Weierstrass Institute for
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Moment asymptotics for branching random walks in random environment

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Goal of this talk

- We consider a particle system in $\mathbb{Z}^d$ having three random components:

  migration, site-dependent branching/killing rates, branching/killing.

- We are generally interested in the description of the main flow of the particles at late times, in particular in the influence of the random environment.
- Today, we report on the behaviour of the p-th moment (over the medium) of the n-th moment (over migration and branching/killing) of the particle number.

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- Today, we report on the behaviour of the p-th moment (over the medium) of the n-th moment (over migration and branching/killing) of the particle number.
- Earlier work [ALBEVERIO, BOGACHEV, MOLCHANOV, YAROVAYA 2000] derived a recursive formula for the n-th moment and the first asymptotic term.
- We present a direct formula and find also the second term.
- Main tools for deriving the moment formula in [ABMY00]: PDEs. Here: spines.
- Main tools for deriving the large-time asymptotics: Large deviations like in [GÄRTNER/MOLCHANOV 1998].
- We will be able to use the vast knowledge on the parabolic Anderson model.

Ingredients:

- random migration like continuous-time simple random walk in $\mathbb{Z}^d$ with generator

$$\Delta f(x) = \sum_{y \sim x} [f(y) - f(x)]$$

- random, space-dependent i.i.d. killing rates $\xi_0 = (\xi_0(z))_{z \in \mathbb{Z}^d}$ and binary branching rates $\xi_2 = (\xi_2(z))_{z \in \mathbb{Z}^d}$
- killing and branching of the particles present at z with rates $\xi_0(z)$ resp. $\xi_2(z)$.

Probability measure and expectation: P_x and E_x, starting from precisely one particle at x.

$$\begin{aligned} \eta(t,y) &= \text{particle number at time } t \text{ in } y. \\ \eta(t) &= \sum_{y \in \mathbb{Z}^d} \eta(t,y) \quad \text{global particle number at time } t. \end{aligned}$$

Quenched moments:

$$m_n(t,x,y) = E_x[\eta(t,y)^n] \qquad \text{and} \qquad m_n(t,x) = E_x[\eta(t)^n],$$

for fixed branching/killing environment (ξ_2, ξ_0)..

Connection with the parabolic Anderson model

Denote $u(t,y)=m_1(t,0,y)$, when starting with one particle at the origin. Then it is well-known (see [GÄRTNER/MOLCHANOV 1990] for background) that u is the unique positive solution of the Cauchy problem for the heat equation with random potential $\xi=\xi_2-\xi_0$:

$$\frac{\partial}{\partial t}u(t,y)=\Delta u(t,y)+\xi(y)u(t,y),$$

with initial condition $u(0,\cdot)=\delta_0(\cdot)$.

Feynman-Kac formula : $$u(t,y)=\mathbb{E}_0\Big[\exp\Big\{\int_0^t \xi(X_s)\,\mathrm{d}s\Big\}\delta_y(X_t)\Big].$$

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Moment Asymptotics [GÄRTNER/MOLCHANOV 1998]

If ξ has double-exponential tails, i.e., $\mathrm{Prob}(\xi(x) > r) \approx \exp\{-\mathrm{e}^{r/\rho}\}$ as $r \to \infty$, then

$$\langle u^p(t,x)\rangle = \mathrm{e}^{H(pt)}\,\mathrm{e}^{-2d\chi(\rho)pt + o(t)}, \qquad p \in \mathbb{N},$$

where $H(t) = \log\langle \mathrm{e}^{t\xi(0)}\rangle \approx \rho t \log t - \rho t$ and

$$\chi(\rho) = \frac{1}{2} \inf_{\mu \in \mathcal{P}(\mathbb{Z})} [\mathcal{S}(\mu) + \rho \mathcal{I}(\mu)].$$

Comments

The two functions appearing in the variational formula are

$$\mathcal{S}(\mu) = \sum_{x\in\mathbb{Z}} \left(\sqrt{\mu(x+1)} - \sqrt{\mu(x)}\right)^2 \qquad \text{and} \qquad \mathcal{I}(\mu) = -\sum_{x\in\mathbb{Z}} \mu(x) \log \mu(x).$$

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The result implies that

$$\langle m_1^p(t,x)\rangle = \langle m_1(tp,x)\rangle \mathrm{e}^{o(t)}, \qquad t\to\infty,$$

which can easily be guessed from an eigenvalue expansion for the Anderson operator $\Delta+\xi$ in some large box,

$$m_1(t,x) \approx \sum_{k=0}^{\infty} \mathrm{e}^{t\lambda_k(\xi)} \langle \mathrm{e}_k, \mathbb{1}\rangle \mathrm{e}_k(x),$$

which shows that

$$m_1^p(t,x) \approx \mathrm{e}^{pt\lambda_0(\xi)}$$

for the principal eigenvalue $\lambda_0(\xi)$.

Recursive moment formula for $n \geq 2$

[ABMY00] derived the following inhomogeneous Cauchy problem:

$$\frac{\partial}{\partial t} m_n(t,x,y) = \kappa \Delta m_n(t,x,y) + \xi(x) m_n(t,x,y) + \xi_2(x) h_n[m_1, \dots, m_{n-1}](t,x,y),$$

with initial condition $m_n(0,\cdot,y) = \delta_y(\cdot)$, where

$$h_n[m_1, \dots, m_{n-1}] = \sum_{i=1}^{n-1} \binom{n}{i} m_i m_{n-i}.$$

Proof idea: Use the generator of the particle system to derive an equation for the Laplace transform of $\eta(t,y)$ and differentiate n times with respect to the parameter, and put it zero.

Moment asymptotics for Weibull tails [ABMY00]

If ξ has Weibull tails, i.e., $\mathrm{Prob}(\xi(x) > r) \approx \exp\{-Cr^\alpha\}$ as $r \to \infty$, with $\alpha > 1, C > 0$, then

$$\langle m_n(t,x)^p \rangle = \mathrm{e}^{C'(npt)^{\alpha'}(1+o(1))}, \qquad n, p, \in \mathbb{N},$$

where $\frac{1}{\alpha} + \frac{1}{\alpha'} = 1$ and $C' = C'(C,\alpha)$ is explicit.

Main auxiliary object: a branching random walk (BRW) in $\mathbb{Z}^d$ with time interval $[0, t]$ with $\leq n - 1$ splitting events. Ingredients:

- a tree $\mathcal{T}$ with $k \in \{0, \dots, n-1\}$ splits that expresses the branching structure,
- a monotonous numbering I of the splitting vertices of the tree to express their order in time,
- a time duration attached to each bond,
- a simple random walk bridge attached to each bond.

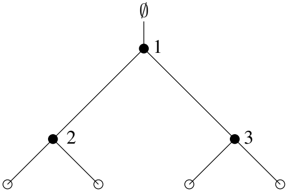


Figure: Two monotonous numberings of the splitting vertices of a tree with three splits.

Now equip the monotonously numbered tree with a time vector

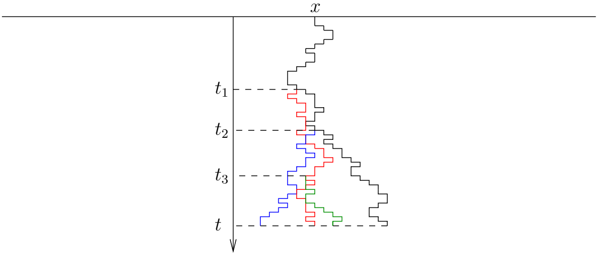
$$\widehat{t} = (t_0, \dots, t_{k+1}), \qquad \text{where} \qquad 0 = t_0 < t_1 < \cdots < t_k < t_{k+1} = t.$$

For a bond b, let

$$Y^{(b,\widehat{t})} = \left(Y_r^{(b,\widehat{t})} : r \in [t_{I(u)}, t_{I(v)}]\right)$$

be a continuous-time simple random walk on $\mathbb{Z}^d$ with generator Δ, starting from zero, independent over b. Define the branching random walk now by putting, for any leaf l,

$$X_r^{(l)} := x + \sum_{m=1}^{i-1} Y_{t_{I(u_m)}}^{(b_m,\widehat{t})} + Y_r^{(b_i,\widehat{t})}, \qquad r \in [t_{I(u_{i-1})}, t_{I(u_i)}], i \in \{1, \dots, j\}.$$



Direct moment formula

Abbreviate

$$\Phi_x(\mathcal{T}, I, t, y) = \int_{0=t_0<t_1<\cdots<t_k<t_{k+1}=t} \mathrm{d}\widehat{t}\,\mathbb{E}_x^{(\mathcal{T}, I, \widehat{t})}\Big[\exp\Big(\sum_{(u,v)\in E} \int_{t_{I(u)}}^{t_{I(v)}} \xi(X_r^{(u,v)})\,\mathrm{d}r\Big) \Big(\prod_{v\in S} \xi_2(X_{t_{I(v)}}^{(u,v)})\Big) \sum_{l\in L} \mathbb{1}\{X_t^{(l)} = y\}\Big].$$

Moment formula, [Gün/K./Sekulovic (2012)]

For any $n \in \mathbb{N}$,

$$m_n(t,x,y) = \sum_{k=0}^{n-1} \sum_{\mathcal{T}\in T_k} \sum_{I\in\mathcal{N}(\mathcal{T})} c_{k,n}\Phi_x(\mathcal{T}, I, t, y),$$

and an analogous formula for $m_n(t,x)$. The constants are defined by $c_{0,n} = 1$ for all $n \in \mathbb{N}$ and

$$c_{k,n} = \sum_{i=1}^{n-k} \binom{n}{i} c_{k-1,n-i}, \qquad k = 1, \dots, n-1.$$

There is an analogous formula for more general branching mechanisms, but it is cumbersome.

Some remarks on the proof

Consider the following variant of the branching process, which introduces spines to some of the particles. The differences are:

1. We start with one particle at x, which carries n marks (called spines) $1, 2, \dots, n$.
2. A particle at position y carrying j marks branches at rate $2^j \xi_2(y)$ into two particles.
3. At such a branching event of a particle carrying j marks, each mark chooses independently and uniformly at random one of the two particles to follow.

We now apply the many-to-few lemma [HARRIS/ROBERTS 2011]:

$$m_n(t,x) = \mathbb{Q}_x^{(n)}\Big[\exp\Big(\sum_{v\in \mathrm{skel}(t)} \int_{\sigma_v\wedge t}^{\tau_v\wedge t} \Big((2^{D(v)}-1)\xi_2(Z_r^{(v)}) - \xi_0(Z_r^{(v)})\Big)\,\mathrm{d}r\Big)\Big],$$

where

- $\mathrm{skel}(t)$ is the collection of particles that have carried at least one mark up to time t
- $D(v)$ is the number of marks carried by a particle v,
- σ_u and τ_u are the birth time and the death time of particle u,
- $Z_s^{(u)}$ is the position of the unique ancestor of u alive at time $s \in [0,t]$.

Now one evaluates the above expectation, using an induction on n.

Moment asymptotics, [GÜN/K./SEKULOVIC (2012)]

Suppose that $\xi_2(0)$ is doubly-exponentially distributed with parameter ρ. In the case $\rho = \infty$, we also assume, for any $k \in \mathbb{N}$,

$$\left\langle \xi_2(0)^k \, \mathrm{e}^{\xi_2(0)t} \right\rangle \le \langle \mathrm{e}^{\xi_2(0)t} \rangle \mathrm{e}^{o(t)} \quad \text{as } t \to \infty.$$

Then, for any $n, p \in \mathbb{N}$,

$$\langle m_n^p(t,x) \rangle = \exp\Big(H(npt) \, - 2d\chi(\rho)npt + o(t) \Big), \qquad t \to \infty.$$

In particular,

$$\langle m_n^p(t,x) \rangle = \langle m_1^{np}(t,x) \rangle \mathrm{e}^{o(t)} = \langle m_1(tnp,x) \rangle \mathrm{e}^{o(t)}, \qquad t \to \infty.$$

Explanation for $n = 2$: The moment formula gives $m_2 = m_1 + \widetilde{m}_2$, where

$$\widetilde{m}_2(t,x) = \int_0^t \mathbb{E}_x\Big[\exp\Big\{\int_0^s \xi(X_r)\,\mathrm{d}r + \int_s^t \xi(X'_r)\,\mathrm{d}r + \int_s^t \xi(X''_r)\,\mathrm{d}r\Big\} 2\xi_2(X_s)\Big]\,\mathrm{d}s,$$

where $(X_r)_{r\in[0,s]}$ and $(X'_r)_{r\in[s,t]}$ and $(X''_r)_{r\in[s,t]}$ are independent walks, starting at $X_0 = x$, and $X'_s = X''_s = X_s$.

- The term $2\xi_2(X_s)$ should have hardly any influence (guaranteed by our additional assumption).
- Guessing from [GÄRTNER/MOLCHANOV (1998)], the leading term should be $\mathrm{e}^{H(2t-s)}$, since $2t - s = s + (t-s) + (t-s)$ is the total time that the three random walks spend in the random environment.
- Since $H(t) \gg t$, the leading contribution comes from $s \approx 0$.
- But then we have basically a product of the contribution of two independent walks X' and X'' on $[0,t]$.
- This means that $\widetilde{m}_2 \approx m_1^2$.

Comments and outlook

- In the proof, one separates the exponential term from the polynomial term by a careful application of Hölder's inequality and redoes some parts of the proof of [GÄRTNER/MOLCHANOV 1998].
- The same should be possible for much more general branching mechanisms, but will be cumbersome to formulate.
- Hence, in the moment formula, all the branchings should happen as soon as possible, to optimize the total time spent by the BRW in the random environment.
- This effect should be entirely due to the positivity of $\mathrm{esssup}\xi(0)$.
- In the case $\mathrm{esssup}\xi(0) \leq 0$, it is easy to guess that $\langle m_n(t,0)^p \rangle \approx \langle m_1(t,0)^p \rangle$, since all the branchings should happen as late as possible.
- The almost sure asymptotics of $m_n(t,0)$ should be doable as well; the expected picture is easy to guess.
- Possibly, one can refine the techniques to derive useful asymptotics for all the p-th moments of $m_n(t,0)/\langle m_n(t,0)\rangle$. (May be, for the Pareto distribution?).
- Future work will be devoted to the analysis of the large-n limit of $m_n(1,0)$ and of $\langle m_n(1,0)\rangle$; this corresponds to the upper tails of $\eta(1)$.