# A Two-Cities Theorem for the Parabolic Anderson Model

Wolfgang König

Universität Leipzig

*Based on joint work with Hubert Lacoin (Paris), Peter Mörters (Bath) and Nadia Sidorova (London)*

# The Parabolic Anderson Model



We consider the Cauchy problem for the heat equation with random coefficients and localised initial datum:

$$\frac{\partial}{\partial t} u(t,z) = \Delta^{\mathrm{d}} u(t,z) + \xi(z) u(t,z), \qquad \text{for } (t,z) \in (0,\infty) \times \mathbb{Z}^d, \tag{1}$$

$$u(0,z) = \mathbb{1}_0(z), \qquad \text{for } z \in \mathbb{Z}^d, \tag{2}$$

where

- $\xi = (\xi(z) \colon z \in \mathbb{Z}^d)$ i.i.d. random potential, $[-\infty, \infty)$-valued.
- $\Delta^{\mathrm{d}} f(z) = \sum_{y \sim z} [f(y) - f(z)]$ discrete Laplacian
- $\Delta^{\mathrm{d}} + \xi$ Anderson Hamiltonian

Some Remarks:

- The solution $u(t, \cdot)$ is a random time-dependent shift-invariant field.
- Its a.s. existence is guaranteed under a mild moment condition on the potential.
- It has all moments finite if all positive exponential moments of $\xi(0)$ are finite.

# Motivations and Goals



Interpretations / Motivations:

- Random mass transport through a random field of sinks and sources.
- Expected particle number in a branching random walk model in a field of random branching and killing rates.

Background literature and surveys:

[MOLCHANOV 1994], [CARMONA/MOLCHANOV 1994], [SZNITMAN 1998], [GÄRTNER/K. 2005].

Main Goal:

Describe the large-$t$ behavior of the solution $u(t,\cdot)$.

In particular:

Where does the main bulk of the total mass stem from?

Total mass of the solution:

$$U(t) = \sum_{z\in\mathbb{Z}^d} u(t,z), \qquad \text{for } t > 0.$$

# Intermittency

UNIVERSITÄT LEIPZIG

Much work is devoted to a thorough understanding of the effect of

Intermittency: Asymptotically as $t \to \infty$, the main contribution to $U(t)$ comes from few small remote islands.

- These islands are randomly located, $t$-dependent, not too far from the origin.
- Both the solution $u(t,\cdot)$ and the potential $\xi(\cdot)$ are exceptionally large in these islands.
- The large-$t$ behavior is determined by the largest eigenvalue of the Anderson Hamiltonian $\Delta^{\rm d} + \xi$ (i.e., by the bottom of the spectrum of $-\Delta^{\rm d} - \xi$) in large $t$-dependent boxes.
- This in turn is determined by the extreme values of the potential $\xi$.
- Hence, only the upper tails of $\xi(0)$ matter.

# Finite Exponential Moments



Under the assumption

$$\langle e^{t\xi(0)} \rangle < \infty \qquad \text{for any } t > 0,$$

the asymptotic behavior of the PAM could be classified in four universality classes [vd Hofstad/K./Mörters 2007], according to the asymptotics of the upper tails of $\xi(0)$:

- bounded from above: island size quickly diverges,
- almost bounded: island size slowly diverges,
- 'double-exponential' tails: islands are discrete, size does not diverge,
- thicker tails: islands are single-point.

The asymptotic shapes of the potential and of the solution in the islands are characterised by four respective variational problems, respectively by their duals.

The two leading terms in the asymptotics of the total mass are deterministic.

The number of islands and their precise locations are unknown in general (only rough bounds available).

# Heavy-Tailed Potentials



From now, we consider the Pareto-distribution:

$$\mathrm{Prob}(\xi(0) > r) = r^{-\alpha}, \qquad r \in [1, \infty), \qquad \text{(parameter } \alpha \in (d, \infty)\text{)}.$$

Then the total mass $U(t)$ has no moments.

It is known that large values of a sum of i.i.d. heavy tailed random variables are most easily realised by having just one of the values extremely large (and the others of moderate size).

We will see that this principle is implicitly reflected in the behaviour of the parabolic Anderson model.

# Asymptotics of the Total Mass



**Weak Asmptotics of the Total Mass** [van der Hofstad/Mörters/Sidorova 2008]. We have the weak convergence

$$\lim_{t\to\infty} \text{Prob}\Big(\Big(\frac{t}{\log t}\Big)^{-\frac{d}{\alpha-d}}\frac{1}{t}\log U(t) \leq x\Big) = \exp\big(-\mu x^{d-\alpha}\big),$$

where $\mu \in (0,\infty)$ is some suitable, explicit constant.

- Note that the leading term in the asymptotics of $U(t)$ is random.
- Some a.s. limsup and liminf results for $\log(\frac{1}{t}\log U(t))$ are also contained in [HMS08].
- $\frac{1}{t}\log U(t)$ has the same weak asymptotics as the maximum of $t^d$ independent Pareto $(\alpha-d)$-distributed random variables. Apparantly, the potential's random fluctuations dominate the smoothing effect of $\Delta^{\mathrm{d}}$.
- [HMS08] also contains analogous results for stretched-exponential upper tails (Weibull distribution).
- Main tools of [HMS08]: extreme value theory, point process techniques, Borel-Cantelli lemmas.

# Complete Localisation



The Pareto-distributed potential admits a strong intermittency effect on two points almost surely ...

**Two-Cities Theorem.** [Lacoin/K./Mörters/Sidorova 2007]. There are two processes $(Z_t^{(1)})_{t>0}$ and $(Z_t^{(2)})_{t>0}$ in $\mathbb{Z}^d$ such that

$$\lim_{t\to\infty} \frac{u(t,Z_t^{(1)})+u(t,Z_t^{(2)})}{U(t)} = 1 \qquad \text{almost surely.}$$

... and on just one point in probability:

**One-point Localisation in Probability** [LKMS 2007]. With $Z_t = Z_t^{(1)}$, we have

$$\lim_{t\to\infty} \frac{u(t,Z_t)}{U(t)} = 1 \qquad \text{in probability.}$$

Furthermore, $Z_t(\log t/t)^{\alpha/(\alpha-d)}$ converges in distribution to some non-degenerate random variable.

# Comments

UNIVERSITÄT LEIPZIG

- Hence, there is precisely one intermittent island at large deterministic times, and this island is single-point. Such a strong localisation could not be established for other potentials yet.
- In almost sure sense, the localisation on two sites, since the process $(Z_t)_{t>0}$ jumps.
- The proof combines the Feynman-Kac formula and results and tools from [HMS08] (extreme value theory, point process techniques, asymptotics for $\frac{1}{t}\log U(t)$) and [GKM07] (a spectral decomposition approach).
- There is some hope to prove a related localisation for potentials with finite exponential moments as well.