Percolation of the SINR Secrecy Graph (SSG)

Following Vaze and Iyer 2014

Andreas Preiß

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Definition

Let the points in $\Phi, \Phi_E \subset \mathbb{R}^2$ be distributed according to independent poisson point processes of intensity λ, λ_E. We call Φ the set of legitimate nodes and Φ_E the set of eavesdropper nodes and define

$$\mathrm{SINR}_{xy} := \frac{l(d_{xy})}{\gamma \sum_{z \in \Phi, z \neq x} l(d_{zy}) + 1}$$

for all $x, y \in \Phi$ and

$$\mathrm{SINR}_{xe} := \frac{l(d_{xe})}{\gamma_E \sum_{z \in \Phi, z \neq x} l(d_{ze}) + 1}$$

for all $x \in \Phi$, $e \in \Phi_E$.

Definition

We say that the signal attenuation function $l : [0, \infty) \to [0, \infty)$ fulfills standard assumptions if l is strictly decreasing on its support and $\int_0^\infty xl(x)\,\mathrm{d}x < \infty$.
Furthermore, we say that l fulfills the additional decay condition if for all $c > 0$ there is $M > 0$ such that $\forall\, x \geq 0 : \; l(x + M) \leq cl(x)$.

Definition

The maximum rate of secure communication [Wyner 1975] between $x, y \in \Phi$ is given by

$$R_{xy}^{\mathrm{SINR}} := 0 \vee \min_{e \in \Phi_E} \log_2 \left(\frac{1 + \mathrm{SINR}_{xy}}{1 + \mathrm{SINR}_{xe}} \right).$$

Definition

- For $\theta \geq 0$ we define the SINR secrecy graph $\mathrm{SSG}(\theta) := \{\Phi, \mathcal{E}\}$, where $\mathcal{E} := \{(x, y) : R_{xy}^{\mathrm{SINR}} > \theta\}$.
- We call $x \in \Phi$ connected to $y \in \Phi$ if $(x, y) \in \mathcal{E}$.
- If there is a sequence of edges from $x \in \Phi$ to $z \in \Phi$ we speak of a path from x to z and write $x \to z$.
- The connected component, also called cluster, of $x \in \Phi$ is given by $C_x := \{z \in \Phi : x \to z\}$.

In the following, we will only consider $\mathrm{SSG} := \mathrm{SSG}(0)$ with edge set $\mathcal{E} := \{(x, y) : \mathrm{SINR}_{xy} > \mathrm{SINR}_{xe}\ \forall e \in \Phi_E\}$.

Theorem

Let P^0 *be the palm distribution of* Φ *and* Φ_E *with respect to* $0 \in \Phi$. *Let* l *be a signal attenuation function fulfilling standard assumptions. For all* $\lambda_E \in (0, \infty)$ *and* $\gamma_E \in [0, 1]$,

1. *there is* $\lambda_1 \in (0, \infty), \gamma_1 \in (0, 1)$ *such that*
   $\forall\, \lambda > \lambda_1, \gamma < \gamma_1 : \; P^0(|C_0| = \infty) > 0$ *(supercritical regime, i.e. percolation occurs),*
2. *if* l *satisfies the additional decay condition, there is* $\lambda_2 \in (0, \infty)$ *such that*
   $\forall\, \lambda < \lambda_2, \gamma \in [0, 1] : \; P^0(|C_0| = \infty) = 0$ *(subcritical regime, i.e. percolation does not occur).*

For the proof of the first part, it is sufficient to consider the case of $\gamma_E = 0$, as percolation in this case implies percolation for arbitrary $\gamma_E \in [0, 1]$. The edge set then reduces to $\mathcal{E} = \{(x, y) : \mathrm{SINR}_{xy} > l(d_{xe})\ \forall\, e \in \Phi_E\}$.

Definition

- Let $\mathbf{S}$ be the square lattice with side $s > 0$ with a vertex at the origin and $\mathbf{S}' := \mathbf{S} + (s/2, s/2)$ be the dual lattice. For an edge $\mathbf{a}$ of $\mathbf{S}$ let $\mathbf{a}'$ be the edge of $\mathbf{S}'$ which crosses $\mathbf{a}$.
- Choose $\alpha(s) > 0$ such that $l(3s) < \frac{l(\sqrt{5}s)}{1+\alpha(s)}$. For an edge $\mathbf{a}$ of $\mathbf{S}$ let $S_1(\mathbf{a})$ and $S_2(\mathbf{a})$ be its two adjent squares and $Y(\mathbf{a})$ the $7s \times 8s$ rectangle of $\mathbf{S}$ which contains a $3s$ surrounding of $S_1(\mathbf{a}) \cup S_2(\mathbf{a})$.

Definition

For any edge $\mathbf{a}$ of $\mathbf{S}$ consider indicator variables $A(\mathbf{a}), B(\mathbf{a}), C(\mathbf{a})$ given by

1. $A(\mathbf{a}) = 1$ iff $S_1(\mathbf{a}) \cap \Phi \neq \emptyset$ and $S_2(\mathbf{a}) \cap \Phi \neq \emptyset$,
2. $B(\mathbf{a}) = 1$ iff $Y(\mathbf{a}) \cap \Phi_E = \emptyset$,
3. $C(\mathbf{a}) = 1$ iff for all $x, y \in (S_1(\mathbf{a}) \cup S_2(\mathbf{a})) \cap \Phi$ we have $I_{xy} := \sum_{z \in \Phi, z \neq x} l(d_{zy}) \leq \frac{\alpha(s)}{\gamma}$.

Then $\mathbf{a}$ and $\mathbf{a}'$ are defined to be open edges if $D(\mathbf{a}) := A(\mathbf{a})B(\mathbf{a})C(\mathbf{a}) = 1$ and closed edges otherwise.

Lemma

If an edge $\mathbf{a}$ *of* $\mathbf{S}$ *is open, then* $(x,y) \in \mathcal{E}$ *for all* $x, y \in (S_1(\mathbf{a}) \cup S_2(\mathbf{a})) \cap \Phi$*.*

Proof.

Let $x, y \in (S_1(\mathbf{a}) \cup S_2(\mathbf{a})) \cap \Phi$ be arbitrary.
Then, $d_{xy} \leq \sqrt{5}s$ and $I_{xy} \leq \frac{\alpha(s)}{\gamma}$ by property $C(\mathbf{a}) = 1$, hence $\mathrm{SINR}_{xy} \geq \frac{l(\sqrt{5}s)}{1+\alpha(s)}$.
For all $e \in \Phi_E$, by property $B(\mathbf{a}) = 1$ we have $d_{xe} > 3s$ and via $l(3s) < \frac{l(\sqrt{5}s)}{1+\alpha(s)}$ we get $\mathrm{SINR}_{xy} > \mathrm{SINR}_{xe}$. □

Theorem

Any finite open cluster of $\mathbf{S}$ *is surrounded by a closed circuit of* $\mathbf{S}'$ [Grimmett 1999, page 284][Kesten 1982, page 386].

Lemma

Let $\{\mathbf{a}_i\}_{1\leq i\leq n}$ *be a collection of distinct edges in* $\mathbf{S}$. *Then,*

1. $P^0(A(\mathbf{a}_i) = 0\ \forall\, 1 \leq i \leq n) \leq {p_1}^n$ *where* $p_1 := \sqrt[6]{1-(1-\exp(-\lambda s^2))^2}$,
2. $P^0(B(\mathbf{a}_i) = 0\ \forall\, 1 \leq i \leq n) \leq {p_2}^n$ *where* $p_2 := \sqrt[449]{1-\exp(-56s^2\lambda_E)}$,
3. [Dousse et al. 2006] $P^0(C(\mathbf{a}_i) = 0\ \forall\, 1 \leq i \leq n) \leq {p_3}^n$ *where* $p_3 := \exp\left(\frac{2\lambda}{K}\int_0^\infty x l(x)\,\mathrm{d}x + \frac{l(0)}{K} - \frac{\alpha(s)}{\gamma K}\right)$ *and* $K > 0$ *only depends on* l *and* s.

Lemma

Let $\{\mathbf{a}_i\}_{1\leq i\leq n}$ *be a collection of distinct edges in* $\mathbf{S}$*. Then,* $P^0(A(\mathbf{a}_i) = 0\ \forall\, 1 \leq i \leq n) \leq {p_1}^n$ *where* $p_1 := \sqrt[7]{1 - (1 - \exp(-\lambda s^2))^2}$.

Proof.

There is $T \subseteq \{\mathbf{a}_i\}_{1\leq i\leq n}$ with $|T| \geq \frac{n}{7}$ such that the interiors of $\{S_1(\mathbf{a}) \cup S_2(\mathbf{a})\}_{\mathbf{a}\in T}$ do not overlap.
Hence, the variables $\{A(\mathbf{a})\}_{\mathbf{a}\in T}$ are independent and $P^0(A(\mathbf{a}) = 0) \leq 1 - (1 - \exp(-\lambda s^2))^2$ for all $\mathbf{a} \in T$. □

Lemma

Let $\{\mathbf{a}_i\}_{1\le i\le n}$ *be a collection of distinct edges in* $\mathbf{S}$*. Then,*
$P^0(B(\mathbf{a}_i)=0\ \forall\, 1\le i\le n)\le {p_2}^n$ *where*
$p_2 := \sqrt[449]{1-\exp(-56s^2\lambda_E)}$.

Proof.

There is $Q\subseteq\{\mathbf{a}_i\}_{1\le i\le n}$ with $|Q|\ge\frac{n}{449}$ such that the interiors of $\{Y(\mathbf{a})\}_{\mathbf{a}\in Q}$ do not overlap.
Hence, the variables $\{B(\mathbf{a})\}_{\mathbf{a}\in Q}$ are independent and as all $Y(\mathbf{a})$ consist of 56 squares of $\mathbf{S}$, we get
$P^0(B(\mathbf{a}))=0)=1-\exp(-56s^2\lambda_E)$ for all $\mathbf{a}\in Q$. □

Lemma

Let $\{\mathbf{a}_i\}_{1 \leq i \leq n}$ *be a collection of distinct edges in* $\mathbf{S}$*. Then,* $P^0(D(\mathbf{a}_i) = 0 \; \forall \, 1 \leq i \leq n) \leq q^n$ *where* $q := \sqrt{p_1} + \sqrt[4]{p_2} + \sqrt[4]{p_3}$*.*

Lemma

For small enough $q > 0$, the probability of having a closed circuit in S' surrounding the origin is lower than 1.

Proof.

The number of possible circuits of length n around the origin is lower than $n3^{n-2}$. Hence, by the previous lemma, the probability in question is lower than $\sum_{n=1}^{\infty} n3^{n-2}q^n = \frac{1}{3(1-3q)^2}$. □

As having no closed circuit in $\mathbf{S}'$ surrounding the origin implies percolation in $\mathbf{S}$ implies percolation of SSG, we have $P^0(|C_0| = \infty) > 0$ for small enough $q > 0$.
Let $\varepsilon > 0$ be such that all $q \leq \varepsilon$ are small enough.
Choose

- $s > 0$ small enough such that $p_2 = \sqrt[449]{1 - \exp(-56 s^2 \lambda_E)} \leq (\varepsilon/3)^4$,
- $\lambda \in (0, \infty)$ large enough such that $p_1 = \sqrt[7]{1 - (1 - \exp(-\lambda s^2))^2} \leq (\varepsilon/3)^2$,
- γ small enough such that $p_3 = \exp\left(\frac{2\lambda}{K} \int_0^\infty x l(x)\,\mathrm{d}x + \frac{l(0)}{K} - \frac{\alpha(s)}{\gamma K}\right) \leq (\varepsilon/3)^4$.

Then we have $q = \sqrt{p_1} + \sqrt[4]{p_2} + \sqrt[4]{p_3} \leq \varepsilon$.

For the proof of the second part, it is sufficient to consider the case of $\gamma = 0$ and $\gamma_E = 1$, as percolation in this case implies percolation for arbitrary $\gamma, \gamma_E \in [0, 1]$.
The edge set then reduces to
$\mathcal{E} = \{(x, y) : l(d_{xy}) > \mathrm{SINR}_{xe} \; \forall \, e \in \Phi_E\}$ where

$$\mathrm{SINR}_{xe} = \frac{l(d_{xe})}{\sum_{z \in \Phi, z \neq x} l(d_{ze}) + 1}.$$

Definition

- For initially arbitrary $m > 0$ and $c > 0$ fix $M(m,c) > 9m$ such that $l(d + \frac{1}{9}M(m,c)) \leq \frac{l(d)}{1+c}$ for all $d \geq M(m,c)$.
- Let $\mathbf{M}$ be the square lattice with side $M(m,c)$ with a vertex at the origin and $\mathbf{M}'$ be the dual lattice.
- For an edge $\mathbf{a}$ of $\mathbf{M}$ let $S_1(\mathbf{a})$ and $S_2(\mathbf{a})$ be its two adjent squares and $T_i(\mathbf{a})$ be the square with side m with the same center as $S_i(\mathbf{a})$.

Definition

For any edge $\mathbf{a}$ of $\mathbf{M}$ consider indicator variables $\tilde{A}(\mathbf{a}), \tilde{B}(\mathbf{a}), \tilde{C}(\mathbf{a})$ given by

1. $\tilde{A}(\mathbf{a}) = 1$ iff $T_1(\mathbf{a}) \cap \Phi_E \neq \emptyset$ and $T_2(\mathbf{a}) \cap \Phi_E \neq \emptyset$,
2. $\tilde{B}(\mathbf{a}) = 1$ iff $(S_1(\mathbf{a}) \cup S_2(\mathbf{a})) \cap \Phi = \emptyset$,
3. $\tilde{C}(\mathbf{a}) = 1$ iff for all $e \in (T_1(\mathbf{a}) \cup T_2(\mathbf{a})) \cap \Phi_E$ we have $I_e := \sum_{z \in \Phi} l(d_{ze}) \leq c$.

Then $\mathbf{a}$ and $\mathbf{a}'$ are defined to be open edges iff $\tilde{D}(\mathbf{a}) := \tilde{A}(\mathbf{a})\tilde{B}(\mathbf{a})\tilde{C}(\mathbf{a}) = 1$.

Lemma

Edges of SSG *cannot cross open edges of* $\mathbf{M}$*.*

Proof.

Assume we have $x, y \in \Phi$ such that the straight line between x and y crosses an open edge $\mathbf{a}$ of $\mathbf{M}$.
By properties $\tilde{A}(\mathbf{a}) = 1$ and $\tilde{B}(\mathbf{a}) = 1$ and $M(m, c) > 9m$ there is an $e \in \Phi_E$ such that we have $d_{xy} > d_{xe} + \frac{1}{9}M(m, c)$.
As $l(d_{xe} + \frac{1}{9}M(m, c)) \leq \frac{l(d_{xe})}{1+c}$ and by property $\tilde{C}(\mathbf{a}) = 1$ also $I_e \leq c$, we get $\mathrm{SINR}_{xy} \leq \mathrm{SINR}_{xe}$ and hence $(x, y) \notin \mathcal{E}$. □

Lemma

Let $\{\mathbf{a}_i\}_{1\leq i\leq n}$ be a collection of distinct edges in $\mathbf{M}$ which do not contain the origin. Then,

1. $P^0(\tilde{A}(\mathbf{a}_i) = 0\ \forall\, 1 \leq i \leq n) \leq {r_1}^n$ *where* $r_1 := \sqrt[7]{1-(1-\exp(-\lambda_E\, m^2))^2}$,
2. $P^0(\tilde{B}(\mathbf{a}_i) = 0\ \forall\, 1 \leq i \leq n) \leq {r_2}^n$ *where* $r_2 := \sqrt[7]{1-\exp(-2\lambda M^2)}$,
3. [Dousse et al. 2006] $P^0(\tilde{C}(\mathbf{a}_i) = 0\ \forall\, 1 \leq i \leq n) \leq {r_3}^n$ *where* $r_3 := \exp\left(\frac{4\lambda\pi}{K}\int_0^\infty x l(x)\,\mathrm{d}x + \frac{l(0)}{K} - \frac{c}{K}\right)$ *and* $K > 0$ *only depends on* l *and* M,
4. $P^0(\tilde{D}(\mathbf{a}_i) = 0\ \forall\, 1 \leq i \leq n) \leq {r_s}^n$ *where* $r_s := \sqrt{r_1} + \sqrt[4]{r_2} + \sqrt[4]{r_3}$.

Lemma

For small enough $q > 0$*, the probability of having an open circuit in* $\mathbf{M}$ *surrounding the origin is equal to* 1*.*

As having an open circuit in $\mathbf{M}$ surrounding the origin implies that the cluster C_0 is finite, we have $P^0(|C_0| = \infty) = 0$ for small enough $r_s > 0$.
Let $\varepsilon > 0$ be such that all $r_s \leq \varepsilon$ are small enough.
Choose

- $m \in (0, \infty)$ large enough such that
  $r_1 = \sqrt[7]{1 - (1 - \exp(-\lambda_E\, m^2))^2} \leq (\varepsilon/3)^2$
- $c \in (0, \infty)$ large enough such that
  $r_3' := \exp\left(\frac{4\pi}{K}\int_0^\infty xl(x)\,\mathrm{d}x + \frac{l(0)}{K} - \frac{c}{K}\right) \leq (\varepsilon/3)^4$,
- $\lambda \in (0, 1)$ small enough such that
  $r_2 := \sqrt[7]{1 - \exp(-2\lambda M(m, c)^2)} \leq (\varepsilon/3)^4$.

Then we have $r_3 = \exp\left(\frac{4\lambda\pi}{K}\int_0^\infty xl(x)\,\mathrm{d}x - \frac{c}{K}\right) \leq r_3'$ and $r_s = \sqrt{r_1} + \sqrt[4]{r_2} + \sqrt[4]{r_3} \leq \varepsilon$.

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Finally,

THE END.