Potential Confinement Property in the Parabolic Anderson Model

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The Parabolic Anderson Model

We consider the Cauchy problem for the heat equation with random coefficients:

$$\begin{aligned} \partial_t u(t,z) &= \Delta^{\mathrm{d}} u(t,z) + \xi(z) u(t,z), && \text{for } (t,z) \in (0,\infty) \times \mathbb{Z}^d, && (1) \\ u(0,z) &= \mathbb{1}_0(z), && \text{for } z \in \mathbb{Z}^d. && (2) \end{aligned}$$

- $\xi = (\xi(z) \colon z \in \mathbb{Z}^d)$ i.i.d. random potential, $[-\infty, \infty)$-valued.
- $\Delta^{\mathrm{d}} f(z) = \sum_{y \sim z} \big[f(y) - f(z) \big]$ discrete Laplacian
- $\Delta^{\mathrm{d}} + \xi$ Anderson Hamiltonian
- $u(t, \cdot)$ is a random time-dependent (not shift-invariant) field
- $U(t) = \sum_{z \in \mathbb{Z}^d} u(t,z)$ total mass

Interpretations / Motivations:

- Random mass transport through a random field of sinks and sources.
- Expected particle number in a branching random walk model in a field of random branching and killing rates.

Moment assumption: $H(t) = \log \langle \mathrm{e}^{t\xi(0)} \rangle < \infty$ for any $t > 0$.

The universality classes

According to [VAN DER HOFSTAD/K./MÖRTERS 2006], there are, under some regularity assumption, precisely four universality classes:

(i) *double-exponential case:* $\log \mathrm{Prob}(\xi(0) > r) \approx -\mathrm{e}^{r/\rho}$ as $r \to \infty$ with $\rho \in (0, \infty)$, see [GÄRTNER/MOLCHANOV 1998].

(ii) *double-exponential case with* $\rho = \infty$, see [GÄRTNER/MOLCHANOV 1998].

(iii) *almost bounded potentials* (see below)

(iv) *bounded potentials:* $\log \mathrm{Prob}(\xi(0) > -x) \approx -x^{\gamma/(1-\gamma)}$ as $x \downarrow 0$, with $\gamma \in [0, 1)$, see [BISKUP/K. 2001].

In this talk, we consider almost bounded potentials:

Assumption: For some $\rho \in (0, \infty)$ and some scale function $\kappa(t) = o(t)$,

$$\lim_{t \to \infty} \frac{H(ty) - yH(t)}{\kappa(t)} = -\rho y \log y, \qquad y > 0.$$

Introduce another scale function $1 \ll \alpha(t) = t^{o(1)}$ by

$$\kappa\Big(\frac{t}{\alpha(t)^d}\Big) = \frac{1}{\alpha(t)^{d+2}}, \qquad t \gg 1.$$

Characteristic variational problem

Informally, $\alpha(t)$ is the order of the diameter of the area from which the main contribution to the total expected mass stems, i.e.,

$$\langle U(t)\rangle \approx \Big\langle \sum_{|z|\leq R\alpha(t)} u(t,z)\Big\rangle, \qquad t \gg 1 \text{ and then } R\to\infty.$$

Our goal: Describe the shapes of ξ that give the biggest contribution to $\langle U(t)\rangle$.

The moment asymptotics are given as follows [VAN DER HOFSTAD/K./MÖRTERS 2006]:

$$\frac{1}{t}\log\langle U(t)\rangle = \frac{H\big(t\alpha(t)^{-d}\big)}{t\alpha(t)^{-d}} - \frac{\chi + o(1)}{\alpha(t)^2}, \qquad \text{where} \qquad \chi = \min_{\psi\in\mathcal{C}(\mathbb{R}^d)} \Big[\mathcal{L}(\psi) - \lambda(\psi)\Big].$$

Here $\lambda(\psi)$ is the top of the spectrum of $\Delta+\psi$ in $L^2(\mathbb{R}^d)$, and $\mathcal{L}(\psi) = \frac{\rho}{\mathrm{e}}\int_{\mathbb{R}^d} \mathrm{e}^{\psi(x)/\rho}\,\mathrm{d}x$.

The minimiser is the parabola $\psi_\rho(x) = \rho + \rho\frac{d}{2}\log\frac{\rho}{\pi} - \rho^2|x|^2$
($\Longrightarrow$ logarithmic Sobolev inequality).

The principal eigenfunction of $\Delta+\psi_\rho$ is g_ρ,
where g_ρ^2 is the Gaussian density $g_\rho^2(x) = (\frac{\rho}{\pi})^{d/2}\mathrm{e}^{-\rho|x|^2} = \frac{1}{\mathrm{e}}\mathrm{e}^{\frac{1}{\rho}\psi_\rho(x)}$.

Explanation of the moment asymptotics

In terms of the shifted and rescaled potential

$$\overline{\xi}_t(x) = \alpha(t)^2 \Big[\xi\big(\lfloor \alpha(t) x \rfloor\big) - \frac{H\big(t\alpha(t)^{-d}\big)}{t\alpha(t)^{-d}} \Big],$$

the total mass may be written

$$U(t)\, \mathrm{e}^{-H(t/\alpha(t)^d)\alpha(t)^d} \approx \exp\Big\{ \frac{t}{\alpha(t)^2} \lambda(\overline{\xi}_t) \Big\},$$

using an eigenvalue expansion and scaling properties of the eigenvalue. The shifted and rescaled potential satisfies the large-deviation principle

$$\mathrm{Prob}(\overline{\xi}_t \approx \psi) \approx \exp\Big\{ -\frac{t}{\alpha(t)^2} \mathcal{L}(\psi) \Big\}.$$

Combining this with Varadhan's lemma, suggests the moment asymptotics.

Interpretation: The main contribution to the expectation of the total mass $U(t)$ comes from

- those ξ which make $\overline{\xi}_t$ resemble the perfect parabola ψ_ρ.
- those solutions $u(t,\cdot)$ that resemble the perfect Gaussian density g_ρ^2, up to spatial rescaling and vertical shifting.

Goal: Give mathematical substance to this.

Potential confinement

Let $\mathrm{dist}(\cdot,\cdot)$ be a metric for L^1-convergence on compact subsets of $\mathbb{R}^d$. Our main result is a law of large numbers for $\overline{\xi}_t$ towards the parabola ψ_ρ:

Theorem [GRÜNINGER/K. (2007)]: For any $\varepsilon > 0$,

$$\lim_{t\to\infty} \frac{\langle U(t) 1\!\!1_{\Gamma_{t,\varepsilon}}(\overline{\xi}_t)\rangle}{\langle U(t)\rangle} = 0,$$

where

$$\Gamma_{t,\varepsilon} = \bigcap_{M>0} \quad \bigcap_{x\in\mathbb{Z}^d\colon |x|\leq t\log t} \quad \Big\{\psi\colon \mathrm{dist}\big(\mathrm{e}^{\frac{1}{\rho}\psi(x+\,\cdot)\wedge M}, \mathrm{e}^{\frac{1}{\rho}\psi_\rho(\cdot)}\big) \geq \varepsilon\Big\}.$$

- In words, the totality of all ξ such that every shift of $\mathrm{e}^{\frac{1}{\rho}\overline{\xi}_t\wedge M}$ is, for any $M > 0$, away from the Gaussian $g_\rho^2 = \mathrm{e}^{\frac{1}{\rho}\psi_\rho}$, gives a negligible contribution.
- An almost sure version of the potential confinement property was proved in [GÄRTNER/K./MOLCHANOV (2007)] for the double-exponential distribution and in [SZNITMAN (1992)] for the related model of Brownian motion among Poisson traps.
- A path confinement property (for the Brownian motion in the Feynman-Kac representation of $u(t,z)$) was proved in [SZNITMAN (1992)], [BOLTHAUSEN (1992)], [POVEL (1999)] for the trap problem.

Comments on the proof

Functional analytic side: A key step is to prove the strictness of the minimisation in the following strong sense for a variant of the formula for χ:

$$-\chi = \sup_{\psi}\Big[\lambda(\psi) - \rho\log\Big(\frac{\mathrm{e}}{\rho}\mathcal{L}(\psi)\Big)\Big] > \sup_{\psi\in\Gamma_{t,\varepsilon}}\Big[\lambda(\psi) - \rho\log\Big(\frac{\mathrm{e}}{\rho}\mathcal{L}(\psi)\Big)\Big] = -\chi_\varepsilon.$$

Probabilistic side: Derive effective estimates for the contribution to $U(t)$ from ξ's such that $\overline{\xi}_t$ is away from ψ_ρ in the above sense.

As in the above heuristics, we have

$$(*) := \frac{\langle U(t)1\!\!1_{\Gamma_{t,\varepsilon}}(\overline{\xi}_t)\rangle}{\langle U(t)\rangle} \approx \Big\langle \exp\Big\{\frac{t}{\alpha(t)^2}\lambda(\overline{\xi}_t)\Big\}1\!\!1_{\Gamma_{t,\varepsilon}}(\overline{\xi}_t)\Big\rangle \mathrm{e}^{\frac{t}{\alpha(t)^2}\chi}.$$

Now add and subtract the term $\frac{t}{\alpha(t)^2}\rho\log(\frac{\mathrm{e}}{\rho}\mathcal{L}(\overline{\xi}_t))$ in the exponent. The difference term is estimated against the variational formula $-\chi_\varepsilon$. Hence,

$$(*) \le \mathrm{e}^{-\frac{t}{\alpha(t)^2}(\chi_\varepsilon-\chi)}\Big\langle \exp\Big\{\frac{t}{\alpha(t)^2}\rho\log\Big(\frac{\mathrm{e}}{\rho}\mathcal{L}(\overline{\xi}_t)\Big)\Big\}\Big\rangle.$$

Combinatorial techniques show that the exponential rate of the last term vanishes, i.e., $(*) \to 0$ exponentially fast.

Remark: The combinatorics only work after cutting the potential at some large level, i.e., after replacing $\overline{\xi}_t$ by $\overline{\xi}_t \wedge M$, which must be incorporated beforehand.