Mathematical Institute
University Leipzig
Summer term 2005

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ODE for Physicists - Homework 9

Due: June 14, 2005

25. (4 pts.) Use the method of power series to find the solution to the initial value problem $y'' + x^2 y' + 2xy = 0$, $y(0) = 1$, $y'(0) = 0$.

26. (3 pts.) Let $a > 0$

    (i) Show that
    $$\mathcal{L}\left(\frac{\sin at}{2a^3} - \frac{t\cos at}{2a^2}\right)(s) = \frac{1}{(s^2+a^2)^2}.$$

    (ii) Find a function $t \mapsto f(t)$ such that
    $$\mathcal{L}(f)(s) = \frac{s}{(s^2+a^2)^2}.$$

27. (3 pts.) What function has the Laplace transform (i) $s \mapsto (s-4)^{-3}$?

28. (3 pts.) Let $F(s) = \mathcal{L}(f)(s)$. Suppose that $f(t)/t$ has a limit as t approaches zero. Prove that
    $$\mathcal{L}\left(f(t)/t\right)(s) = \int_s^{+\infty} F(\zeta)\, d\zeta.$$
    Remark: The assumption that $f(t)/t$ has a limit as $t \downarrow 0$ guarantees that the integral on the right-hand side exists.

29. (3 pts.)

(i) Let f be periodic with period p, i.e. $f(t+p) = f(p)$ for every $t \geq 0$. Show that
$$\mathcal{L}(f)(s) = \frac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t)\, dt.$$

(ii) Use the formula in (i) to find the Laplace transform of $|\sin|$.

(iii) Use the formula in (i) to find the Laplace transform of the step function that is periodic with period one and is equal to one in $[0, \frac{1}{2})$ and equal to two in $[\frac{1}{2}, 1)$.