Mathematical Institute
University Leipzig
Summer term 2005

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ODE for Physicists - Homework 5

Due: May 10, 2005

12. (4 pts.) (Variation of constants.)

    (a) Let $a, c \in (0, \infty)$ and $b \in \mathbb{R}$. Show that every solution of $y' + ay = be^{-cx}$, approaches 0 as $x \to +\infty$.

    (b) Solve the following IVP:

    $$y' + y = \frac{1}{1+x^2}, \qquad y(2) = 3.$$

13. (2 pts.) (Application of first order equations.)

    A cylindrical tank of radius r_0 and height h_0 is filled with water. The water leaves the tank through a round hole of radius δ_0 in the bottom of the tank, where $r_0 > \delta_0 > 0$. The velocity of the flow is $v = \sqrt{h}$, where h is the depth of water of the tank. Find the time required to empty the tank.

    Hint: The volume of water that runs out of the tank in time $\mathrm{d}t$ is $\mathrm{d}V = \pi\delta_0^2\sqrt{h}\,\mathrm{d}t$. What is the expression of $\mathrm{d}V$ in terms of $\mathrm{d}h$?

14. (8 pts.) (Integrating factor.)

    (a) Find the general solution to $\frac{1}{2}y^2 + 2y\mathrm{e}^x + (y + \mathrm{e}^x)y' = 0$.

    (b) Find the general solution to $y' + y = y^2(\cos x - \sin x)$.
    Hint: Substitute $v = 1/y$.

    (c) Solve $1 + (1 + xy)\mathrm{e}^{xy} + (1 + x^2\mathrm{e}^{xy})y' = 0$.

    (d) Find all the functions $\psi(x)$ such that the differential equation $y^2 \sin x + y\psi(x)y' = 0$ is exact. Solve the equation for these ψ's.

15. (2 pts.) (Ansatz of the type of the right hand side.)

    Find the general solution to:

    (a) $y' + 3y = \mathrm{e}^{2x} + x^2$.

    (b) $y' - 2y = \mathrm{e}^x \sin x$.