Weierstrass Institute for
Applied Analysis and Stochastics

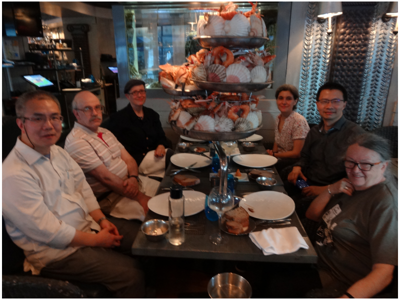
Leibniz Association

Weakly self-avoiding random walk in a Pareto-distributed random potential

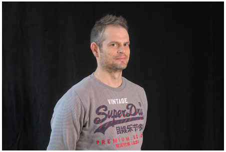
Wolfgang König
TU Berlin and WIAS

based on joint work with Nicolas Pétrélis, Renato dos Santos, Willem van Zuijlen, *AoP*, to appear

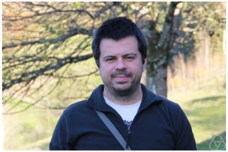
CONGRATULATION AND ALL OUR BEST WISHES, DEAR ZHAN!



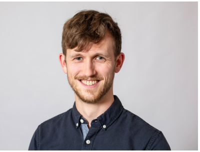
(from another birthday conference (2016))



Nicolas Pétrélis
(Nantes)



Renato dos Santos
(Belo Horizonte)



Willem van Zuijlen
(WIAS)

- We study a continuous-time random walk on $\mathbb{Z}^d$ with attraction to randomly located sites, but also under self-repellent interaction.
- The first effect is in the spirit of the parabolic Anderson model, where it is known that the path typically moves quickly to one site and stays there for the rest of the time (highly localised behaviour).
- The self-repellence (properly fine-tuned) forces the random walker to quickly move to many other sites as well.
- We express this comprehensively in terms of a random variational problem.

$X=(X_s)_{s\geq 0}$ continuous-time simple random walk on $\mathbb{Z}^d$.

$\xi=(\xi(z))_{z\in\mathbb{Z}^d}$ i.i.d. random potential *Pareto-distributed* with parameter $\alpha>d$:

$$F(r)=\mathbf{P}[\,\xi(z)\leq r\,]=1-r^{-\alpha},\qquad r\geq 1.$$

Hamiltonian $$H_t^{(\xi,\beta)}(X)=\int_0^t \xi(X_s)\,\mathrm{d}s-\beta\int_0^t\int_0^t \mathbb{1}_{\{X_s=X_u\}}\,\mathrm{d}s\,\mathrm{d}u,\qquad \beta\in[0,\infty).$$

Transformed path measure $$\frac{\mathrm{d}\mathbb{P}_t^{(\xi,\beta)}}{\mathrm{d}\mathbb{P}}(X)=\frac{\mathrm{e}^{H_t^{(\xi,\beta)}(X)}}{Z_t^{(\xi,\beta)}},\qquad \text{where } Z_t^{(\xi,\beta)}=\mathbb{E}[\mathrm{e}^{H_t^{(\xi,\beta)}}].$$

Special case: $\beta=0$. Here $Z_t^{(\xi,0)}$ is the parabolic Anderson model (PAM). The walk is attracted to the sites with the best compromise between a high value of ξ and being not too far from the origin. It runs quickly to the best site and stays there until time t.

The PAM with Pareto-distributed potential was studied in [K., LACOIN, MÖRTERS, SIDOROVA 2009]. See [MÖRTERS, ORTGIESE, SIDOROVA 2011] for an extension and [MÖRTERS 2011] for a survey on the PAM with heavy-tailed potential. Furthermore, see [BISKUP, K., DOS SANTOS 2018] for localisation in the PAM for the double-exponential distribution, and see [ASTRAUSKAS 2016] and [K. 2016] for surveys on the PAM. Localisation for some bounded potential has been proved in [DING, FUKUSHIMA, SUN, XU 2020].

$$\text{local times} \qquad \ell_t(z) = \int_0^t \mathbb{1}\{X_s = z\}\,\mathrm{d}s, \qquad z \in \mathbb{Z}^d. \tag{1}$$

Then

$$H_t^{(\xi,\beta)}(X) = \sum_{z\in\mathbb{Z}^d} \xi(z)\ell_t(z) - \beta \sum_{z\in\mathbb{Z}^d} \ell_t(z)^2 = \langle \xi, \ell_t\rangle - \beta\|\ell_t\|_2^2.$$

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Three influences:

- attraction to sites of high ξ-peaks,
- self-repellence,
- probability costs.

Interesting only if all three contribute on same scale. Hence, we rescale the self-repellence: replace β by β_t such that

$$\langle \xi, \ell_t\rangle \asymp \beta_t\|\ell_t\|_2^2 \qquad \text{under } \mathbb{P}_t^{(\xi,\beta_t)}.$$

How do we find the asymptotic scale of ξ under $\mathbb{P}_t^{(\xi,\beta_t)}$?

Since we consider large-t exponential moments, of ξ, this comes from the highest peaks of ξ, according to spatial extreme-value theory. For Pareto-distributed ξ, the highest peaks are single sites, and their heights and locations are approached by a PPP as follows.

spatial length scale : $\qquad r_t := \left(\frac{t}{\log t}\right)^{1+q}, \qquad$ where $q = \frac{d}{\alpha - d}$.

rescaled point process: $\qquad \Pi_t = \sum_{z\in\mathbb{Z}^d} \delta_{\left(\frac{\xi(z)}{r_t^{d/\alpha}}, \frac{z}{r_t}\right)} \Longrightarrow \Pi \qquad$ as $t \to \infty$,

with reference Poisson point processs

$$\Pi \sim \mathrm{PPP}\big((0,\infty)\times\mathbb{R}^d, \alpha f^{-1-\alpha}\,\mathrm{d}f \otimes \mathrm{d}y\big).$$

Hence, we expect the rescaled local times $\frac{1}{t}\ell(\lfloor y r_t \rfloor)$ to be concentrated on $y \approx$ the points of Π (i.e., its space-part). Then

$$H_t^{(\xi)}(X) = t r_t^{d/\alpha} \int \frac{\xi(x r_t)}{r_t^{d/\alpha}}\, r_t^d \mathrm{d}x - \beta_t t^2 \int \Big(\frac{\ell_t(x r_t)}{t}\Big)^2 r_t^d \mathrm{d}x.$$

Hence, the central object is the rescaled local times as density w.r.t. Π_t:

$$w(f,y) = \frac{\mathrm{d}W_t^{(\xi,X)}}{\mathrm{d}\Pi_t}(f,y) = \frac{\ell_t(\lfloor y r_t \rfloor)}{t}, \qquad (f,y) \in (0,\infty) \times \mathbb{R}^d.$$

Then $\mathrm{d}W_t^{(\xi,X)}$ is a random variable with values in $\mathcal{W}$, the set of subprobability measures on $(0,\infty) \times \mathbb{R}^d$. With the vague topology, it is compact and Polish.

With a parameter $\theta \in (0,\infty)$, we put $\beta_t := \theta \dfrac{t^{q-1}}{(\log t)^q}$.

Then

$$H_t^{(\xi)}(X) = r_t \log t \int_{(0,\infty)\times\mathbb{R}^d} \left[f w(f,y) - \theta w(f,y)^2 \right] \, \mathrm{d}\Pi_t(f,y).$$

(q and β_t are decreasing in α. If $\alpha > 2d$, then $\beta_t \to 0$ as $t \to \infty$.)

Abbreviate $H_t^{(\xi)} := H_t^{(\xi,\beta_t)}, \qquad Z_t^{(\xi)} := Z_t^{(\xi,\beta_t)}, \qquad \mathbb{P}_t^{(\xi)} := \mathbb{P}_t^{(\xi,\beta_t)}.$

Interpret a point measure μ on $(0,\infty)\times\mathbb{R}^d$ as the rescaled measure of the points where the walk spends $\asymp t$ time; assume that $\mu \ll \mathcal{P}$. (Later, we take $\mathcal{P}=\Pi$, the PPP.)
Put $w=\frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{P}}$ and define

the energy functional
$$\Phi_{\mathcal{P}}(\mu)=\int_{(0,\infty)\times\mathbb{R}^d}\left[fw(f,y)-\theta w(f,y)^2\right]\,\mathrm{d}\mathcal{P}(f,y),$$

and the entropy functional
$$\mathcal{D}_{\mathcal{P}}(\mu)=\sup_{Y\Subset\mathrm{supp}_{\mathbb{R}^d}\mu} D_0(Y),$$

where $D_0(Y)$ is the smallest possible length of a path from the origin that reaches all points in Y (in the spirit of the traveling-salesman problem).

Variational problem:
$$\Xi(\mathcal{P})=\sup_{\mu\in\mathcal{W}}\big(\Phi_{\mathcal{P}}(\mu)-q\mathcal{D}_{\mathcal{P}}(\mu)\big),\qquad \mathcal{P}\in\mathcal{M}_{\mathrm{p}}((0,\infty)\times\mathbb{R}^d).$$

Theorem: Weak convergences

For $\theta \in (0,\infty)$ and $\alpha \in (d,\infty)$,

1.
$$\frac{1}{r_t \log t} \log Z_t^{(\xi)} \overset{t\to\infty}{\Longrightarrow} \Xi(\Pi) \qquad \text{and} \qquad \mathbf{P}\big[\,\Xi(\Pi) \in [0,\infty)\,\big] = 1.$$

2. if $\alpha \in (2d,\infty)$, then

   2.1 $\mathbf{P}$-almost surely,
   $$[\Phi_\Pi - q\mathcal{D}_\Pi](\mu^*) = \Xi(\Pi) \qquad \text{for a unique random } \mu^*.$$

   2.2 Almost surely, μ^* is a probability measure with finite support and $\mu^* \ll \Pi$. For all $k \in \mathbb{N}$, $\mathbf{P}[\,\#\operatorname{supp}\mu^* = k\,] > 0$. Consequently, $\mathbf{P}\big[\,\Xi(\Pi) \in (0,\infty)\,\big] = 1$.

   2.3 $W_t^{(\xi,X)} \overset{t\to\infty}{\Longrightarrow} \mu^*$ in $\mathcal{W}$ in the sense that
   $$\mathbf{E}\big[\mathbb{E}_t^{(\xi)}[g(W_t^{(\xi,X)})]\big] \to \mathbf{E}[g(\mu^*)], \qquad g \in C_{\mathrm{b}}(\mathcal{W}).$$

Actually, 2.3 is implied by the stronger statement that $\mathbb{P}_t^{\xi} \circ (W_t^{(\xi,X)})^{-1}$ converges weakly on $\mathcal{W}$ under $\mathbf{E}$ towards δ_{μ^*}, which we actually prove.

Recall that

$$H_t^{(\xi)}(X) = (r_t \log t)\, \Phi_{\Pi_t}(W_t^{(\xi,X)}) \qquad \text{and} \qquad Z_t^{(\xi)} = \mathbb{E}\big[\mathrm{e}^{(r_t \log t)\, \Phi_{\Pi_t}(W_t)}\big].$$

Hence, Varadhan's lemma suggests the main result, if we can explain why and how $\mathcal{D}$ describes the random walk ($\Longrightarrow$ later).

Relevance of the minimizer μ^*:

There exist (random) $k^* \in \mathbb{N}$ and $(f_1^*, y_1^*), \dots, (f_{k^*}^*, y_{k^*}^*) \in \mathrm{supp}(\Pi)$ and $w_1^*, \dots, w_{k^*}^* \in (0,1]$ satisfying $\sum_{i=1}^{k^*} w_i^* = 1$ such that

$$\mu^* = \sum_{i=1}^{k^*} w_i^* \delta_{(f_i^*, y_i^*)}.$$

Hence, a bit loosely speaking, the typical path under $\mathbb{P}_t^{(\xi)}$ spends $\sim w_i^* t$ time units in the site $\sim \lfloor y_i^* r_t \rfloor$ with value $\xi(\lfloor y_i^* r_t \rfloor) \sim f_i^* r_t^{d/\alpha}$ for any $i \in \{1, \dots, k^*\}$ and elsewhere only $o(t)$ time units.

No assertion about the order of the visited sites! This is still to be optimized in the spirit of the traveling salesman problem.

Denote by $\mathcal{A}_t^{z,s}$ the event that the walker wanders on some fixed shortest path to a site z during the time interval $[0, st)$ and stays at z during $[st, t]$. Then

$$\mathbb{P}(\mathcal{A}_t^{z,s}) = \mathrm{Poi}_{2dst}(|z|)(2d)^{-|z|}\mathrm{e}^{-(1-s)2dt}\#\{\text{shortest paths } 0 \longleftrightarrow z\}$$
$$\approx \exp\left\{|z|\left[\log\frac{t}{|z|} + \log s\right]\right\},$$

and hence

$$Z_t^{(\xi)} \geq \exp\left\{r\left[\log\frac{t}{r} + \log s\right]\right\}\mathrm{e}^{tr^{d/\alpha}}\mathrm{e}^{-str^{d/\alpha}},$$

The optimal $s \in [0, 1]$ for the second and the last term is $s \approx \frac{1}{t}r^{1-d/\alpha}$:

$$Z_t^{(\xi)} \geq \exp\left\{r\log\frac{t}{r} + tr^{d/\alpha} - \log\left(tr^{d/\alpha-1}\right)\right\} = \exp\left\{tr^{d/\alpha} - \frac{d}{\alpha}r\log r\right\}.$$

The maximal r satisfies $tr^{d/\alpha-1} = 1 + \log r$, i.e., $r = r_t = (t/\log t)^{1+q}$ with $q = \frac{d}{\alpha-d}$
Then both the energy term $tr^{d/\alpha}$ and the entropy term $-\frac{d}{\alpha}r\log r \approx -qr_t\log t$ are on the scale $r_t\log t$.
Hence, the functional $q\,\mathcal{D}_{\mathcal{P}}(\mu) = q\,\sup_{Y\in\mathrm{supp}_{\mathbb{R}^d}\mu} D_0(Y)$ describes the exponential probabilistic cost payed by the simple random walk.

So far, this is without self-repellence (like in the PAM) and without specifying the cost (borrowed from [K., LACOIN, MÖRTERS, SIDOROVA 2009]).

Because of $\beta_t t^2 = \theta r_t \log t$, the self-repellence term is on the same order, but the functional $\Phi_{\mathcal{P}}(\mu) = \int_{(0,\infty)\times\mathbb{R}^d} \left[f w(f,y) - \theta w(f,y)^2 \right] \,\mathrm{d}\mathcal{P}(f,y)$ is not minimal if w is concentrated on just one site. There are many sites with potential values $\asymp r_t^{d/\alpha}$, and their distances are all of order $\asymp r_t$.

Elementary (but cumbersome) optimizing techniques, jointly with PPP-calculations, derive the assertions of the theorem. Main tools:

- Gamma-convergence
- Skorohod-embedding
- compactification of the variational problem (vague topology).

We first coin the term *goodness* of a given rescaled point process, and then show that, for $\alpha > 2d$, both Π_t and Π have this property.

We also show that, almost surely,

$$\alpha > d \implies \Xi(\Pi) < \infty \quad \text{and} \quad \alpha > 2d \implies \Xi(\Pi) > 0.$$

The latter assertion seems crucial for the behaviour of the path in the random potential. Apparently the PPP possesses sufficiently many sufficiently high potential values with not too large distances between them, such that trajectories exist for which it is worth paying the travels in order to profit from spending time in those large potentials.

Conjecturally different for $\alpha \in (d, 2d)$! We think that both $\{\Xi(\Pi) = 0\}$ and $\{\Xi(\Pi) > 0\}$ have positive probability here. With positive probability the PPP has the above property, but also the complementary property, leading to no such preferable locations as it is not worth travelling that far to profit from the large potential. We conjecture that intermediate order statistics need to be considered.