Mathematical Institute
University Leipzig
Summer term 2005

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ODE for Physicists - Homework 11

Due: June 28, 2005

32. (2 points) Let the following two functions be given:

$$(i) \quad f_1(x,y) = 5x\cos(\pi y), \qquad\qquad (ii) \quad f_2(x,y) = \frac{1}{x}\mathrm{e}^{-y^2}.$$

Determine what the maximal domains $\subset \mathbb{R}^2$ look like in which f_1 resp. f_2 is (a) Lipschitz continuous with respect to y, (b) locally Lipschitz continuous with respect to y.

33. (4 pts.) Use the method of Picard iteration to find the solution to the IVP $y'(x) = -4y(x) + 2$, $y(0) = 1$.

34. (4 pts.) Prove the following fixed point theorem:

Let (X, d) be a complete metric space and $A\colon X \to X$ a map that satisfies $\mathrm{d}(A^n(x_1), A^n(x_2)) \le \alpha_n \mathrm{d}(x_1, x_2)$ for any $n \in \mathbb{N}$ and $x_1, x_2 \in X$, where A^n is the n-th iterate of A (i.e., $A^1 = A$ and $A^{n+1} = A \circ A^n$ for any $n \in \mathbb{N}_0$), and α_n is a given sequence of positive numbers such that $\sum_{n\in\mathbb{N}} \alpha_n < \infty$. Then A possesses a unique fixed point in X. Furthermore, for any initial value $x_0 \in X$, the iterating sequence $x_{n+1} = A(x_n)$ converges towards the fixed point. Give an error estimate for the distance between the fixed point and x_n.

35. (6 pts.) Use problem 34 to prove the following existence and uniqueness theorem:

Let $a, b > 0$ and $R = [-a, a] \times [-b, b]$ a rectangle, and let $f\colon R \to \mathbb{R}$ be continuous. Put $M = \max_R |f|$ and assume that f is Lipschitz continuous with respect to y with Lipschitz constant $L > 0$. Then, for any $0 < a_1 < \min\{a, \frac{b}{M}\}$, there is a unique solution $y\colon [-a_1, a_1] \to [-b, b]$ of the first-order IVP $y' = f(x, y)$, $y(0) = 0$.

Remark. This shows that the interval on which we construct a solution does not have to depend on the Lipschitz constant of f.

Hint. Use Lemma 4.2.1 and repeat part of the proof of Lemma 4.2.5. In particular, you do not have to repeat the proof that $A(y)(x) = \int_0^x f(t, y(t))\,\mathrm{d}t$ defines a map from $\mathcal{C}_{0,b}$ into itself, where $\mathcal{C}_{0,b}$ is the set of continuous functions $[-a_1, a_1] \to [-b, b]$ having value 0 at 0.

Information: The mathematical content of the written exam on Saturday, 9 July, is contained in the material of problems 1 up to and including 34.