbb{1} \rangle^2 \Big)^p \Big\rangle \\ &\geq e^{o(t\alpha(pt)^{-2})} \Big\langle \sum_k e^{pt\lambda_k} \langle \mathrm{e}_k, \mathbb{1} \rangle^{2p} \Big\rangle. \end{aligned} \tag{3.12}$$

Now we use Jensen's inequality as follows

$$\begin{aligned} \Big\langle \sum_k e^{pt\lambda_k} \langle \mathrm{e}_k, \mathbb{1} \rangle^{2p} \Big\rangle &= \frac{\big\langle \sum_k e^{pt\lambda_k} \langle \mathrm{e}_k, \mathbb{1} \rangle^{2p} \big\rangle}{\big\langle \sum_k e^{pt\lambda_k} \big\rangle} \Big\langle \sum_k e^{pt\lambda_k} \Big\rangle \geq \left(\frac{\big\langle \sum_k e^{pt\lambda_k} \langle \mathrm{e}_k, \mathbb{1} \rangle^2 \big\rangle}{\big\langle \sum_k e^{pt\lambda_k} \big\rangle} \right)^p \Big\langle \sum_k e^{pt\lambda_k} \Big\rangle \\ &= \Big\langle \sum_{z \in B_r} u_r(pt,z) \Big\rangle \left(\frac{\big\langle \sum_k e^{pt\lambda_k} \big\rangle}{\big\langle \sum_k e^{pt\lambda_k} \langle \mathrm{e}_k, \mathbb{1} \rangle^2 \big\rangle} \right)^{1-p} \geq \big\langle u_r(pt,0) \big\rangle. \end{aligned} \tag{3.13}$$

In the last step, we have used the eigenfunction expansions in (3.7) and (3.8) to see that the ratio is not bigger than one. Combining (3.12) and (3.13) completes the proof of (3.9) also in the case $p \in [1,\infty)$. $\square$

Proof of the upper bound in (3.4). A main ingredient in our proof is the following preparatory lemma, which provides, for any potential V, an estimate of $u^V(t,0)$ in terms of the maximal principal eigenvalue of $\Delta^{\mathrm{d}} + V$ in small subboxes ('microboxes') of a 'macrobox'. For $z \in \mathbb{Z}^d$ and $R > 0$, we denote by $\lambda^{\mathrm{d}}_{z;R}(V)$ the principal eigenvalue of the operator $\Delta^{\mathrm{d}} + V$ with Dirichlet boundary conditions in the *shifted* box $z + B_R$.

Lemma 3.2. *Let $r\colon (0,\infty) \to (0,\infty)$ such that $r(t)/t \uparrow \infty$. For $R, t > 0$ let $B_R(t) = B_{r(t)+2\lfloor R \rfloor}$. Then there is a constant $C > 0$ such that, for any sufficiently large R, t and any potential $V\colon \mathbb{Z}^d \to [-\infty,\infty)$,*

$$u^V(t,0) \leq \mathbb{E}\big[e^{2\int_0^t V(X_s)\,ds}\big]^{1/2} e^{-r(t)} + e^{Ct/R^2} \big(3r(t)\big)^d \exp\Big\{ t \max_{z \in B_R(t)} \lambda^d_{z;2R}(V) \Big\}. \tag{3.14}$$

Proof. This is a modification of the proof of [BK01, Proposition 4.4], which refers to nonpositive potentials V only. The proof of [BK01, Proposition 4.4] consists of [BK01, Lemma 4.5] and [BK01, Lemma 4.6]. The latter states that

$$u^V_{r(t)}(t,0) \quad\le\quad e^{Ct/R^2}\big(3r(t)\big)^d \exp\Big\{t \max_{z\in B_R(t)} \lambda^{\mathrm{d}}_{z;2R}(V)\Big\}. \tag{3.15}$$

A careful inspection of the proof shows that no use is made of nonpositivity of V and hence (3.15) applies in the present setting.

In order to estimate $u^V(t,0)-u^V_{r(t)}(t,0)$, we introduce the exit time $\tau_R=\inf\{t>0\colon X(t)\notin B_R\}$ from the box B_R and use the Cauchy-Schwarz inequality to obtain

$$\begin{aligned} u^V(t,0)-u^V_{r(t)}(t,0) &= \mathbb{E}\Big[\exp\Big\{\int_0^t V(X(s))\,ds\Big\}1\!\!1\{\tau_{r(t)}\le t\}\Big] \\ &\le \mathbb{E}\big[e^{2\int_0^t V(X_s)\,ds}\big]^{1/2}\mathbb{P}\{\tau_{r(t)}\le t\}^{1/2}. \end{aligned}$$

According to [GM98, Lemma 2.5(a)], for any $r>0$,

$$\mathbb{P}\{\tau_r\le t\}\le 2^{d+1}\exp\Big\{-r\Big(\log\frac{r}{dt}-1\Big)\Big\}.$$

Hence, we may estimate $\mathbb{P}\{\tau_{r(t)}\le t\}^{1/2}\le e^{-r(t)}$, for sufficiently large t, completing the proof. □

We now complete the proof of the upper bound in (3.4), subject to Proposition 3.1. Let $p\in(0,\infty)$ and fix $R>0$. First, notice that the second term in (3.14) can be estimated in terms of a sum,

$$\exp\Big\{pt\max_{z\in B_R(t)}\lambda^{\mathrm{d}}_{z;2R}(V)\Big\}\le\sum_{z\in B_R(t)} e^{pt\lambda^{\mathrm{d}}_{z;2R}(V)}. \tag{3.16}$$

Thus, applying (3.14) to $u^{\xi_{pt}}(t,0)$ with R replaced by $R\alpha(pt)$, raising both sides to the p-th power, and using (3.16), we get

$$\begin{aligned} u^{\xi_{pt}}(t,0)^p \le 2^p\Big(&\mathbb{E}\big[e^{2\int_0^t\xi_{pt}(X(s))\,ds}\big]^{p/2}e^{-pr(t)} \\ &+e^{Cpt/(R^2\alpha(pt)^2)}\big(3r(t)\big)^{pd}\sum_{z\in B_{R\alpha(pt)}(t)}e^{pt\lambda^{\mathrm{d}}_{z;2R\alpha(pt)}(\xi_{pt})}\Big). \end{aligned}$$

Next we take the expectation with respect to ξ and note that, by the shift-invariance of ξ, the distribution of $\lambda^{\mathrm{d}}_{z;2R\alpha(pt)}(\xi)$ does not depend on $z\in\mathbb{Z}^d$. This gives

$$\begin{aligned} \langle u^{\xi_{pt}}(t,0)^p\rangle \le 2^p\Big(&\Big\langle\mathbb{E}\big[e^{2\int_0^t\xi_{tp}(X(s))\,ds}\big]^{p/2}\Big\rangle e^{-pr(t)} \\ &+e^{Cpt/(R^2\alpha(pt)^2)}\big(3r(t)\big)^{pd+d}\big\langle e^{pt\lambda^{\mathrm{d}}_{0;2R\alpha(pt)}(\xi_{pt})}\big\rangle\Big). \end{aligned} \tag{3.17}$$

In order to show that the first term on the right is negligible, estimate, in the case $p\ge2$, with the help of Jensen's inequality and Fubini's theorem,

$$\begin{aligned} \Big\langle\mathbb{E}\big[e^{2\int_0^t\xi_{tp}(X(s))\,ds}\big]^{p/2}\Big\rangle &\le \mathbb{E}\big[\big\langle e^{\frac1t\int_0^t pt\xi(X(s))\,ds}\big\rangle\big]\exp\Big\{-\frac{H(pt\alpha(pt)^{-d})}{\alpha(pt)^{-d}}\Big\} \\ &\le \mathbb{E}\Big[\Big\langle\frac1t\int_0^t e^{pt\xi(X(s))}\,ds\Big\rangle\Big]\exp\Big\{-\frac{H(pt\alpha(pt)^{-d})}{\alpha(pt)^{-d}}\Big\} \\ &= e^{H(pt)}\exp\Big\{-\frac{H(pt\alpha(pt)^{-d})}{\alpha(pt)^{-d}}\Big\}. \end{aligned}$$

In the case $p<2$, a similar calculation shows that

$$\Big\langle \mathbb{E}\big[e^{2\int_0^t \xi_{tp}(X(s))\,ds}\big]^{p/2}\Big\rangle \le e^{\frac{p}{2}H(2t)}\exp\Big\{-\frac{H(pt\alpha(pt)^{-d})}{\alpha(pt)^{-d}}\Big\}.$$

Hence, for the choice $r(t)=t^2$, the first term on the right hand side of (3.17) satisfies

$$\limsup_{t\uparrow\infty}\frac{\alpha(pt)^2}{pt}\log\Big(\Big\langle \mathbb{E}\big[e^{2\int_0^t \xi_{tp}(X(s))\,ds}\big]^{p/2}\Big\rangle e^{-pr(t)}\Big)=-\infty, \tag{3.18}$$

where we use that $H(t)/t$ and $\alpha(t)$ are slowly varying.

In (3.17), take the logarithm, multiply by $\alpha(pt)^2/(pt)$ and let $t\uparrow\infty$. Then we have that

$$\limsup_{t\uparrow\infty}\frac{\alpha(pt)^2}{pt}\log\big\langle u^{\xi_{pt}}(t,0)^p\big\rangle \le \frac{C}{R^2}+\limsup_{t\uparrow\infty}\frac{\alpha(pt)^2}{pt}\log\big\langle \exp\{pt\lambda^{\mathrm{d}}_{0;2R\alpha(pt)}(\xi_{pt})\}\big\rangle, \tag{3.19}$$

where we also used that $r(t)^{pd+d}=e^{o(t\alpha(pt)^{-2})}$ as $t\uparrow\infty$. Now we estimate the right hand side of (3.19). We denote by $\lambda^{\mathrm{d},k}_{0;R\alpha(pt)}(\xi_{pt})$ the k^{th} eigenvalue of $\Delta^{\mathrm{d}}+\xi_{pt}$ in the box $B_{R\alpha(pt)}$ with zero boundary condition. Using an eigenfunction expansion as in (3.7), we get

$$\begin{aligned}
\Big\langle \exp\big\{pt\lambda^{\mathrm{d}}_{0;R\alpha(pt)}(\xi_{pt})\big\}\Big\rangle &\le \Big\langle \sum_k \exp\big\{pt\lambda^{\mathrm{d},k}_{0;R\alpha(pt)}(\xi_{pt})\big\}\Big\rangle = \sum_{x\in B_{R\alpha(pt)}}\big\langle p^{\xi_{pt}}_{R\alpha(pt)}(pt,x,x)\big\rangle\\
&\le \sum_{x\in B_{R\alpha(pt)}}\big\langle u^{\xi_{pt}}_{R\alpha(pt)}(pt,x)\big\rangle\\
&\le \sum_{x\in B_{R\alpha(pt)}}\Big\langle \mathbb{E}_x\Big[e^{\int_0^{pt}\xi_{pt}(X(s))\,ds}\mathbb{1}\{\operatorname{supp}(\ell_{pt})\subseteq x+B_{2R\alpha(pt)}\}\Big]\Big\rangle\\
&\le |B_{R\alpha(pt)}|\big\langle u^{\xi_{pt}}_{2R\alpha(pt)}(pt,0)\big\rangle,
\end{aligned}\tag{3.20}$$

where we also used the shift-invariance. Recall that $|B_{R\alpha(pt)}|\le e^{o(t\alpha(pt)^{-2})}$. We finally use Proposition 3.1(i) for pt instead of t to complete the proof of the upper bound in (3.4). □

3.2 Proof of Proposition 3.1(i)

Recall the local times of the walk, $\ell_t(z)=\int_0^t \mathbb{1}\{X(s)=z\}\,ds$. Note that $\int_0^t V(X(s))\,ds=\langle V,\ell_t\rangle$, where $\langle\cdot,\cdot\rangle$ stands for the inner product on $\ell^2(\mathbb{Z}^d)$. From (3.5) with $V=\xi_t$, we have

$$\big\langle u^{\xi_t}_{R\alpha(t)}(t,0)\big\rangle=\big\langle \mathbb{E}_0\big[e^{\langle\xi_t,\ell_t\rangle}\mathbb{1}\{\operatorname{supp}(\ell_t)\subseteq B_{R\alpha(t)}\}\big]\big\rangle. \tag{3.21}$$

Recall from (1.5) that $\langle e^{l\xi(x)}\rangle=e^{H(l)}$ for any $l\in\mathbb{R}$ and $x\in\mathbb{Z}^d$. We carry out the expectation with respect to the potential, and obtain, using Fubini's theorem and the independence of the potential variables,

$$\begin{aligned}
\big\langle u^{\xi_t}_{R\alpha(t)}(t,0)\big\rangle &= e^{-\alpha(t)^dH(t/\alpha(t)^d)}\mathbb{E}_0\Big[\Big\langle e^{\sum_{x\in R\alpha(t)}\ell_t(x)\xi(x)}\Big\rangle\mathbb{1}\{\operatorname{supp}(\ell_t)\subseteq B_{R\alpha(t)}\}\Big]\\
&=\mathbb{E}_0\Big[\exp\Big\{\sum_{x\in B_{R\alpha(t)}}\Big[H(\ell_t(x))-\ell_t(x)\tfrac{\alpha(t)^d}{t}H(t/\alpha(t)^d)\Big]\Big\}\mathbb{1}\{\operatorname{supp}(\ell_t)\subseteq B_{R\alpha(t)}\}\Big],
\end{aligned}\tag{3.22}$$

where we also use that $\sum_{x\in\mathbb{Z}^d}\ell_t(x)=t$. We now split the sum in the exponent into a part where we have some control over the size of the local times, and a part with very large local times. Introducing

$$\begin{aligned}\mathcal{H}_M^{(t)}(\ell_t) &= \frac{\alpha(t)^2}{t}\sum_{x\in B_{R\alpha(t)}}\Big[H(\ell_t(x))-\ell_t(x)\tfrac{\alpha(t)^d}{t}H(t/\alpha(t)^d)\Big]\mathbb{1}\{\ell_t(x)\le\tfrac{Mt}{\alpha(t)^d}\}, &(3.23)\\ \mathcal{R}_M^{(t)}(\ell_t) &= \sum_{x\in B_{R\alpha(t)}}\Big[H(\ell_t(x))-\ell_t(x)\tfrac{\alpha(t)^d}{t}H(t/\alpha(t)^d)\Big]\mathbb{1}\{\ell_t(x)>\tfrac{Mt}{\alpha(t)^d}\}, &(3.24)\end{aligned}$$

we have

$$\langle u^{\xi_t}_{R\alpha(t)}(t,0)\rangle=\mathbb{E}_0\Big[\exp\Big\{\frac{t}{\alpha(t)^2}\mathcal{H}_M^{(t)}(\ell_t)+\mathcal{R}_M^{(t)}(\ell_t)\Big\}\mathbb{1}\{\mathrm{supp}\,(\ell_t)\subseteq B_{R\alpha(t)}\}\Big],$$

We will see that $\mathcal{H}_M^{(t)}$ gives the main term and $\mathcal{R}_M^{(t)}$ a small remainder in the limit $t\to\infty$, followed by $M\to\infty$. To separate the two factors coming from this split, we use Hölder's inequality. For any small $\eta>0$, we have

$$\begin{aligned}\langle u^{\xi_t}_{R\alpha(t)}(t,0)\rangle\le\mathbb{E}_0\Big[\exp\Big\{(1+\eta)\frac{t}{\alpha(t)^2}\mathcal{H}_M^{(t)}(\ell_t)\Big\}\mathbb{1}\{\mathrm{supp}\,(\ell_t)\subseteq B_{R\alpha(t)}\}\Big]^{\frac{1}{1+\eta}}\\ \times\,\mathbb{E}_0\Big[\exp\Big\{\tfrac{1+\eta}{\eta}\mathcal{R}_M^{(t)}(\ell_t)\Big\}\mathbb{1}\{\mathrm{supp}\,(\ell_t)\subseteq B_{R\alpha(t)}\}\Big]^{\frac{\eta}{1+\eta}}.\end{aligned}\tag{3.25}$$

We show later that the second factor is asymptotically negligible, more precisely, we show that

$$\limsup_{M\to\infty}\limsup_{t\to\infty}\frac{\alpha(t)^2}{t}\log\mathbb{E}_0\Big[\exp\Big\{C\mathcal{R}_M^{(t)}(\ell_t)\Big\}\mathbb{1}\{\mathrm{supp}\,(\ell_t)\subseteq B_{R\alpha(t)}\}\Big]=0,\qquad\text{for } C>0.\tag{3.26}$$

Let us first focus on the first term. Recall the definition of $\alpha(t)$ in (1.9) and the uniform convergence claimed in Proposition 1.1. For every $\varepsilon>0$ and all sufficiently large times t, we obtain the upper bound

$$\begin{aligned}\mathcal{H}_M^{(t)}(\ell_t)&\le\tfrac{\alpha(t)^2}{t}\kappa(\tfrac{t}{\alpha(t)^d})\,\rho\sum_{x\in B_{R\alpha(t)}}\tfrac{\ell_t(x)}{t/\alpha(t)^d}\log\big(\tfrac{\ell_t(x)}{t/\alpha(t)^d}\big)\mathbb{1}\{\ell_t(x)\le\tfrac{Mt}{\alpha(t)^d}\}+\varepsilon\,(2R)^d\alpha(t)^d\,\tfrac{\alpha(t)^2}{t}\kappa(\tfrac{t}{\alpha(t)^d})\\ &\le\rho\sum_{x\in B_{R\alpha(t)}}\tfrac1t\ell_t(x)\log\big(\tfrac1t\ell_t(x)\,\alpha(t)^d\big)+\varepsilon\,(2R)^d=G_t(\tfrac1t\ell_t)+\varepsilon\,(2R)^d,\end{aligned}\tag{3.27}$$

where we dropped the indicator, which we can do for $M\ge1$ since $y\log y\ge0$ for $y>M$, and let

$$G_t(\mu)=\rho\sum_{x\in B_{R\alpha(t)}}\mu(x)\log\big(\alpha(t)^d\mu(x)\big),\qquad\text{for }\mu\in\mathcal{M}(\mathbb{Z}^d).\tag{3.28}$$

The further analysis makes crucial use of an inequality derived in [BHK05]. In [BHK05], the law of the local times are investigated, and an explicit formula is derived for the density of the local times on the range of the random walk. This explicit formula makes it possible to give strong upper bounds on exponential functionals:

Proposition 3.3. *For any finite set $B\subseteq\mathbb{Z}^d$ and any measurable functional $F\colon\mathcal{M}_1(B)\to\mathbb{R}$,*

$$\mathbb{E}_0\Big[e^{tF(\frac1t\ell_t)}\,\mathbb{1}\{\mathrm{supp}\,(\ell_t)\subseteq B\}\Big]\le\exp\Big\{t\sup_{\mu\in\mathcal{M}_1(B)}\Big[F(\mu)-\tfrac12\sum_{x\sim y}\big(\sqrt{\mu(x)}-\sqrt{\mu(y)}\big)^2\Big]\Big\}\,(2dt)^{|B|}|B|.\tag{3.29}$$

We substitute (3.27) into (3.25) and apply (3.29) for $F=(1+\eta)G_t/\alpha(t)^2$ and $B=B_{R\alpha(t)}$ and note that $(2dt)^{|B_{R\alpha(t)}|}\le e^{o(t/\alpha(t)^2)}$. Hence, we obtain that the first term on the right hand side of (3.25) can be estimated by

$$\begin{aligned} &\mathbb{E}_0\Big[\exp\Big\{(1+\eta)\frac{t}{\alpha(t)^2}\mathcal{H}_M^{(t)}(\ell_t)\Big\}1\!\!1\{\operatorname{supp}(\ell_t)\subseteq B_{R\alpha(t)}\}\Big] \\ &\qquad\le e^{o(t/\alpha(t)^2)}\exp\Big\{-t\Big[\chi^{\rm d}\big(\tfrac{\widetilde\rho}{\alpha(t)^2}\big)-\tfrac{1}{\alpha(t)^2}\big(d\widetilde\rho\log\alpha(t)+\varepsilon\,(2R)^d\big)\Big]\Big\}, \end{aligned} \tag{3.30}$$

where we abbreviated $\widetilde\rho=(1+\eta)\rho$ and introduced

$$\chi^{\rm d}(\delta)=\inf_{\mu\in\mathcal{M}_1(\mathbb{Z}^d)}\Big[\tfrac12\sum_{x\sim y}\big(\sqrt{\mu(x)}-\sqrt{\mu(y)}\big)^2-\delta\sum_{x\in\mathbb{Z}^d}\mu(x)\log\mu(x)\Big],\qquad\text{for }\delta>0, \tag{3.31}$$

the discrete variant of $\chi(\rho)$ in (1.23), which was studied in Gärtner and den Hollander [GH99]. In Proposition 3 and the subsequent remark they show that

$$\chi^{\rm d}(\delta)=\frac{d\delta}{2}\Big(\log\frac{\pi e^2}{\delta}+o(1)\Big),\qquad\text{as }\delta\downarrow 0.$$

Substituting this into (3.30), we obtain

$$\begin{aligned} &\limsup_{t\to\infty}\frac{\alpha(t)^2}{t}\log\mathbb{E}_0\Big[\exp\Big\{(1+\eta)\frac{t}{\alpha(t)^2}\mathcal{H}_M^{(t)}(\ell_t)\Big\}1\!\!1\{\operatorname{supp}(\ell_t)\subseteq B_{R\alpha(t)}\}\Big] \\ &\qquad\le-\frac{\widetilde\rho d}{2}\log\frac{\pi e^2}{\widetilde\rho}+\varepsilon(2R)^d=-\chi(\widetilde\rho)+\varepsilon(2R)^d, \end{aligned} \tag{3.32}$$

as can be seen from Proposition 1.11. Using (3.32) together with (3.26) in (3.25) and letting $M\to\infty$, $\varepsilon\downarrow 0$ and $\eta\downarrow 0$, gives the desired upper bound and finishes the proof of Proposition 3.1(i) subject to the proof of (3.26).

It remains to investigate the second term in (3.25), i.e., to prove (3.26). We first estimate $\mathcal{R}_M^{(t)}(\ell_t)$ (recall (3.24)) from above in terms of a nice functional of ℓ_t. Since we have to work uniformly for arbitrarily large local times, it is not possible to estimate against a functional of the form $\sum_x\ell_t(x)\log\ell_t(x)$, but we succeed in finding an upper bound of the form $(\sum_x\ell_t(x)^q)^{1/q}$ for some $q>1$ close to 1. Then Proposition 2.1 can be applied and yields (3.26).

We fix $\delta\in(0,\frac12]$ and note that there exist $A>1$, $t_0>0$ such that

$$\frac{H(ty)-yH(t)}{\kappa(t)}\le Ay^{1+\delta^2/3}\qquad\text{for any }y\ge 1\text{ and }t>t_0. \tag{3.33}$$

Indeed, this follows from [BGT87, Theorem 3.8.6(a)]. Therefore, we obtain that

$$H(\ell_t(x))-\ell_t(x)\tfrac{\alpha(t)^d}{t}H\big(\tfrac{t}{\alpha(t)^d}\big)\le A\kappa(t\alpha(t)^{-d})\Big(\frac{\ell_t(x)}{t\alpha(t)^{-d}}\Big)^{1+\delta^2/3}. \tag{3.34}$$

We pick now $\varepsilon>0$ such that

$$1+\delta^2/3-\varepsilon=1/(1+\delta). \tag{3.35}$$

For any $\mu \in \mathcal{M}_1(\mathbb{Z}^d)$, we use Jensen's inequality together with (3.35) as follows:

$$\begin{aligned}
\sum_{x\colon \mu(x)>M} \mu(x)^{1+\delta^2/3} &= \Big(\sum_{x\colon \mu(x)>M} \mu(x)^{\varepsilon}\Big) \sum_{x\colon \mu(x)>M} \frac{\mu(x)^{\varepsilon}}{\sum_{x\colon \mu(x)>M}\mu(x)^{\varepsilon}} \mu(x)^{1+\delta^2/3-\varepsilon} \\
&\le \Big(\sum_{x\colon \mu(x)>M} \mu(x)^{\varepsilon}\Big)\Big(\sum_{x\colon \mu(x)>M} \frac{\mu(x)^{1+\varepsilon}}{\sum_{x\colon \mu(x)>M}\mu(x)^{\varepsilon}}\Big)^{\frac{1}{1+\delta}} \\
&= \Big(\sum_{x\colon \mu(x)>M} \mu(x)^{\varepsilon}\Big)^{1-\frac{1}{1+\delta}}\Big(\sum_{x\colon \mu(x)>M} \mu(x)^{1+\varepsilon}\Big(\frac{\mu(x)}{M}\Big)^{\delta-\varepsilon}\Big)^{\frac{1}{1+\delta}} \\
&\le M^{\frac{\delta}{1+\delta}(\varepsilon-1)} M^{\frac{\varepsilon-\delta}{1+\delta}}\Big(\sum_x \mu(x)^{1+\delta}\Big)^{\frac{1}{1+\delta}} = M^{\varepsilon-\frac{2\delta}{1+\delta}}\Big(\sum_x \mu(x)^{1+\delta}\Big)^{\frac{1}{1+\delta}},
\end{aligned} \tag{3.36}$$

where we used in the last step that in the first integral on the right, $\mu^{\varepsilon} \le M^{\varepsilon-1}\mu$ on $\{\mu > M\}$, and hence the first term on the right is not bigger than one, as the exponent is positive and $\mu \in \mathcal{M}_1(\mathbb{Z}^d)$. We write $q = 1+\delta$. We apply the above to $\mu = \frac{1}{t}\ell_t$ and M replaced by $\frac{M}{\alpha(t)^d}$, to obtain that

$$\begin{aligned}
\sum_x \Big(\frac{\ell_t(x)}{t\alpha(t)^{-d}}\Big)^{1+\delta^2/3} \mathbb{1}\{\ell_t(x) > \tfrac{Mt}{\alpha(t)^d}\} &\le \alpha(t)^{d(1+\delta^2/3)}\big(\tfrac{M}{\alpha(t)^d}\big)^{\varepsilon-\frac{2\delta}{1+\delta}} t^{-1}\|\ell_t\|_q \\
&= M^{\varepsilon-\frac{2\delta}{1+\delta}}\alpha(t)^{d(1+\frac{\delta}{1+\delta})} t^{-1}\|\ell_t\|_q.
\end{aligned} \tag{3.37}$$

We recall (3.24), use (3.34) and the definition of $\alpha(t)$ in (1.9). With the help of (3.37) we arrive at

$$\begin{aligned}
\mathcal{R}_M^{(t)}(\ell_t) &\le \kappa(\tfrac{t}{\alpha(t)^d}) \sum_x \Big(\frac{\ell_t(x)}{t\alpha(t)^{-d}}\Big)^{1+\delta^2/3} \mathbb{1}\{\ell_t(x) > A\tfrac{Mt}{\alpha(t)^d}\} \\
&\le AM^{\varepsilon-\frac{2\delta}{1+\delta}}\alpha(t)^{-(2+d)+d(1+\frac{\delta}{1+\delta})}\|\ell_t\|_q \\
&= AM^{\varepsilon-\frac{2\delta}{1+\delta}}\alpha(t)^{-\frac{1}{q}[d+(2-d)q]}\|\ell_t\|_q,
\end{aligned}$$

where we recall that $q = 1+\delta$ and therefore $-(2+d)+d(1+\frac{\delta}{1+\delta}) = -\frac{1}{q}[d+(2-d)q]$. Put $\theta = AM^{\varepsilon-\frac{2\delta}{1+\delta}}$, and observe that $\theta \downarrow 0$ as $M \uparrow \infty$ for $\delta > 0$ small enough, since $\varepsilon - \frac{2\delta}{1+\delta} = \frac{\delta^2}{3} - \frac{\delta}{1+\delta} < 0$ for $\delta > 0$ small enough. Hence, (3.26) follows immediately from Proposition 2.1. This completes the proof of Proposition 3.1(i).

3.3 Proof of Proposition 3.1(ii)

Recall from (1.20) the rescaled version, $\overline{\xi}_t$, of the vertically shifted potential, ξ_t, defined in (3.3). Furthermore, introduce the normalised, scaled version of the random walk local times,

$$L_t(x) := \frac{\alpha(t)^d}{t}\,\ell_t\big(\lfloor x\alpha(t)\rfloor\big), \qquad \text{for } x \in \mathbb{R}^d,$$

and note that L_t is an L^1-normalised random step function. Note that $\operatorname{supp}(L_t) \subseteq Q_R$ if $\operatorname{supp}(\ell_t) \subseteq B_{R\alpha(t)}$ where we abbreviated $Q_R = [-R,R]^d$. We start from (3.22). Let

$$\widehat{H}_t(y) = \frac{H\big(y\frac{t}{\alpha(t)^d}\big) - yH\big(\frac{t}{\alpha(t)^d}\big)}{\kappa\big(\frac{t}{\alpha(t)^d}\big)}, \qquad \text{for } t, y > 0,$$

and recall that $\widehat{H}_t$ converges to $\widehat{H}$, uniformly on all compact sets. Now the exponent on the right hand side of (3.22) can be rewritten as follows.

$$\begin{aligned}-\alpha(t)^d H\big(\tfrac{t}{\alpha(t)^d}\big) + \sum_{z\in B_{R\alpha(t)}} H(\ell_t(z)) &= -\alpha(t)^d \int_{Q_R} L_t(x)\, H\big(\tfrac{t}{\alpha(t)^d}\big)\, dx + \alpha(t)^d \int_{Q_R} H\big(\tfrac{t}{\alpha(t)^d} L_t(x)\big)\, dx \\ &= \alpha(t)^d \kappa\big(\tfrac{t}{\alpha(t)^d}\big) \int_{Q_R} \widehat{H}_t\big(L_t(x)\big)\, dx = \tfrac{t}{\alpha(t)^2} \mathcal{H}_R^{(t)}(L_t),\end{aligned}$$

where we use the definition of $\alpha(t)$ in (1.9) and introduce the functional

$$\mathcal{H}_R^{(t)}(f) = \int_{Q_R} \widehat{H}_t\big(f(x)\big)\, dx.$$

Hence,

$$\big\langle u^{\xi_t}_{R\alpha(t)}(t,0)\big\rangle = \mathbb{E}_0\Big[\exp\Big(\frac{t}{\alpha(t)^2}\mathcal{H}_R^{(t)}(L_t)\Big)\mathbb{1}\{\mathrm{supp}\,(L_t)\subseteq Q_R\}\Big]. \tag{3.38}$$

A key ingredient in the proof of Proposition 3.1(ii) is the large deviation principle for $(L_t\colon t>0)$ as formulated in the following proposition:

Proposition 3.4. *Fix $R>0$. Under $\mathbb{P}_0\{\,\cdot\,,\mathrm{supp}\,(L_t)\subseteq Q_R\}$, the rescaled local times process $(L_t\colon t>0)$ satisfies a large deviation principle as $t\uparrow\infty$ on the set of L^1-normalized functions $Q_R\to\mathbb{R}$, equipped with the weak topology induced by test integrals against all continuous functions, where the speed of the large deviation principle is $t\alpha(t)^{-2}$, and the rate function is $g^2\mapsto\|\nabla g\|_2^2$, on the set of all $g\in H^1(\mathbb{R}^d)$ with* $\mathrm{supp}\,(g)\subseteq Q_R$, *and is equal to ∞ outside this set.*

Proof. This large deviation principle is stated in [GKS04, Lemma 3.2] in the discrete-time case, and is proved in [GKS04, Section 6]. The proof in the continuous-time case is very similar. □

In order to apply the large deviation principle in Proposition 3.4 to obtain a lower bound for the right hand side of (3.38), we need the lower-bound half of Varadhan's lemma, and we have to replace $\mathcal{H}_R^{(t)}$ by its limiting version

$$\mathcal{H}_R(f) = \rho\int_{Q_R} f(x)\log f(x)\, dx. \tag{3.39}$$

However, the latter is technically not so easy. Inserting the indicator on the event $\{\|L_t\|_\infty<M\}$ for any $M>1$ would make it possible to use the locally uniform convergence of $\widehat{H}_t(y)$ towards $\rho y\log y$, but this event is not open in the topology of the large deviation principle. Therefore, similarly to the proof of the upper bound, we have to split $\mathcal{H}_R^{(t)}(L_t)$ into the sum of $\mathcal{H}_R(L_t)$ and a remainder term, separate these two from each other by the use of Hölder's inequality and apply Proposition 2.1 to the remainder term. Let us turn to the details.

Since H is convex with $H(0)=0$, we have $H(yt)\geq yH(t)$ for all $t>0$ and all $y\geq1$. Therefore, $\widehat{H}_t(f(x))\geq0$ on $\{x\colon f(x)>M\}$ for any $M>1$. Hence, we may estimate

$$\begin{aligned}\mathcal{H}_R^{(t)}(f) &\geq \int_{Q_R}\mathbb{1}\{f(x)\leq M\}\widehat{H}_t\big(f(x)\big)\, dx = \rho\int_{Q_R}\mathbb{1}\{f(x)\leq M\}f(x)\log f(x)\, dx + o(1)\\ &= \mathcal{H}_R(f) - \rho\int_{Q_R}\mathbb{1}\{f(x)>M\}f(x)\log f(x)\, dx + o(1).\end{aligned}$$

The remainder can be estimated, for any $\delta > 0$, as follows. For any $f\colon Q_R \to [0,\infty)$ satisfying $\int f = 1$,

$$\begin{aligned}\int_{f>M} f\log f &= \frac{2}{\delta}\Big(\int_{f>M} f\Big)\int_{f>M}\frac{f}{\int_{f>M} f}\log f^{\delta/2} \le \frac{2}{\delta}\Big(\int_{f>M} f\Big)\log\frac{\int_{f>M} f^{1+\delta/2}}{\int_{f>M} f}\\ &\le \frac{2}{\delta}\Big(\int_{f>M} f\Big)\log\frac{M^{-\delta/2}\int_{f>M} f^{1+\delta}}{\int_{f>M} f} \le \frac{2}{\delta}\Big(\int_{f>M} f\Big)\Big(\frac{M^{-\delta/2}\int_{f>M} f^{1+\delta}}{\int_{f>M} f}\Big)^{\frac{1}{1+\delta}}\\ &= \frac{2}{\delta}M^{-\frac{\delta}{2+2\delta}}\Big(\int_{f>M} f\Big)^{\frac{\delta}{1+\delta}}\|f\|_q \le \frac{2}{\delta}M^{-\frac{\delta}{2+2\delta}}\|f\|_q,\end{aligned}$$

where we put $q = 1+\delta$. Altogether, we have, abbreviating $\theta = 2\frac{\varrho}{\delta}M^{-\frac{\delta}{2+2\delta}}$,

$$\big\langle u^{\xi_t}_{R\alpha(t)}(t,0)\big\rangle \ge \mathbb{E}_0\Big[\exp\Big(\frac{t}{\alpha(t)^2}\big(\mathcal{H}_R(L_t) - \theta\|L_t\|_q\big)\Big)1\!\!1\{\mathrm{supp}\,(L_t)\subseteq Q_R\}\Big]e^{o(t/\alpha(t)^2)}. \tag{3.40}$$

Similarly to the proof of the upper bound, the main contribution will turn out to come from $\mathcal{H}_R$, and the q-norm is a small remainder. In order to separate the two from each other, we use Hölder's inequality to estimate, for some small $\eta > 0$,

$$\begin{aligned}&\mathbb{E}_0\Big[\exp\Big(\frac{t}{\alpha(t)^2}(1-\eta)\mathcal{H}_R(L_t)\Big)1\!\!1\{\mathrm{supp}\,(L_t)\subseteq Q_R\}\Big]\\ &\quad\le \mathbb{E}_0\Big[\exp\Big(\frac{t}{\alpha(t)^2}\big(\mathcal{H}_R(L_t)-\theta\|L_t\|_q\big)\Big)1\!\!1\{\mathrm{supp}\,(L_t)\subseteq Q_R\}\Big]^{1-\eta}\\ &\qquad\times \mathbb{E}_0\Big[\exp\Big(\frac{t}{\alpha(t)^2}\frac{1-\eta}{\eta}\theta\|L_t\|_q\Big)1\!\!1\{\mathrm{supp}\,(L_t)\subseteq Q_R\}\Big]^{\eta}.\end{aligned} \tag{3.41}$$

This effectively yields a lower bound on the expected value in (3.40) of the form

$$\begin{aligned}\big\langle u^{\xi_t}_{R\alpha(t)}(t,0)\big\rangle &\ge \mathbb{E}_0\Big[\exp\Big(\frac{t}{\alpha(t)^2}(1-\eta)\mathcal{H}_R(L_t)\Big)1\!\!1\{\mathrm{supp}\,(L_t)\subseteq Q_R\}\Big]^{\frac{1}{1-\eta}}\\ &\quad\times \mathbb{E}_0\Big[\exp\Big(\frac{t}{\alpha(t)^2}\frac{1-\eta}{\eta}\theta\|L_t\|_q\Big)1\!\!1\{\mathrm{supp}\,(L_t)\subseteq Q_R\}\Big]^{-\frac{\eta}{1-\eta}}e^{o(t/\alpha(t)^2)}.\end{aligned} \tag{3.42}$$

From Proposition 2.1 it follows that the second expectation on the right is negligible in the limit $t\to\infty$, followed by $M\to\infty$, i.e., $\theta\downarrow 0$. Hence, we can concentrate on the first term. To apply the lower-bound half of Varadhan's lemma, see [DZ98, Lemma 4.3.4], we need the following lower semi-continuity property of the function $\mathcal{H}_R$:

Lemma 3.5. *Let $f\colon Q_R\to[0,\infty)$ be continuous. Then $\mathcal{H}_R$ is lower semi-continuous in f in the topology induced by pairing with all continuous functions $Q_R\to[0,\infty)$.*

Proof. Let $(f_n\colon n\in\mathbb{N})$ be a family in $L^1(Q_R)$ such that $\langle f_n,\psi\rangle \to \langle f,\psi\rangle$ as $n\to\infty$ for any continuous function $\psi\colon Q_R\to\mathbb{R}$. We have to show that $\liminf_{n\to\infty}\mathcal{H}_R(f_n)\ge\mathcal{H}_R(f)$.

For any $s\in(0,\infty)$ we denote by g_s the tangent to $y\mapsto\phi(y) := \rho y\log y$ in s, i.e., $g_s(y) = \rho(1+\log s)y - \rho s$, for all $y\in\mathbb{R}$. By convexity we have $g_s\le\phi$ for any $s\in(0,\infty)$. Therefore, for any $0<\varepsilon<1/e$,

$$\mathcal{H}_R(f_n) = \int_{Q_R}\phi\big(f_n(x)\big)\,dx \ge \int_{Q_R} g_{f(x)\vee\varepsilon}\big(f_n(x)\big)\,dx = \rho\big\langle 1+\log(f\vee\varepsilon), f_n\big\rangle - \rho\big\langle f\vee\varepsilon, f_n\big\rangle.$$

Letting $n \to \infty$, we obtain, using the boundedness and continuity of $\log(f \vee \varepsilon)$,

$$\begin{aligned}\liminf_{n\to\infty} \mathcal{H}_R(f_n) &\geq \rho\big\langle 1+\log(f\vee\varepsilon), f\big\rangle - \rho\big\langle f\vee\varepsilon, f\big\rangle\\ &\geq \rho \int_{Q_R} f(x)\log f(x) 1\!\!1_{\{f(x)>\varepsilon\}}\,dx + \int_{Q_R} g_\varepsilon(f(x)) 1\!\!1_{\{f(x)\leq\varepsilon\}}\,dx\\ &\geq \rho\int_{Q_R} f(x)\log f(x)\,dx + \int_{Q_R}\big(f(x)(1+\log\varepsilon)-\varepsilon\big)1\!\!1_{\{f(x)\leq\varepsilon\}}\,dx.\end{aligned}$$

The second summand is bounded from below by $\mathrm{Leb}(Q_R)\varepsilon\log\varepsilon$, which converges to zero as $\varepsilon \downarrow 0$. This completes the proof. ☐

Now we can apply [DZ98, Lemma 4.3.4] and obtain

$$\begin{aligned}\liminf_{t\to\infty}\frac{\alpha(t)^2}{t}\log\mathbb{E}_0\Big[\exp\Big(\frac{t}{\alpha(t)^2}(1-\eta)\mathcal{H}_R(L_t)\Big)1\!\!1\{\mathrm{supp}\,(L_t)\subseteq Q_R\}\Big]\\ \geq -\inf\Big\{\|\nabla g\|_2^2 - (1-\eta)\mathcal{H}_R(g^2)\colon g\in H^1(\mathbb{R}^d)\cap\mathcal{C}(\mathbb{R}^d), \|g\|_2=1, \mathrm{supp}\,(g)\subseteq Q_R\Big\}.\end{aligned}$$

Letting $\eta \downarrow 0$ and $R \uparrow \infty$, it is easy to see that the right hand side tends to $-\chi(\rho)$ defined in (1.23). Indeed, use appropriate continuous cut-off versions $g^{(R)}_{(1-\eta)\rho}$ of the minimiser $g_{(1-\eta)\rho}$ in (1.26) to verify this claim. Using this on the right hand side of (3.42) and recalling Proposition 2.1, we see that the proof of the lower bound in Proposition 3.1(ii) is finished.

4. The almost-sure asymptotics: Proof of Theorem 1.7

We again derive upper and lower bounds, following the strategy in [BK01, Section 5]. Recall the scale function $\beta(t)$ defined in (1.15) and let

$$\xi_{\beta(t)}(z) = \xi(z) - H\Big(\frac{\beta(t)}{\alpha(\beta(t))^d}\Big)\frac{\alpha(\beta(t))^d}{\beta(t)} \tag{4.1}$$

denote the appropriately vertically shifted potential (compare to (3.3)). Then Theorem 1.7 is equivalent to the assertion

$$\lim_{t\uparrow\infty}\frac{\alpha(\beta(t))^2}{t}\log u^{\xi_{\beta(t)}}(t,0) = -\widetilde{\chi}(\rho), \qquad \text{almost surely,} \tag{4.2}$$

where $\widetilde{\chi}(\rho) = \rho(d - \frac{d}{2}\frac{\rho}{\pi} + \log\frac{\rho}{e}) = -\sup\{\lambda(\psi)\colon \psi\in\mathcal{C}(\mathbb{R}^d), \mathcal{L}(\psi)\leq 1\}$, see Section 1.6.2.

4.1 Proof of the upper bound in (4.2)

Let $r(t) = t\log t$ and apply Lemma 3.2 with $V = \xi_{\beta(t)}$ and with R replaced by $R\alpha(\beta(t))$. Furthermore, take logarithms, multiply with $\alpha(\beta(t))^2/t$ and let $t\uparrow\infty$. As in (3.18), one shows that the first term is negligible. Hence, we obtain that

$$\limsup_{t\uparrow\infty}\frac{\alpha(\beta(t))^2}{t}\log u^{\xi_{\beta(t)}}(t,0) \leq \frac{C}{R^2} + \limsup_{t\uparrow\infty}\Big[\alpha(\beta(t))^2\max_{z\in B(t)}\lambda_{z;2R\alpha(\beta(t))}(\xi_{\beta(t)})\Big],$$

where $B(t) = B_{R\alpha(\beta(t))}(t)$ (recall the definition $B_R(t) = B_{r(t)+\lfloor 2R\rfloor}$ from Lemma 3.2). Let $(\lambda_i(t)\colon i=1,\dots,N(t))$, with $N(t) = |B_R(t)|$, be a deterministic enumeration of the random variables $\lambda_{z;2R\alpha(\beta(t))}(\xi_{\beta(t)})$ with $z\in B(t)$. Note that these random variables are identically distributed (but not independent) and that, by (3.20) and Proposition 3.1(i), their exponential moments are estimated by

$$\limsup_{t\uparrow\infty} \frac{\alpha(\beta(t))^2}{\beta(t)} \log \left\langle e^{\beta(t)\,\lambda_1(t)} \right\rangle \le -\chi(\rho). \tag{4.3}$$

We next show that, for any $\varepsilon > 0$, almost surely,

$$\limsup_{t\uparrow\infty} \alpha(\beta(t))^2 \max_{i=1}^{N(t)} \lambda_i(t) \le -\widetilde{\chi}(\rho) + \varepsilon, \tag{4.4}$$

which completes the proof of the upper bound in (4.2).

To prove (4.4), one first realizes that is suffices to show (4.4) only for $t \in \{e^n : n \in \mathbb{N}\}$, since the functions $t \mapsto \alpha(t)$, $t \mapsto \beta(t)$, and $t \mapsto H(t)/t$ are slowly varying, and $t \mapsto N(t)$, $R \mapsto \lambda_R(\xi_{\beta(t)})$ are increasing. Let

$$p_n = \mathrm{Prob}\Big\{ \max_{i=1}^{N(e^n)} \lambda_i(e^n) \ge \frac{-\widetilde{\chi}(\rho)+\varepsilon}{\alpha(\beta(e^n))^2} \Big\}.$$

We recall that $\beta(e^n)\alpha(\beta(e^n))^{-2} \sim dn$. Using Chebyshev's inequality and (4.3), we estimate, for any $k > 0$,

$$\begin{aligned} p_n &\le N(e^n)\mathrm{Prob}\Big\{ e^{k\beta(e^n)\lambda_1(e^n)} \ge e^{-k\beta(e^n)\alpha(\beta(e^n))^{-2}(\widetilde{\chi}(\rho)-\varepsilon)} \Big\} \\ &\le e^{n(d+o(1))} e^{nkd(\widetilde{\chi}(\rho)-\varepsilon)} \Big\langle e^{k\beta(e^n)\lambda_1(e^n)} \Big\rangle. \end{aligned} \tag{4.5}$$

In order to evaluate the last expectation, we intend to apply (4.3) with $\beta(t)$ replaced by $k\beta(t)$. For this purpose, we note that we can replace $\alpha(\beta(t))$ by $\alpha(k\beta(t))$ in (4.3), since α is slowly varying. Also,

$$k\beta(t)\lambda_{R\alpha(k\beta(t))}(\xi_{\beta(t)}) = k\beta(t)\lambda_{R\alpha(k\beta(t))}(\xi_{k\beta(t)}) - k\beta(t)\Big[\frac{H(\beta(t)\alpha(\beta(t))^{-d})}{\beta(t)\alpha(\beta(t))^{-d}} - \frac{H(k\beta(t)\alpha(k\beta(t))^{-d})}{k\beta(t)\alpha(k\beta(t))^{-d}} \Big],$$

where we use that by (4.1), the field $\xi_{\beta(t)} - \xi_{k\beta(t)}$ is constant and deterministic. Now we use (1.6) and (1.9), to see that the deterministic term is equal to

$$\begin{aligned} k\beta(t)&\Big[\frac{H(\beta(t)\alpha(\beta(t))^{-d})}{\beta(t)\alpha(\beta(t))^{-d}} - \frac{H(k\beta(t)\alpha(k\beta(t))^{-d})}{k\beta(t)\alpha(k\beta(t))^{-d}} \Big] \\ &= \alpha(\beta(t))^d \Big(k\,H(\beta(t)\alpha(\beta(t))^{-d}) - H(k\beta(t)\alpha(\beta(t))^{-d}) \Big) + o(n) \\ &= -\alpha(\beta(t))^d \big(\widehat{H}(k) + o(1) \big)\, \kappa\big(\beta(t)\alpha(\beta(t))^{-d}\big) \, + o(n) \\ &= -\frac{\beta(t)}{\alpha(\beta(t))^2} \big(\rho k \log k + o(1)\big)(1+o(1)) \, + o(n) \\ &= -nd\,(\rho k \log k)(1+o(1)). \end{aligned}$$

Hence,

$$\Big\langle e^{k\beta(e^n)\lambda_1(e^n)} \Big\rangle \le \exp\Big\{ -nd\big(k\chi(\rho) - \rho k \log k + o(1) \big) \Big\}.$$

Using this in (4.5), we arrive at

$$p_n \le \exp\Big\{ nd\big(1 + k(\widetilde{\chi}(\rho)-\varepsilon) - k\chi(\rho) + \rho k \log k + o(1) \big) \Big\}.$$

Choosing $k = \frac{1}{\rho}$, we see that $p_n \le e^{-nd(k\varepsilon + o(1))}$. This is summable over $n \in \mathbb{N}$, and the Borel-Cantelli lemma yields that (4.4) holds almost surely. This completes the proof of the upper bound in (4.2).

4.2 Proof of the lower bound in (4.2)

Our proof of the lower bound in (4.2) follows the strategy of [BK01, Sect. 5.2]. First we establish that, with probability one, for any sufficiently large t, there is, inside a 'macrobox' of radius roughly t, centred at the origin, some 'microbox' of radius $R\alpha(\beta(t))$ in which the random field $\xi_{\beta(t)}$ has some shape with optimal spectral properties. Then we obtain a lower bound for the Feynman-Kac formula in (3.2) by requiring that the random walk moves quickly to that box and stays there for approximately t time units. As a result, the contribution from that strategy is basically given by the largest eigenvalue of $\Delta^{\mathrm{d}} + \xi$ in that microbox. Rescaling and letting $R \uparrow \infty$, the lower bound is derived from this.

Let us go to the details. We pick an increasing auxiliary scale function $t \mapsto \gamma_t$ satisfying

$$\begin{aligned} &\gamma_t = t^{1-o(1)}, \qquad t - \gamma_t = t(1+o(1)), \\ &\gamma_t = o\Big(\frac{t}{\alpha(\beta(t))^2}\Big), \qquad \gamma_t \frac{H(\beta(t)\alpha(\beta(t))^{-d})}{\beta(t)\alpha(\beta(t))^{-d}} = o\Big(\frac{t}{\alpha(\beta(t))^2}\Big). \end{aligned} \tag{4.6}$$

(Note that the second requirement follows from the third.) For example, $\gamma_t = t\alpha(\beta(t))^{-2}\varepsilon_t$ with some suitable $\varepsilon_t \downarrow 0$ as a small inverse power of $\log t$ satisfies (4.6). This is obvious in the case where $\lim_{s\uparrow\infty} H(s)/s = 0$, and in the case where $\lim_{s\uparrow\infty} H(s)/s = \infty$, it is also clear since $H(s)/s$ diverges only subpolynomially in s, while $\beta(t) = (\log t)^{1+o(1)}$ and α is slowly varying.

The crucial step is to show that, in the 'macrobox' B_{γ_t}, we find an appropriate 'microbox'. To fix some notation, let $Q_R = [-R, R]^d$ and let $\mathcal{C}(Q_R)$ denote the set of continuous functions $Q_R \to \mathbb{R}$. We need finite-space versions of the functionals $\mathcal{H}, \mathcal{L}$ and λ defined in (1.24) and (1.25). Recall the definition of $\mathcal{H}_R$ from (3.39) and define its Legendre transform $\mathcal{L}_R \colon \mathcal{C}(Q_R) \to (-\infty, \infty]$ by

$$\mathcal{L}_R(\psi) = \sup\big\{\langle f, \psi\rangle - \mathcal{H}_R(f) \colon f \in \mathcal{C}(Q_R),\, f \geq 0,\, \operatorname{supp} f \subseteq \operatorname{supp}\psi\big\}. \tag{4.7}$$

As in the proof of Proposition 1.11 one can see that $f = e^{\psi/\rho - 1}$ is the unique maximizer in (4.7) with

$$\mathcal{L}_R(\psi) = \frac{\rho}{e}\int_{Q_R} e^{\psi(x)/\rho}\, dx.$$

Proposition 4.1 (Existence of an optimal microbox)**.** *Fix $R > 0$ and let $\psi \in \mathcal{C}(Q_R)$ satisfy $\mathcal{L}_R(\psi) < 1$. Let $\varepsilon > 0$. Then, with probability one, there exists $t_0 > 0$, depending also on ξ, such that, for all $t > t_0$, there is $y_t \in B_{\gamma_t}$, depending on ξ, such that*

$$\xi_{\beta(t)}(y_t + z) \geq \frac{1}{\alpha(\beta(t))^2}\psi\Big(\frac{z}{\alpha(\beta(t))}\Big) - \frac{\varepsilon}{\alpha(\beta(t))^2}, \qquad \text{for } z \in B_{R\alpha(\beta(t))}. \tag{4.8}$$

The proof of Proposition 4.1 is deferred to the end of this section.

Now we finish the proof of the lower bound in (4.2) subject to Proposition 4.1. Let $R, \varepsilon > 0$, and let $\psi \in \mathcal{C}(Q_R)$ be twice continuously differentiable with $\mathcal{L}_R(\psi) < 1$. Fix ξ not belonging to the exceptional set of Proposition 4.1, i.e., let t_0 and $(y_t \colon t > t_0)$ in B_{γ_t} be chosen such that (4.8) holds for every $t > t_0$. Fix $t > t_0$. In the Feynman-Kac formula

$$u^{\xi_{\beta(t)}}(t, 0) = \mathbb{E}_0 \exp\Big\{\int_0^t \xi_{\beta(t)}(X(s))\, ds\Big\},$$

we obtain a lower bound by requiring that the random walk is at y_t at time γ_t and remains within the microbox

$$B_{y_t,t} = y_t + B_{R\alpha(\beta(t))}$$

during the time interval $[\gamma_t, t]$. Using the Markov property at time γ_t, we obtain by this the lower bound

$$\begin{aligned} u^{\xi_{\beta(t)}}(t,0) \geq \mathbb{E}_0\Big[\exp\Big\{\int_0^{\gamma_t} \xi_{\beta(t)}(X(s))\,ds\Big\}\delta_{y_t}(X(\gamma_t))\Big] \\ \times \mathbb{E}_{y_t}\Big[\exp\Big\{\int_0^{t-\gamma_t} \xi_{\beta(t)}(X(s))\,ds\Big\}\mathbb{1}\{\tau_{y_t,t} > t-\gamma_t\}\Big], \end{aligned} \tag{4.9}$$

where $\tau_{y_t,t} = \inf\{s > 0 \colon X(s) \notin B_{y_t,t}\}$ denotes the exit time from the microbox $B_{y_t,t}$. In the first expectation on the right side of (4.9), we estimate ξ from below by its minimum $K = \operatorname{essinf} \xi(0) > -\infty$, and in the second expectation we use (4.8) and shift spatially by y_t to obtain

$$\begin{aligned} u^{\xi_{\beta(t)}}(t,0) \geq \exp\Big\{\gamma_t\Big[K - \frac{H(\beta(t)\alpha(\beta(t))^{-d})}{\beta(t)\alpha(\beta(t))^{-d}}\Big]\Big\}\mathbb{P}_0\{X(\gamma_t) = y_t\} \\ \times e^{-\varepsilon(t-\gamma_t)\alpha(\beta(t))^{-2}}\mathbb{E}_0\Big[\exp\Big\{\int_0^{t-\gamma_t} \psi_t(X(s))\,ds\Big\}\mathbb{1}\{\tau_{0,t} > t-\gamma_t\}\Big], \end{aligned} \tag{4.10}$$

where we have denoted $\psi_t(\cdot) = \alpha(\beta(t))^{-2}\psi(\cdot\,\alpha(\beta(t))^{-1})$. By our choice in (4.6), the first term on the right side of (4.10) is $e^{o(t\alpha(\beta(t))^{-2})}$. Now, by choosing a path from the origin to y_t consisting of k steps for $k = \lfloor\gamma_t\rfloor$ or $k = \lfloor\gamma_t\rfloor + 1$,

$$\mathbb{P}_0\{X(\gamma_t) = y_t\} \geq \big(\tfrac{1}{2d}\big)^k\, \mathbb{P}\{\sigma(1) + \cdots + \sigma(k) \leq \gamma_t < \sigma(1) + \cdots + \sigma(k+1)\},$$

where $\sigma(1), \sigma(2), \dots$ are independent exponential random variables with mean $1/2d$. Using that

$$\mathbb{P}\big\{\sigma(1) + \cdots + \sigma(k) \leq \gamma_t < \sigma(1) + \cdots + \sigma(k+1)\big\} \geq \mathbb{P}\big\{\sigma(1) + \cdots + \sigma(k) \in [\tfrac{\gamma_t}{2}, \gamma_t)\big\}\,\mathbb{P}\big\{\sigma(0) \geq \tfrac{\gamma_t}{2}\big\},$$

and Cramér's theorem, we obtain the lower bound

$$\mathbb{P}_0\{X(\gamma_t) = y_t\} \geq e^{-\mathcal{O}(\gamma_t)} = e^{-o(t\alpha(\beta(t))^{-2})}.$$

By an eigenfunction expansion we have that

$$\begin{aligned} &\mathbb{E}_0\Big[\exp\Big\{\int_0^{t-\gamma_t} \psi_t(X(s))\,ds\Big\}\mathbb{1}\{\tau_{0,t} > t-\gamma_t\}\Big] \\ &\qquad \geq \mathbb{E}_0\Big[\exp\Big\{\int_0^{t-\gamma_t} \psi_t(X(s))\,ds\Big\}\mathbb{1}\{\tau_{0,t} > t-\gamma_t, X(t-\gamma_t) = 0\}\Big] \\ &\qquad \geq \exp\Big\{(t-\gamma_t)\lambda^{\mathrm{d}}(t)\Big\}\mathrm{e}_t(0)^2, \end{aligned}$$

where $\lambda^{\mathrm{d}}(t)$ is the principal eigenvalue of $\Delta^{\mathrm{d}} + \psi_t$ in the box $B_{R\alpha(\beta(t))}$ with zero boundary condition, and e_t is the corresponding positive ℓ^2-normalized eigenvector. Summarising these estimates and recalling from (4.6) that $t - \gamma_t = t(1 + o(1))$, we obtain, almost surely,

$$\liminf_{t\uparrow\infty} \frac{\alpha(\beta(t))^2}{t}\log u^{\xi_{\beta(t)}}(t,0) \geq -\varepsilon + \liminf_{t\uparrow\infty} \alpha(\beta(t))^2\lambda^{\mathrm{d}}(t) + \liminf_{t\uparrow\infty} \frac{\alpha(\beta(t))^2}{t}\log \mathrm{e}_t(0)^2. \tag{4.11}$$

We now define the continuous counterpart λ_R of $\lambda^{\mathrm{d}}(t)$, which is the finite-space version of the spectral radius defined in (1.25):

$$\lambda_R(\psi) = \sup\big\{\langle\psi, g^2\rangle - \|\nabla g\|_2^2 \colon g \in H^1(\mathbb{R}^d),\ \|g\|_2 = 1, \operatorname{supp} g \subseteq Q_R\big\}. \tag{4.12}$$

According to [BK01, Lemma 5.3],

$$\liminf_{t\uparrow\infty} \alpha(\beta(t))^2\lambda^{\mathrm{d}}(t) \geq \lambda_R(\psi) \qquad \text{and} \qquad \liminf_{t\uparrow\infty} \frac{\alpha(\beta(t))^2}{t}\log \mathrm{e}_t(0)^2 \geq 0.$$

Using this in (4.11), we obtain

$$\liminf_{t\uparrow\infty} \frac{\alpha(\beta(t))^2}{t} \log u^{\xi_{\beta(t)}}(t,0) \geq -\varepsilon + \lambda_R(\psi), \tag{4.13}$$

for any $\varepsilon > 0$ and for any twice continuously differentiable function $\psi \in \mathcal{C}^2(Q_R)$ satisfying $\mathcal{L}_R(\psi) < 1$. Hence,

$$\liminf_{t\uparrow\infty} \frac{\alpha(\beta(t))^2}{t} \log u^{\xi_{\beta(t)}}(t,0) \geq -\widetilde{\chi}_R,$$

where

$$\widetilde{\chi}_R = \inf\big\{-\lambda_R(\psi) \colon \psi \in \mathcal{C}^2(Q_R) \text{ and } \mathcal{L}_R(\psi) < 1\big\}. \tag{4.14}$$

It remains to show that, for any $\rho > 0$, we have $\limsup_{R\uparrow\infty} \widetilde{\chi}_R \leq \widetilde{\chi}(\rho)$. This can be seen as follows: By Proposition 1.13 the variational problem in (1.32) has a minimizer ψ^*, a parabola with $\mathcal{L}(\psi^*) = 1$. Pick $\psi_R = \varepsilon_R + \psi^*|_{Q_R}$, where $\varepsilon_R > 0$ is chosen such that $\mathcal{L}_R(\psi_R) = 1 - \frac{1}{R}$. Obviously $\varepsilon_R \downarrow 0$. It is easy to show, using the explicit principal eigenfunction of $\Delta + \psi^*$ that $\lim_{R\to\infty} \lambda_R(\psi_R) = \lambda(\psi^*)$. This completes the proof of the lower bound in (4.2) subject to Proposition 4.1.

We finally prove Proposition 4.1:

Proof of Proposition 4.1. This is very similar to the proof of [BK01, Prop. 5.1]. Recall that $\psi_t(\cdot) = \alpha(\beta(t))^{-2}\psi(\cdot\,\alpha(\beta(t))^{-1})$. Consider the event

$$A_y^{(t)} = \bigcap_{z\in B_{R\alpha(\beta(t))}} \Big\{\xi_{\beta(t)}(y+z) \geq \psi_t(z) - \frac{\varepsilon}{2\alpha(\beta(t))^2}\Big\}, \qquad \text{for } y \in \mathbb{Z}^d.$$

Note that the distribution of $A_y^{(t)}$ does not depend on y. Our first goal is to show that, for every $\varepsilon > 0$,

$$\operatorname{Prob}\big(A_0^{(t)}\big) \geq t^{-d\mathcal{L}_R(\psi) - C\varepsilon + o(1)}, \qquad \text{as } t \uparrow \infty, \tag{4.15}$$

where $C > 0$ depends only on R and ψ, but not on ε.

It is convenient to abbreviate

$$s_t = \beta(t)\alpha(\beta(t))^{-d}. \tag{4.16}$$

Let $f \in \mathcal{C}(Q_R)$ be some positive auxiliary function (to be determined later), and consider the tilted probability measure

$$\operatorname{Prob}_{t,z}(\,\cdot\,) = \big\langle e^{f_t(z)\xi_{\beta(t)}(z)} 1\!\!1\{\xi(z) \in \,\cdot\,\}\big\rangle e^{-H(f_t(z)) + f_t(z)H(s_t)/s_t}, \qquad \text{for } z \in \mathbb{Z}^d,$$

where $f_t(z) = s_t f(z\alpha(\beta(t))^{-1})$ is the scaled version of f. The purpose of this tilting is to make the event $A_0^{(t)}$ typical. We denote the expectation with respect to $\operatorname{Prob}_{t,z}$ by $\langle\,\cdot\,\rangle_{t,z}$. Consider the event

$$D_t(z) = \Big\{\frac{\varepsilon}{2\alpha(\beta(t))^2} \geq \xi_{\beta(t)}(z) - \psi_t(z) \geq -\frac{\varepsilon}{2\alpha(\beta(t))^2}\Big\}.$$

Using that $\bigcap_{z\in B_{R\alpha(\beta(t))}} D_t(z) \subseteq A_0^{(t)}$ and the left inequality in the definition of $D_t(z)$, we obtain

$$\begin{aligned}\operatorname{Prob}\big(A_0^{(t)}\big) \geq \exp\Big\{ &\sum_{z\in B_{R\alpha(\beta(t))}} \Big[H(f_t(z)) - f_t(z)\Big(\frac{H(s_t)}{s_t} + \psi_t(z) + \frac{\varepsilon}{2\alpha(\beta(t))^2}\Big)\Big]\Big\}\\ &\times \prod_{z\in B_{R\alpha(\beta(t))}} \operatorname{Prob}_{t,z}\big(D_t(z)\big).\end{aligned} \tag{4.17}$$

Since $\beta(t)\alpha(\beta(t))^{-2} = d\log t$, it is clear from a Riemann sum approximation that

$$\begin{aligned}
&\exp\Big\{\sum_{z\in B_{R\alpha(\beta(t))}}\Big[-f_t(z)\Big(\psi_t(z)+\frac{\varepsilon}{2\alpha(\beta(t))^2}\Big)\Big]\Big\}\\
&\qquad= \exp\Big\{-\frac{\beta(t)}{\alpha(\beta(t))^2}\frac{1}{\alpha(\beta(t))^d}\sum_{z\in B_{R\alpha(\beta(t))}} f\big(\tfrac{z}{\alpha(\beta(t))}\big)\Big(\psi\big(\tfrac{z}{\alpha(\beta(t))}\big)+\tfrac{\varepsilon}{2}\Big)\Big\} \qquad (4.18)\\
&\qquad= t^{-d\langle f,\psi\rangle - d\frac{\varepsilon}{2}\langle f,1\!\!1\rangle + o(1)}, \qquad \text{as } t\uparrow\infty.
\end{aligned}$$

We use the uniformity of the convergence in (1.6), the definitions (1.9) of $\alpha(\,\cdot\,)$ and (1.15) of $\beta(t)$, and a Riemann sum approximation to obtain

$$\begin{aligned}
\sum_{z\in B_{R\alpha(\beta(t))}}\Big[H(f_t(z)) - f_t(z)\tfrac{H(s_t)}{s_t}\Big] &= \sum_{z\in B_{R\alpha(\beta(t))}}\Big[H\Big(s_t f\big(\tfrac{z}{\alpha(\beta(t))}\big)\Big) - f\big(\tfrac{z}{\alpha(\beta(t))}\big)H(s_t)\Big]\\
&= \kappa(s_t)\Big[\sum_{z\in B_{R\alpha(\beta(t))}}\rho f\big(\tfrac{z}{\alpha(\beta(t))}\big)\log f\big(\tfrac{z}{\alpha(\beta(t))}\big) + o(\alpha\circ\beta(t)^d)\Big] \qquad (4.19)\\
&= \big(\mathcal{H}_R(f)+o(1)\big)\,(1+o(1))\,\frac{\beta(t)}{\alpha(\beta(t))^2} = \mathcal{H}_R(f)\,\big(d(\log t)+o(1)\big).
\end{aligned}$$

Using (4.18) and (4.19) in (4.17), we arrive at

$$\operatorname{Prob}\big(A_0^{(t)}\big) \ge t^{d\big(\mathcal{H}_R(f)-\langle f,\psi\rangle - \frac{\varepsilon}{2}\langle f,1\!\!1\rangle\big)+o(1)}\prod_{z\in B_{R\alpha(\beta(t))}}\operatorname{Prob}_{t,z}\big(D_t(z)\big), \qquad \text{as } t\uparrow\infty.$$

Recall from (4.7) that $\mathcal{L}_R$ is the Legendre transform of $\mathcal{H}_R$. We choose f as the minimizer on the right of (4.7), i.e., such that $\mathcal{H}_R(f)-\langle f,\psi\rangle = -\mathcal{L}_R(\psi)$. Hence, to show that (4.15) holds, it is sufficient to show that

$$\prod_{z\in B_{R\alpha(\beta(t))}}\operatorname{Prob}_{t,z}\big(D_t(z)\big) \ge t^{o(1)}, \qquad \text{as } t\uparrow\infty. \tag{4.20}$$

To show this, note that

$$\begin{aligned}
\operatorname{Prob}_{t,z}\big(D_t(z)\big) = 1 &- \operatorname{Prob}_{t,z}\Big\{\xi_{\beta(t)}(z) > \psi_t(z)+\frac{\varepsilon}{2\alpha(\beta(t))^2}\Big\}\\
&- \operatorname{Prob}_{t,z}\Big\{\xi_{\beta(t)}(z) < \psi_t(z)-\frac{\varepsilon}{2\alpha(\beta(t))^2}\Big\}.
\end{aligned} \tag{4.21}$$

Since both terms are handled in the same way, we treat only the second term. For any $a>0$ we use the exponential Chebyshev inequality to bound

$$\begin{aligned}
&\operatorname{Prob}_{t,z}\Big\{\xi_{\beta(t)}(z) < \psi_t(z)-\frac{\varepsilon}{2\alpha(\beta(t))^2}\Big\}\\
&\quad\le e^{-H(f_t(z))+f_t(z)H(s_t)/s_t}\Big\langle \exp\Big\{f_t(z)\xi_{\beta(t)}(z) + a\Big[\psi_t(z)-\xi_{\beta(t)}(z)-\frac{\varepsilon}{2\alpha(\beta(t))^2}\Big]\Big\}\Big\rangle\\
&\quad= e^{H(f_t(z)-a)-H(f_t(z))+aH(s_t)/s_t}e^{a[\psi_t(z)-\varepsilon/2\alpha(\beta(t))^2]}.
\end{aligned}$$

We pick $a=\delta_t f_t(z)$ with some $\delta_t\downarrow 0$. Then the terms involving H can be treated similarly to (4.19). Indeed, abbreviating $\widetilde{f} = f(z\alpha(\beta(t))^{-1})$, we obtain

$$\begin{aligned}
H(f_t(z)-a) - H(f_t(z)) + a\frac{H(s_t)}{s_t} &= H\big(s_t(1-\delta_t)\widetilde{f}\big) - (1-\delta_t)\widetilde{f}\,H(s_t) - \Big[H\big(s_t\widetilde{f}\big) - \widetilde{f}\,H(s_t)\Big]\\
&= \kappa(s_t)(1+o(1))\Big[\widehat{H}\big((1-\delta_t)\widetilde{f}\big) - \widehat{H}\big(\widetilde{f}\big)\Big] = -\frac{\rho+o(1)}{\alpha(\beta(t))^d}\widetilde{f}\,\delta_t\,d\,\log t\,\big[1+\log\widetilde{f}\big],
\end{aligned}$$

where we also used the approximation $\log(1-\delta_t) = -\delta_t(1+o(1))$. Hence, we obtain

$$\mathrm{Prob}_{t,z}\Big\{\xi_{\beta(t)}(z) < \psi_t(z) - \frac{\varepsilon}{2\alpha(\beta(t))^2}\Big\} \leq t^{\delta_t\alpha(\beta(t))^{-d}\widetilde{f}[\widetilde{\psi}-\varepsilon/2-\rho(1+\log\widetilde{f})](d+o(1))}, \qquad \text{as } t\uparrow\infty,$$

where we recall (4.1) and abbreviate $\widetilde{\psi} = \psi(z\alpha(\beta(t))^{-1})$. Recall that we chose f optimally in (4.7), which in particular means that $\log f(x) = \psi(x)/\rho - 1$. Hence, for some $C>0$, not depending on t nor on z, we have, for $t>1$ large enough,

$$\mathrm{Prob}_{t,z}\Big\{\xi_{\beta(t)}(z) < \psi_t(z) - \frac{\varepsilon}{2\alpha(\beta(t))^2}\Big\} \leq t^{-Cd\varepsilon\delta_t\alpha(\beta(t))^{-d}} \leq \frac{1}{4}.$$

Going back to (4.21) and assuming that the first probability term satisfies the same bound, we have

$$\prod_{z\in B_{R\alpha(\beta(t))}} \mathrm{Prob}_{t,z}\big(D_t(z)\big) \geq \big(1-\tfrac{1}{2}\big)^{|B_{R\alpha(\beta(t))}|} = e^{C(R\alpha(\beta(t)))^d} = e^{o(\log t)} = t^{o(1)}, \tag{4.22}$$

where we use that α is slowly varying and $\beta(t) = (\log t)^{1-o(1)}$, so that $\alpha(\beta(t))^d \leq \beta(t)^{d\eta} = o(\log t)$ for $t\to\infty$. This proves (4.20), and therefore (4.15).

We finally complete the proof of Proposition 4.1. As in the proof of [BK01, Prop. 5.1] it suffices to prove the almost sure existence of a (random) $n_0\in\mathbb{N}$ such that, for any $n\geq n_0$, there is a $y_n\in B_{\gamma_{e^n}}$ such that the event $A_{y_n}^{(e^{n+1})}$ occurs. In the following, we abbreviate $t=e^n$. Let $M_t = B_{\gamma_t}\cap\lfloor 3R\alpha(\beta(et))\rfloor\mathbb{Z}^d$. Note that $|M_t|\geq t^{d-o(1)}$ as $t\uparrow\infty$ and that the events $A_y^{(et)}$ with $y\in M_t$ are independent. It suffices to show the summability of

$$p_t = \mathrm{Prob}\Big\{\sum_{y\in M_t}\mathbb{1}\{A_y^{(et)}\} \leq \tfrac{1}{2}|M_t|\mathrm{Prob}(A_0^{(et)})\Big\}$$

on $t\in e^{\mathbb{N}}$. Indeed, since, by (4.15),

$$|M_t|\mathrm{Prob}(A_0^{(et)}) \geq t^{d-d\mathcal{L}_R(\psi)-C\varepsilon-o(1)} \tag{4.23}$$

tends to infinity if $\varepsilon>0$ is small enough (recall that $\mathcal{L}_R(\psi)<1$), the summability ensures, via the Borel-Cantelli lemma, that, for all sufficiently large t, even a growing number of the events $A_y^{(et)}$ with $y\in M_t$ occurs. To show the summability of p_t for $t\in e^{\mathbb{N}}$, we use the Chebyshev inequality to estimate

$$p_t \leq \mathrm{Prob}\Big\{\Big[\sum_{y\in M_t}\mathbb{1}\{A_y^{(et)}\} - \Big\langle\sum_{y\in M_t}\mathbb{1}\{A_y^{(et)}\}\Big\rangle\Big]^2 > \frac{1}{4}\big[|M_t|\mathrm{Prob}(A_0^{(et)})\big]^2\Big\} \leq 4\frac{1-\mathrm{Prob}(A_0^{(et)})}{|M_t|\mathrm{Prob}(A_0^{(et)})}.$$

The summability over all $t\in e^{\mathbb{N}}$ is clear from (4.23). ∎

5. Appendix: Corrected proof of Lemma 4.2 in [BK01]

We use the opportunity to correct an error in the proof of one of the main results of [BK01], the analogue of Theorem 1.4 for case (4) in Section 1.3. In the original proof the large deviation principle of Proposition 3.4 and Varadhan's lemma are applied to the functional $f\mapsto -\int f^\gamma\,dx$, which fails to be continuous in the topology of the large deviation principle. Here we adapt the techniques of the present paper to derive this result. We use the notation of Section 3.

Recall case (4) from Section 1.3. That is, we are in the case where $\operatorname{esssup}\xi(0)=0$, $\gamma\in(0,1)$ and $\kappa^*=0$. The case $\gamma=0$ is easier and can be treated analogously. The main assumption is that $\lim_{t\to\infty}\widetilde{H}_t(x) = -Dx^\gamma$, uniformly in x on compact subsets of $[0,\infty)$, where

$$\widetilde{H}_t(x) = \frac{\alpha(t)^{d+2}}{t}H\Big(x\frac{t}{\alpha(t)^d}\Big), \tag{5.1}$$

and $D > 0$ is a parameter. We have $\alpha(t) = t^{\nu+o(1)}$ as $t \to \infty$, where $\nu = \frac{1-\gamma}{d+2-d\gamma} \in (0, \frac{1}{d+2})$.
The step which needs amendment in [BK01] is the following analogue of Proposition 3.1:

Proposition 5.1.

(i) *For any* $R > 0$ *and* $M > 0$, $\limsup_{t\uparrow\infty} \frac{\alpha(t)^2}{t} \log \langle u^\xi_{R\alpha(t)}(t,0)\rangle \le -\chi^{(M)}$.

(ii) *For any* $R > 0$, $\liminf_{t\uparrow\infty} \frac{\alpha(t)^2}{t} \log \langle u^\xi_{R\alpha(t)}(t,0)\rangle \ge -\chi_R$,

where

$$\chi^{(M)} = \inf_{\substack{g\in H^1(\mathbb{R}^d)\\ \|g\|_2=1}} \Big(\|\nabla g\|_2^2 + D\int \big(g^2(x)\wedge M\big)^\gamma\,dx\Big),$$

$$\chi_R = \inf_{\substack{g\in H^1(\mathbb{R}^d)\\ \|g\|_2=1,\,\mathrm{supp}\,(g)\subseteq Q_R}} \Big(\|\nabla g\|_2^2 + D\int g^{2\gamma}(x)\,dx\Big).$$

Proof. Introduce

$$\mathcal{H}^{(t)}_R(f) = \int_{Q_R} \widetilde{H}_t\big(f(x)\big)\,dx, \qquad \text{for } f\in L^1(Q_R),\ f\ge 0.$$

As in (3.38), we have

$$\langle u^\xi_{R\alpha(t)}(t,0)\rangle = \mathbb{E}_0\Big[\exp\Big(\frac{t}{\alpha(t)^2}\mathcal{H}^{(t)}_R(L_t)\Big)\mathbb{1}\{\mathrm{supp}\,(L_t)\subseteq Q_R\}\Big], \tag{5.2}$$

where we recall the rescaled and normalized local times L_t.
We start with the proof of (i). Fix $M > 0$. With $\mathcal{H}_R(f) = -D\int_{Q_R} f(x)^\gamma\,dx$, we have, uniformly in $f\in L^1(Q_R)$, $f\ge 0$,

$$\limsup_{t\uparrow\infty} \mathcal{H}^{(t)}_R(f) \le \limsup_{t\uparrow\infty} \mathcal{H}^{(t)}_R(f\wedge M) = \mathcal{H}_R(f\wedge M).$$

Note that $\mathcal{H}_R(L_t\wedge M) = \alpha(t)^2 G_t(\frac{1}{t}\ell_t)$, where we introduce

$$G_t(\mu) = -\frac{D}{\alpha(t)^2}\alpha(t)^{-d}\sum_{z\in B_{R\alpha(t)}}\Big((\alpha(t)^d\mu(z))\wedge M\Big)^\gamma, \qquad \text{for } \mu\in\mathcal{M}_1(B_{R\alpha(t)}).$$

We now use Proposition 3.3 for $B = B_{R\alpha(t)}$ and $F = G_t$ to obtain from (5.2) that, for any large t,

$$\begin{aligned}\langle u^\xi_{R\alpha(t)}(t,0)\rangle &\le e^{o(t\alpha(t)^{-2})}\mathbb{E}_0\Big[\exp\Big(t\,G_t(\tfrac{1}{t}\ell_t)\Big)\mathbb{1}\{\mathrm{supp}\,(\ell_t)\subseteq B_{R\alpha(t)}\}\Big]\\ &\le e^{o(t\alpha(t)^{-2})}\exp\Big(-t\chi^{(M)}_t\Big),\end{aligned}$$

where

$$\chi^{(M)}_t = \inf_{\mu\in\mathcal{M}_1(B_{R\alpha(t)})}\Big(\frac12\sum_{x\sim y}\Big(\sqrt{\mu(x)}-\sqrt{\mu(y)}\Big)^2 - G_t(\mu)\Big).$$

The proof of the upper bound is finished as soon as we have shown that

$$\liminf_{t\uparrow\infty}\alpha(t)^2\chi^{(M)}_t \ge \chi^{(M)}. \tag{5.3}$$

This is shown as follows. Let $(t_n : n \in \mathbb{N})$ be a sequence of positive numbers $t_n \to \infty$ along which $\liminf_{t\uparrow\infty} \alpha(t)^2 \chi_t^{(M)}$ is realized. We may assume that its value is finite. Let $(\mu_n : n \in \mathbb{N})$ be a sequence of approximative minimizers, i.e., probability measures on $\mathbb{Z}^d$ having support in $B_{R\alpha(t)}$ such that

$$\liminf_{n\to\infty} \Big[\alpha(t_n)^2 \tfrac{1}{2} \sum_{z\sim y} \Big(\sqrt{\mu_n(z)} - \sqrt{\mu_n(y)}\Big)^2 + D\alpha(t_n)^{-d} \sum_z \Big(\big(\alpha(t_n)^d \mu_n(z)\big) \wedge M\Big)^\gamma\Big]$$

is equal to the left-hand side of (5.3). For any $i \in \{1, \dots, d\}$ consider $g_n^{(i)} : \mathbb{R}^d \to \mathbb{R}$ given by

$$\begin{aligned} g_n^{(i)}(x) &= \sqrt{\alpha(t_n)^d \mu_n\big(\lfloor \alpha(t_n)x \rfloor\big)} \\ &\quad + \Big(x_i - \frac{\lfloor \alpha(t_n)x_i \rfloor}{\alpha(t_n)}\Big)\alpha(t_n)\Big(\sqrt{\alpha(t_n)^d \mu_n\big(\lfloor \alpha(t_n)x \rfloor + \mathrm{e}_i\big)} - \sqrt{\alpha(t_n)^d \mu_n\big(\lfloor \alpha(t_n)x \rfloor\big)}\Big), \end{aligned}$$

where $\mathrm{e}_i \in \mathbb{Z}^d$ is the i^{th} unit vector. For $x = (x_j : j = 1, \dots, d) \in \mathbb{R}^d$, we abbreviate $\widetilde{x}_i = (x_j : j \neq i) \in \mathbb{R}^{d-1}$ and denote $g_{n,\widetilde{x}_i}^{(i)}(x_i) = g_n^{(i)}(x)$. For almost every $\widetilde{x}_i \in \mathbb{R}^{d-1}$, the map $g_{n,\widetilde{x}_i}^{(i)}$ is continuous and piecewise affine, and hence lies in $H^1(\mathbb{R})$ with support in $[-R, R]$. Furthermore,

$$(g_{n,\widetilde{x}_i}^{(i)})'(x_i) = \frac{\partial g_n^{(i)}}{\partial x_i}(x) = \alpha(t_n)\Big(\sqrt{\alpha(t_n)^d \mu_n\big(\lfloor \alpha(t_n)x \rfloor + \mathrm{e}_i\big)} - \sqrt{\alpha(t_n)^d \mu_n\big(\lfloor \alpha(t_n)x \rfloor\big)}\Big).$$

Hence, using Fubini's theorem and Fatou's lemma, we see that

$$\begin{aligned} \infty &> \liminf_{n\to\infty} \alpha(t_n)^2 \frac{1}{2} \sum_{z\sim y} \Big(\sqrt{\alpha(t_n)^d \mu_n(z)} - \sqrt{\alpha(t_n)^d \mu_n(y)}\Big)^2 \\ &= \liminf_{n\to\infty} \sum_{i=1}^d \int_{\mathbb{R}^{d-1}} d\widetilde{x}_i \int_{\mathbb{R}} dx_i \,\big|(g_{n,\widetilde{x}_i}^{(i)})'(x_i)\big|^2 \geq \sum_{i=1}^d \int_{\mathbb{R}^{d-1}} d\widetilde{x}_i \liminf_{n\to\infty} \int_{\mathbb{R}} dx_i \,\big|(g_{n,\widetilde{x}_i}^{(i)})'(x_i)\big|^2. \end{aligned}$$

Since

$$|x_i - \lfloor \alpha(t_n)x_i \rfloor / \alpha(t_n)| \leq \alpha(t_n)^{-1}, \tag{5.4}$$

this also shows that

$$\lim_{n\to\infty} \|g_n^{(i)} - \sqrt{\alpha(t_n)^d \mu_n(\lfloor \alpha(t_n) \cdot \rfloor)}\|_2 = 0. \tag{5.5}$$

In particular, $g_n^{(i)}$ is asymptotically L^2-normalized. Furthermore, it follows that, along a suitable subsequence, for almost all $\widetilde{x}_i \in \mathbb{R}^{d-1}$, $g_{n,\widetilde{x}_i}^{(i)}$ converges to some $g_{\widetilde{x}_i}^{(i)} \in H^1(\mathbb{R})$. The convergence is (i) strong in L^2, (ii) pointwise almost everywhere, and (iii) weak in L^2 for the gradients. The limit satisfies

$$\liminf_{n\to\infty} \alpha(t_n)^2 \frac{1}{2} \sum_{z\sim y} \Big(\sqrt{\alpha(t_n)^d \mu_n(z)} - \sqrt{\alpha(t_n)^d \mu_n(y)}\Big)^2 \geq \sum_{i=1}^d \int_{\mathbb{R}^{d-1}} d\widetilde{x}_i \int_{\mathbb{R}} dx_i \,\big|(g_{\widetilde{x}_i}^{(i)})'(x_i)\big|^2. \tag{5.6}$$

Since $g_{n,\widetilde{x}_i}^{(i)}(x_i) = g_n^{(i)}(x)$ and $\lim_{n\to\infty} \|g_n^{(i)} - \sqrt{\alpha(t_n)^d \mu_n(\lfloor \alpha(t_n) \cdot \rfloor)}\|_2 = 0$, there is $g \in L^2(\mathbb{R}^d)$ such that $g(x) = g_{\widetilde{x}_i}^{(i)}(x_i)$ for almost all $x \in \mathbb{R}^d$. In particular, (a) $g \in H^1(\mathbb{R}^d)$ with (b) $\|g\|_2 = 1$, (c) $\operatorname{supp}(g) \subset Q_R$ and (d)

$$\|\nabla g\|_2^2 \leq \liminf_{n\to\infty} \alpha(t_n)^2 \frac{1}{2} \sum_{z\sim y} \Big(\sqrt{\alpha(t_n)^d \mu_n(z)} - \sqrt{\alpha(t_n)^d \mu_n(y)}\Big)^2.$$

Indeed, (a) follows from (b) and (d). Item (b) follows from (5.5), while item (c) is trivially satisfied. We are left to prove item (d). Since $g_{\widetilde{x}_i}^{(i)}(x_i) = g(x)$ for almost every x, we get

$$(g_{\widetilde{x}_i}^{(i)})'(x_i) = \frac{\partial}{\partial x_i} g_{\widetilde{x}_i}^{(i)}(x_i) = \frac{\partial}{\partial x_i} g(x), \tag{5.7}$$

and hence

$$\sum_{i=1}^d \int_{\mathbb{R}^{d-1}} d\widetilde{x}_i \int_{\mathbb{R}} dx_i \,\big|(g_{\widetilde{x}_i}^{(i)})'(x_i)\big|^2 = \int_{\mathbb{R}^d} dx \sum_{i=1}^d \big|\frac{\partial}{\partial x_i} g(x)\big|^2 = \|\nabla g\|_2^2. \tag{5.8}$$

Therefore, item (d) follows from (5.6).

It remains to show that $\int (g(x)^2 \wedge M)^\gamma \, dx \le \liminf_{n\to\infty} \alpha(t_n)^{-d} \sum_z ((\alpha(t_n)^d \mu_n(z)) \wedge M)^\gamma$. Note that

$$\begin{aligned} \alpha(t_n)^{-d} \sum_z \Big((\alpha(t_n)^d \mu_n(z)) \wedge M \Big)^\gamma &= \int \Big((\alpha(t_n)^d \mu_n(\lfloor \alpha(t_n) x \rfloor)) \wedge M \Big)^\gamma \, dx \\ &= \int \Big(\Big| g_n^{(i)}(x) - \Big(x_i - \frac{\lfloor \alpha(t_n) x_i \rfloor}{\alpha(t_n)} \Big) (g_{n,\widetilde{x}_i}^{(i)})'(x_i) \Big|^{2\gamma} \wedge M^\gamma \Big) \, dx. \end{aligned} \tag{5.9}$$

We next use the inequality $|a-b|^{2\gamma} \ge (|a|^\gamma - |b|^\gamma)^2 \ge |a|^{2\gamma} - 2|ab|^\gamma$ and for the subtracted term use Jensen's inequality and the Cauchy-Schwarz inequality, as well as (5.4), to see that

$$\begin{aligned} &\int \Big| g_n^{(i)}(x) \Big(x_i - \frac{\lfloor \alpha(t_n) x_i \rfloor}{\alpha(t_n)} \Big) (g_{n,\widetilde{x}_i}^{(i)})'(x_i) \Big|^\gamma \, dx \\ &\qquad \le (2R)^d \Big[(2R)^{-d} \int_{Q_R} \big| g_n^{(i)}(x) \big| \Big| \Big(x_i - \frac{\lfloor \alpha(t_n) x_i \rfloor}{\alpha(t_n)} \Big) (g_{n,\widetilde{x}_i}^{(i)})'(x_i) \Big| \, dx \Big]^\gamma \\ &\qquad \le \alpha(t_n)^{-\gamma} \, (2R)^{(1-\gamma)d} \, \big\| g_n^{(i)} \big\|_2^\gamma \, \big\| \tfrac{\partial}{\partial x_i} g_n^{(i)} \big\|_2^\gamma, \end{aligned}$$

which is negligible. Next we use the fact that $g_n^{(i)} \to g$ pointwise and Fatou's lemma to see that the limit inferior of the right hand side of (5.9) is not smaller than $\int (g(x)^{2\gamma} \wedge M^\gamma) \, dx$. This completes the proof of (5.3) and therefore the proof of (i).

We next turn to the proof of (ii). First we show that, for any $f \in \mathcal{C}(Q_R)$ and any family of $L^1(Q_R)$-normalized functions $f_t \in L^1(Q_R)$ satisfying $f_t \to f$ in the weak topology induced by test integrals against all continuous functions,

$$\liminf_{t\uparrow\infty} \mathcal{H}_R^{(t)}(f_t) \ge \mathcal{H}_R(f). \tag{5.10}$$

We fix a large $M > 0$ and estimate $\mathcal{H}_R^{(t)}(f_t) \ge \mathcal{H}_R^{(t)}(f_t \wedge M) + \mathcal{H}_R^{(t)}(f_t \mathbb{1}\{f_t > M\})$. We first handle the first term. Introduce $\phi(x) = x^\gamma$ and let $g_y(x) = (1-\gamma)y^\gamma + \gamma y^{\gamma-1} x$ denote the tangent of ϕ at $y \in (0,\infty)$. By concavity, we have $\phi \le g_y$ on $(0,\infty)$ for any $y > 0$. This implies that, as $t \uparrow \infty$, for any $\varepsilon > 0$,

$$\begin{aligned} \mathcal{H}_R^{(t)}(f_t \wedge M) &= o(1) - D \int_{Q_R} \phi\big(f_t(x) \wedge M\big) \, dx \ge o(1) - D \int_{Q_R} g_{f(x)\vee\varepsilon}\big(f_t(x)\big) \, dx \\ &\ge o(1) - D(1-\gamma) \int_{Q_R} \big(f(x) \vee \varepsilon\big)^\gamma \, dx - D\gamma \int_{Q_R} f_t(x) \big(f(x) \vee \varepsilon\big)^{\gamma-1} \, dx \\ &= o(1) - D(1-\gamma) \int_{Q_R} (f \vee \varepsilon)^\gamma - D\gamma \int_{Q_R} f \, (f \vee \varepsilon)^{\gamma-1}, \end{aligned}$$

where in the last step we used that $(f \vee \varepsilon)^{\gamma-1}$ is continuous and $f_t \to f$. Letting $\varepsilon \downarrow 0$, we see that $\liminf_{t\uparrow\infty} \mathcal{H}_R^{(t)}(f_t \wedge M) \ge \mathcal{H}_R(f)$ for any $M > 0$.

It remains to show that $\liminf_{M\uparrow\infty} \liminf_{t\uparrow\infty} \mathcal{H}_R^{(t)}(f_t \mathbb{1}\{f_t > M\}) \ge 0$. Fix $\delta > 0$ such that $\gamma + \delta < 1$. Recall (5.1). Since H is regularly varying with exponent γ, by [BGT87, Proposition 1.3.6], there is an $M > 0$ such that, for any sufficiently large t,

$$\widetilde{H}_t(x) \ge -x^{\gamma+\delta}, \qquad \text{for any } x > M.$$

Hence,

$$\mathcal{H}_R^{(t)}(f_t 1\!\!1\{f_t > M\}) \geq -\int_{Q_R} f_t(x)^{\gamma+\delta} 1\!\!1\{f_t(x) > M\}\, dx \geq -M^{\gamma+\delta-1} \int_{Q_R} f_t(x)\, dx = -M^{\gamma+\delta-1},$$

since f_t is L^1-normalized. This completes the proof of (5.10).

We complete the proof of Proposition 5.1(ii) by using (5.10) in (5.2) and use the lower bound of Varadhan's lemma in [DZ98, Lemma 4.3.4] to conclude that the assertion in (ii) holds. □

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