Coverage Process of SINR Cells

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General Assumptions

- $\widehat{\Phi}$ is a marked point process (m.p.p.) with points $x_i \in \mathbb{R}^d$ and marks $(m_i, t_i) \in \mathbb{R}^l \times \mathbb{R}^+$
- $\widetilde{\Phi}$ is the projection of $\widehat{\Phi}$ without marks t_i
- $I_{\widetilde{\Phi}}$ shot-noise field on $\mathbb{R}^d$ generated by $\widetilde{\Phi}$ and response function $L : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^l \mapsto \mathbb{R}^+$
- thermal noise field $w(y) \geq 0$ for all $y \in \mathbb{R}^d$

Definition 1 (SINR collection)**.** *Let $n \in \mathbb{N}$ and $(X_i, M_i)_{i=1,\ldots,n} \in \mathbb{R}^d \times \mathbb{R}^l$ be a collection of marked points. The SINR cells of this collection of marked points for shot-noise field $I_{\widetilde{\Phi}}$, threshold $t_i \geq 0$ and thermal noise field $w(y) \geq 0$ is:*

$$\begin{aligned} C_{(X_i,M_i)} &:= C_{(X_i,M_i)}\big(\tilde{\Phi} + \kappa \sum_{j=1, j\neq i}^{n} \varepsilon_{(X_j,M_j)}, w, t_i\big) \\ &= \Big\{ y \in \mathbb{R}^d \ : \ L(y, X_i, M_i) \geq t_i \Big(I_{\tilde{\Phi}}(y) + \kappa \sum_{j=1, j\neq i}^{n} L(y, X_j, M_j) + w(y) \Big) \Big\} \end{aligned}$$

where $0 < \kappa \leq 1$.

When we change the thresholds $t_i' := \frac{t_i}{1+\kappa t_i}$ it follows:

$$C_{(X_i,M_i)}\big(\tilde{\Phi} + \kappa \sum_{j=1, j\neq i}^{n} \varepsilon_{(X_j,M_j)}, w, t_i\big) = C_{(X_i,M_i)}\big(\tilde{\Phi} + \kappa \sum_{j=1}^{n} \varepsilon_{(X_j,M_j)}, w, t_i'\big)$$

Proposition 1. *Let $(C_{(X_i,M_i)})_{i=1,\ldots,n}$ be the SINR cells of the collection of marked points $(X_i, M_i)_{i=1,\ldots,n} \in \mathbb{R}^d \times \mathbb{R}^l$ for thresholds $t_i > 0,\ i = 1, \ldots, n$.*
For any $J \subset \{1, \ldots, n\}$ satisfying $\bigcap_{i\in J} C_{(X_i,M_i)} \neq \emptyset$ we have $\sum_{i\in J} t_i' \leq \frac{1}{\kappa}$.

Consider a collection of marked points $(X_i, M_i)_{i=1,\ldots,n}$ with $X_i = X \in \mathbb{R}^d$ and $t_i = t \in \mathbb{R}^+$ for all $i = 1, \ldots, n$. With the above result it follows:

$$\frac{1}{\kappa} \geq \sum_{i\in J} t_i' \Leftrightarrow \frac{1 + t\kappa}{t\kappa} \geq |J|.$$

The maximal number of cells covering simultaneously a point $y \in \mathbb{R}^d$ is bounded by $\lfloor \frac{1+t\kappa}{t\kappa} \rfloor$. This upper bound is called the *pole capacity*. Note that the pole capacity only depends on the thresholds and not on the marks.

Definition 2 (SINR coverage process)**.** *The SINR coverage process is the following union of SINR cells*

$$\Xi_{\text{SINR}} = \Xi(\widehat{\Phi}, w) = \bigcup_{(x_i,m_i,t_i)\in\widehat{\Phi}} C_{(x_i,m_i)}(\widetilde{\Phi} - \varepsilon_{(x_i,m_i)}, w, t_i).$$

Standard Scenario

- $\widehat{\Phi}$ stationary independent m.p.p. (i.m.p.p.) with points in $\mathbb{R}^2$ and intensity $\lambda > 0$
- marks (m_i, t_i) have for all i the same given distribution $\mathbb{P}(m_i \leq u, t_i \leq v) = G(u, v)$, which does not depend on the location of the point
- thermal noise is constant in space $w(y) = W$, where $W \geq 0$ is a random variable independent of $\widehat{\Phi}$
- response function $L(y, x, m) = \frac{m}{l(|y-x|)}$, l is a omnidirectional-path-loss function

Nearest Transmitter Cell

We want to calculate the probability that the origin (denoted by O) is covered by the cell of the nearest transmitter. For that we consider the standard scenario with a Poisson distributed Φ, exponential distributed marks m_i and deterministic marks $t_i = t > 0$.

Denote $x^0 = \arg\min_{x\in\Phi} |x|$ the nearest transmitter to O (almost sure well defined) with corresponding mark m^0. The coverage probability is

$$\begin{aligned} p_* &:= \mathbb{P}[O \in C_{(x^0,m^0)}(\widetilde{\Phi} - \varepsilon_{(x^0,m^0)}, W, t)] = \mathbb{P}\big[m^0 \geq l(|x^0|)) \cdot t(I_{\widetilde{\Phi}} - \frac{m^0}{l(|x^0|)} + W)\big] \\ &= \int_0^\infty 2\pi\lambda r \cdot e^{-\pi\lambda r^2} \cdot \mathcal{L}_W(\mu t \cdot l(r)) \cdot \exp\Big\{-2\pi\lambda \int_r^\infty \frac{u}{1 + \frac{l(u)}{t\cdot l(r)}}\, \mathrm{d}u\Big\}\, \mathrm{d}r, \end{aligned}$$

where $\mu = \frac{1}{\mathbb{E}[m^0]}$ and $\mathcal{L}_W$ is the Laplace transform of W.

For the proof of these statement we consider $|x^0| = r$ and applying the law of total probability.

Ξ_{SINR} as a Random Closed Set

In stochastic geometry it is customary to require Ξ_{SINR} to be a closed set, but we want to check the stronger property that the number of cells of Ξ_{SINR} which hit a given bounded set is finite. Let $C_i := C_{(x_i,m_i)}(\widetilde{\Phi} - \varepsilon_{(x_i,m_i)}, w, t_i)$ be the i-th cell and K be bounded. Then $N_K := \sum_{(x_i,m_i)\in\widetilde{\Phi}} \mathbb{1}(K \cap C_i \neq \emptyset)$ is the number of cells that hit the given set K.

We will give some conditions such that $\mathbb{E}[N_K]$ is finite for abitrary large K. Therefore we assume that $\widehat{\Phi}$ is an i.m. Poisson p.p. with i.i.d. marks (m, t), which are independent of $\widehat{\Phi}$. Now we consider two different types of response functions.

(A) there exists a finite R such that for all $x, y \in \mathbb{R}^d$ with $|y - x| > R$ it follows $L(y, x, m) = 0$ for all $m \in \mathbb{R}^l$,

(B) there exist $A > 0$ and $\beta > 0$ such that $L(y, x, m) < A\frac{|m|}{|y-x|^\beta}$ for all $y, x \in \mathbb{R}^d,\ m \in \mathbb{R}^l$

Now we get the following proposition.

Proposition 2. *Let $\widehat{\Phi}$ be an i.m. Poisson p.p. with intensity measure Λ and with i.i.d. marks (m,t), which are independent of $\widehat{\Phi}$. We have $\mathbb{E}[N_K] < \infty$ for abitrary large bounded, measureble K if one of the following statements holds:*

i) (A) is satisfied and $w(y) > 0$ for all $y \in \mathbb{R}^d$ almost surely,

ii) (B) is satisfied, $w(y) > W > 0$ for all $y \in \mathbb{R}^d$ almost surely and for all $R > 0$ it holds

$$\mathbb{E}\Big[\Lambda\Big(B\Big(0, R + (\frac{A|m_0|}{t_0 W})^{\frac{1}{\beta}}\Big)\Big)\Big] < \infty,$$

iii) (B) is satisfied, $L(y,x,m) > 0$ almost surely for all $y \in \mathbb{R}^d$ and for all $R > 0$ it holds

$$\int_{\mathbb{R}^d} e^{-\Lambda(B(0,|x|))}\mathbb{E}\Big[\Lambda\Big(B\Big(0, R + (\frac{A|m_1|}{t_1 \inf_{y:|y|<R} L(y,x,m_0)})^{\frac{1}{\beta}}\Big)\Big)\Big]\,\Lambda(\mathrm{d}x) < \infty,$$

where (m_0,t_0) and (m_1,t_1) are independent marks with the same distribution.

Factorial Moments

We abbreviate $N_y := N_{\{y\}}$, where $N_{\{y\}}$ denote the number of cells covering point $y \in \mathbb{R}^d$. For a natural number n the n-th factorial moment of N_y is given by $\mathbb{E}[N_y^{(n)}]$, where $k^{(n)} = k(k-1)^+ \dots (k-n+1)^+, \forall k \in \mathbb{N}$.

If there exists a constant $\rho > 0$ such that $t_i > \rho$ for all marks, then $N_y < \frac{1}{\rho}$ almost surely. With the expansion of the generating function we get

$$\mathbb{P}[N_y = n] = \frac{1}{n!}\sum_{k=0}^{\infty}(-1)^k \frac{\mathbb{E}[N_y^{(n+k)}]}{k!}$$

since N_y is finite. We can see, that if we know something about the expectation of the n-th factorial moment of N_y, then we get informations about the distribution of N_y.

If we assume that $\widehat{\Phi}$ is an i.m. Poisson p.p. with intensity measure Λ, then we get by the refined Campbell theorem

$$\mathbb{E}[N_y^{(n)}] = \int_{\mathbb{R}^d} \dots \int_{\mathbb{R}^d} \mathbb{P}\Big[y \in \bigcap_{k=1}^{n} C_{(x_k,m_k)}(\widetilde{\Phi} + \sum_{i=1, i\neq k}^{n} \varepsilon_{(x_i,m_i)}, w, t_i)\Big]\,\Lambda(\mathrm{d}x_1)\dots\Lambda(\mathrm{d}x_n).$$

References

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