Weierstraß-Institut für
Angewandte Analysis und Stochastik

Leibniz
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Upper tails of self-intersection local times: survey of proof techniques

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- Simple random walk $(S_n)_{n\in\mathbb{N}_0}$ on $\mathbb{Z}^d$
- Local times $\ell_n(z) = \sum_{i=0}^n \mathbb{1}\{S_i = z\}$ for $n \in \mathbb{N}, z \in \mathbb{Z}^d$
- p-norm of local times $\|\ell_n\|_p = (\sum_{z\in\mathbb{Z}^d} \ell_n(z)^p)^{1/p}$

For $p \in \mathbb{N}$, we have the p-fold self-intersection local time (SILT):

$$\|\ell_n\|_p^p = \sum_{i_1,\dots,i_p=0}^{n} \mathbb{1}_{\{S_{i_1}=\dots=S_{i_p}\}},$$

Typical behaviour [Cerny 2007] for $d = 2$ and [Becker/König 2009] for $d \geq 3$:

$$\mathbb{E}[\|\ell_n\|_p^p] \sim Ca(n), \qquad \text{where} \qquad a(n) = \begin{cases} n^{(p+1)/2} & \text{if } d = 1, \\ n(\log n)^{p-1} & \text{if } d = 2, \\ n & \text{if } d \geq 3. \end{cases}$$

Goal

Goal: Asymptotics of

$$\frac{1}{n}\log \mathbb{P}(\|\tfrac{1}{n}\ell_n\|_p \geq r_n), \qquad n \to \infty,$$

for $(nr_n)^p - \mathbb{E}[\|\ell_n\|_p^p] \to \infty$.

- very large deviations: $(nr_n)^p \gg a(n)$
- large deviations: $(nr_n)^p \sim \gamma a(n)$ with $\gamma > C$

What is the best path strategy to produce many self-intersections?

Rough Heuristics (1)

(only very large-deviations case $(nr_n)^p \gg a(n)$)

Strategy to meet $\{\|\frac{1}{n}\ell_n\|_p \geq r_n\}$:

The path fills a ball B_{α_n} of radius $1 \ll \alpha_n \ll n^{1/d}$ within a time interval $[0,t_n] \subset [0,n]$ in order to produce $(nr_n)^p$ self-intersections, and runs freely afterwards.

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$$\ell_n(z) \approx \ell_{t_n}(z) \asymp t_n \alpha_n^{-d} \qquad \text{for } z \in B_{\alpha_n}$$

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$$(nr_n)^p \asymp \|\ell_n\|_p^p \approx \sum_{z \in B_{\alpha_n}} \ell_{t_n}(z)^p \asymp \alpha_n^d (t_n \alpha_n^{-d})^p = t_n^p \alpha_n^{d(1-p)}, \qquad \text{i.e.,} \qquad t_n \asymp nr_n \alpha_n^{d(p-1)/p}.$$

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$$-\log \mathbb{P}(S_{[0,t_n]} \subset B_{\alpha_n}) \asymp \frac{t_n}{\alpha_n^2} \asymp nr_n \alpha_n^{\frac{d}{p}(p-1)-2}.$$

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Optimal choices:

$$t_n \asymp \begin{cases} n & \text{if } d < \frac{2p}{p-1}, \\ nr_n & \text{if } d > \frac{2p}{p-1}, \end{cases} \qquad \text{and} \qquad \alpha_n \asymp \begin{cases} r_n^{\frac{p}{d(1-p)}} & \text{if } d < \frac{2p}{p-1}, \\ 1 & \text{if } d > \frac{2p}{p-1}. \end{cases}$$

Rough Heuristics (2)

Hence, we conjecture

Theorem A.

$$-\frac{1}{n}\log \mathbb{P}(\|\tfrac{1}{n}\ell_n\|_p \geq r_n) \asymp \frac{1}{n}\frac{t_n}{\alpha_n^2} \asymp \frac{1}{\alpha_n^2} \asymp r_n^{\frac{2p}{d(p-1)}\vee 1} \asymp \begin{cases} r_n^{\frac{2p}{d(p-1)}} & \text{if } d < \frac{2p}{p-1}, \\ r_n & \text{if } d > \frac{2p}{p-1}. \end{cases}$$

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- Lower-critical dimension: homogeneous squeezing on a large area.
- Upper-critical dimension: short-time clumping on finitely many sites.

Precise heuristics (1)

First subcritical dimensions $d < \frac{2p}{p-1}$.
Scaled normalized version of ℓ_n:

$$L_n(x) = \frac{\alpha_n^d}{n} \ell_n\big(\lfloor x\alpha_n \rfloor\big), \qquad \text{for } x \in \mathbb{R}^d.$$

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Weak large-deviation principle (in the spirit of DONSKER-VARADHAN) with speed $n\alpha_n^{-2}$ and rate function

$$\mathscr{I}(f) = \frac{1}{2}\big\|\nabla f\big\|_2^2,$$

i.e.,

$$\mathbb{P}(L_n \in \cdot\,) = \exp\Big\{-\frac{n}{\alpha_n^2}\Big[\inf_{f^2\in\cdot} \mathscr{I}(f) + o(1)\Big]\Big\}.$$

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Note that

$$\|\ell_n\|_p = \Big(\sum_{z\in\mathbb{Z}^d} \ell_n(z)^p\Big)^{1/p} = n\alpha_n^{-d}\Big(\sum_{z\in\mathbb{Z}^d} L_n\big(\tfrac{z}{\alpha_n}\big)^p\Big)^{1/p} = n\alpha_n^{\frac{d(1-p)}{p}}\|L_n\|_p = nr_n\|L_n\|_p.$$

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Hence,

$$\{\|\tfrac{1}{n}\ell_n\|_p \geq r_n\} = \big\{\|L_n\|_p \geq 1\big\} \qquad \text{and} \qquad \frac{n}{\alpha_n^2} = nr_n^{\frac{2p}{d(p-1)}}.$$

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Hence, we conjecture, for $d < \frac{2p}{p-1}$,

Theorem B.

$$\lim_{n\to\infty} \frac{r_n^{\frac{2p}{d(1-p)}}}{n} \log \mathbb{P}(\|\tfrac{1}{n}\ell_n\|_p \geq r_n) = -\chi_{d,p},$$

where

$$\chi_{d,p} = \inf\Big\{\frac{1}{2}\|\nabla f\|_2^2 \colon\ f \in H^1(\mathbb{R}^d), \|f^2\|_p = 1 = \|f\|_2\Big\}.$$

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Remark:

$$\chi_{d,p} > 0 \qquad \Longleftrightarrow \qquad d(p-1) \leq 2p$$ [GANTERT/KÖNIG/SHI 2004]

.

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Now supercritical dimensions $d > \frac{2p}{p-1}$. We approximate

$$\{\|\tfrac{1}{n}\ell_n\|_p \geq r_n\} \approx \{\|\ell_{st_n}\|_p \geq nr_n\} = \Big\{\Big\|\frac{1}{st_n}\ell_{st_n}\Big\|_p \geq \frac{1}{s}\Big\}.$$

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In the time-continuous case,

$$\chi_{d,p} = \inf\left\{\frac{1}{2}\|\nabla g\|_2^2 \colon \|g^2\|_p = 1\right\}.$$

Comments

- Alternative formulations in terms of exponential moments of $\|\ell_n\|_p$.
- Continuous-time case very similar.
- Proof of lower bounds quite easy with the help of Hölder's inequality and some approximations.
- Proof of upper bounds much more difficult due to bad continuity properties of the map $f \mapsto \|f\|_p$.
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Now we survey proofs for upper bounds.

Triangular decomposition and smoothing

[Chen 2009, Theorems 8.2.1 and 8.4.2] proves Theorem B for $p = 2$ in dimensions $d \in \{2,3\}$, even for $r_n = \frac{1}{n}(\mathbb{E}[\|\ell_n\|_2^2] + nb_n)^{1/2}$ with $1 \ll b_n \ll n$.

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- triangular decomposition of the number of self-intersections:

$$\|\ell_n\|_2^2 = \sum_{j=1}^{2^N} \eta_j^{(N)} + \sum_{j=1}^{N} \sum_{k=1}^{2^{j-1}} \xi_{j,k}^{(N)},$$

where $N \in \mathbb{N}$ is a large auxiliary parameter and

$$\eta_j^{(N)} = \sum_{(j-1)n2^{-N} < i < i' \le jn2^{-N}} \mathbb{1}\{S_i = S_{i'}\},$$

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- a smoothing technique with the help of a convolution of a smooth approximation of the delta measure,
- a compactness criterion by [de Acosta 1984] (bounds for certain exponential integrals of the Minkowski functional). Requires Hahn-Banach theorem, topological duality between L^p and L^q, and Arzelá-Ascoli's theorem.

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He extends the triangular decomposition to arbitrary $p > 1$ by a bisection technique for $\|\ell_n\|_p^p$, i.e., for a sum of p-th powers of integers:

$$(l_1+l_2)^p \leq l_1^p + l_2^p + 2^p \sum_{i=0}^{\infty} b_{i+1}^{p-2} l_1 l_2 \mathbb{1}\{b_i \leq \max\{l_1,l_2\} < b_{i+1}\} \qquad l_1, l_2 \in \mathbb{N},$$

where $1 = b_0 < b_1 < b_2 < \dots$ defines a suitable partitioning of $[1,\infty)$.

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where $1 = b_0 < b_1 < b_2 < \dots$ defines a suitable partitioning of $[1, \infty)$.

Furthermore, he uses a decomposition of the space into regions where the local times are small, medium-sized or large, and he decomposes the event $\{\|\frac{1}{n}\ell_n\|_p \ge r_n\}$ into various subevents.

Surgery on circuits and clusters

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The ansatz is an upper estimate of $\|\ell_n\|_2^2 - \mathbb{E}[\|\ell_n\|_2^2]$ in terms of $\|1\!\!1_\Lambda \ell_{s\sqrt{n}}\|_2^2$ for many choices of a finite set $\Lambda \subset \mathbb{Z}^d$ on the event $\{S_{s\sqrt{n}} = 0\}$.

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Asselah introduces for infinite-time random walk a map from finite n-dependent boxes to bounded subboxes that compares paths with high values of local times in the large box to those having high local time values in the small box.

Dynkin's isomorphism

The critical value $p = \frac{d}{d-2}$ in dimensions $d \geq 3$ is considered by [Castell 2010]. It turns out that that Theorem B is true with α_n as in the lower-critical dimension and $\chi_{d,p}$ as in the upper-critical dimension. Later [Laurent 2010a] and [Laurent 2010b] extension to a proof of Theorem B for all $p > 1$ in the very-large deviation case.

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Main idea:

The joint law of $\ell_\tau^{(R)}$ in a box B_R with $R \asymp t^{1/d}$, stopped at an independent exponential time τ with parameter $\asymp r_t$ is related to Z^2, the square of a Gaussian process $Z = (Z_x)_{x \in B_R}$ with covariance matrix equal to the Green function, $G_{R,\tau}$.

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Now concentration inequalities for Gaussian integrals can be applied. The tail behaviour of $\|Z\|_{2p} - M$ (with M the median) is equal to that of a Gaussian variable with variance equal to $\sup\{\langle f, G_{R,\tau} f\rangle \colon f \in \ell^{2p}(\mathbb{Z}^d), \|f\|_{2p} = 1\}$, and this converges towards $\chi_{d,p}$.

Polynomial moments

- Expand $\exp\{\theta \alpha_t^{2-d+d/p} \|\ell_t\|_p\}$,
- explicitly write out $\ell_t(z) = \int_0^t \delta_z(S_r)\,\mathrm{d}r$ and the pk-th moments and summarize and transform the arising multi-sum as far as possible,
- use integrability of the p-th power of the Green function of Brownian motion around its singularity,
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- compactify by periodization for the p-th powers of the Green function.
- [VAN DER HOFSTAD/MÖRTERS/K. 2006], lower-critical dimension:

$$\mathbb{E}\big(\|\ell_t\|_p^{pk}\mathbb{1}\{S_{[0,t]}\subset B_{L\alpha_t}\}\big) \le k^{kp}C^k\alpha_t^{k[d+(2-d)p]}, \qquad k \ge \frac{t}{\alpha_t^2},$$

which implies a bit less than Theorem B, but only for $\alpha_t \ll t^{1/(d+1)}$.

Polynomial moments

- Expand $\exp\{\theta\alpha_t^{2-d+d/p}\|\ell_t\|_p\}$,
- explicitly write out $\ell_t(z)=\int_0^t \delta_z(S_r)\,\mathrm{d}r$ and the pk-th moments and summarize and transform the arising multi-sum as far as possible,
- use integrability of the p-th power of the Green function of Brownian motion around its singularity,
- compactify by periodization for the p-th powers of the Green function.

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  which implies a bit less than Theorem B, but only for $\alpha_t\ll t^{1/(d+1)}$.
- [CHEN/MÖRTERS 2008]: The intersection local time I of p random walks in $d>\frac{2p}{p-1}$ satisfies

$$\lim_{a\to\infty}a^{-1/p}\log\mathbb{P}(I>a)=-p\chi_{d,p}.$$

  This is based on [K./MÖRTERS 2002, Lemma 2.1]: For any positive variable X,

$$\lim_{k\to\infty}\frac{1}{k}\log\mathbb{E}\Big[\frac{X^k}{k!^p}\Big]=\kappa\qquad\Longleftrightarrow\qquad\lim_{a\to\infty}a^{-1/p}\log\mathbb{P}(X>a)=-pe^{\kappa/p}.$$

  (Should be extendable to prove Theorem B in supercritical dimensions.)

Density of local times

- explicit formula for the joint density of the local times $(\ell_t(z))_{z\in B}$ [BRYDGES/VAN DER HOFSTAD/K. 2007],
- formula impenetrable, but handy upper bound. A discrete, t-dependent variational formula arises,
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Main steps:

$$\frac{1}{t}\log\mathbb{E}\left(\exp\left\{t\alpha_t^{-2\lambda}\left\|\tfrac{1}{t}\ell_t^{(L\alpha_t)}\right\|_p\right\}\right)\leq\rho_{d,p}^{(\mathrm{d})}(L\alpha_t,\alpha_t^{-2\lambda})+\varepsilon_t,$$

where $\lambda=\frac{2p+d-dp}{2p}\in(0,1)$ and

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Severe restrictions: $d\leq\frac{2}{p-1}$ and $r_t\gg(\log t/t)^{\frac{d(p-1)}{p(d+2)}}$.