# COMPLETE LOCALISATION IN THE PARABOLIC ANDERSON MODEL WITH PARETO-DISTRIBUTED POTENTIAL

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**Abstract:** The parabolic Anderson problem is the Cauchy problem for the heat equation $\partial_t u(t,z) = \Delta u(t,z) + \xi(z)u(t,z)$ on $(0,\infty)\times\mathbb{Z}^d$ with random potential $(\xi(z)\colon z\in\mathbb{Z}^d)$. We consider independent and identically distributed potential variables, such that $\mathrm{Prob}(\xi(z)>x)$ decays polynomially as $x\uparrow\infty$. If $u$ is initially localised in the origin, i.e. if $u(0,x)=\mathbb{1}_0(x)$, we show that, at any large time $t$, the solution is completely localised in a single point with high probability. More precisely, we find a random process $(Z_t\colon t\geq 0)$ with values in $\mathbb{Z}^d$ such that $\lim_{t\uparrow\infty} u(t,Z_t)/\sum_{z\in\mathbb{Z}^d} u(t,z)=1$, in probability. We also identify the asymptotic behaviour of $Z_t$ in terms of a weak limit theorem.

## 1. Introduction and main results

### 1.1 The parabolic Anderson model and intermittency

We consider the heat equation with random potential on the integer lattice $\mathbb{Z}^d$ and study the Cauchy problem with localised initial datum,

$$\begin{aligned} \partial_t u(t,z) &= \Delta u(t,z) + \xi(z)u(t,z), && (t,z)\in(0,\infty)\times\mathbb{Z}^d,\\ u(0,z) &= \mathbb{1}_0(z), && z\in\mathbb{Z}^d, \end{aligned} \tag{1.1}$$

where

$$(\Delta f)(z) = \sum_{y\sim z}[f(y)-f(z)], \qquad z\in\mathbb{Z}^d,\ f\colon\mathbb{Z}^d\to\mathbb{R},$$

is the discrete Laplacian, and the potential $(\xi(z)\colon z\in\mathbb{Z}^d)$ is a collection of independent identically distributed random variables.

The problem (1.1) and its variants are often called the *parabolic Anderson problem*. The elliptic version of this problem originated in the work of the physicist P. W. Anderson on entrapment of electrons in crystals with impurities, see [An58]. The parabolic version of the problem appears in the context of chemical kinetics and population dynamics, and also provides a simplified qualitative approach to problems in magnetism and turbulence. The references [GM90], [Mo94] and [CM94] provide applications, background and heuristics around the parabolic Anderson model. Interesting recent mathematical progress can be found, for example in [BMR05], [HKM06], and [GH06], and [GK05] is a recent survey article.

One main reason for the great interest in the parabolic Anderson problem lies in the fact that it exhibits an *intermittency effect*: It is believed that, at late times, the overwhelming contribution to the total mass of the solution $u$ of the problem (1.1) comes from a small number of widely separated regions of small diameter, which are often called the *relevant*

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*islands*. This effect is believed to get stronger (with a smaller number of relevant islands, which are of smaller size) as the tail of the potential variable at infinity gets heavier. Providing rigorous evidence for intermittency is a major challenge for mathematicians, which has lead to substantial research efforts in the past 15 years.

An approach, which has been proposed in the physics literature, see [ZM+87] or [GK05], suggests to study large time asymptotics of the moments of the total mass

$$U(t) = \sum_{z\in\mathbb{Z}^d} u(t,z), \qquad t>0\,. \tag{1.2}$$

Denoting expectation with respect to $\xi$ by $\langle\,\cdot\,\rangle$, if all exponential moments $\langle \exp(\lambda\xi(z))\rangle$ for $\lambda>0$ exist, then so do all moments $\langle U(t)^p\rangle$ for $t>0$, $p>0$. Intermittency becomes manifest in a faster growth rate of higher moments. More precisely, the model is called intermittent if

$$\limsup_{t\to\infty} \frac{\langle U(t)^p\rangle^{1/p}}{\langle U(t)^q\rangle^{1/q}} = 0, \qquad \text{for } 0<p<q. \tag{1.3}$$

Whenever $\xi$ is nondegenerate random, the parabolic Anderson model is intermittent in this sense, see [GM90, Theorem 3.2]. Further properties of the relevant islands, like their asymptotic size and shape of potential and solution, are reflected (on a heuristical level) in the asymptotic expansion of $\log\langle U(t)^p\rangle$ for large $t$. Recently, in [HKM06], it was argued that the distributions with finite exponential moments can be divided into exactly four different universality classes, with each class having a qualitatively different long-time behaviour of the solution.

It is, however, a much harder mathematical challenge to prove intermittency in the original geometric sense, and to identify asymptotically the number, size and location of the relevant islands. This programme was initiated by Sznitman for the closely related continuous model of a Brownian motion with Poissonian obstacles, and the very substantial body of research he and his collaborators created is surveyed in his monograph [Sz98]. For the problem (1.1) and two universality classes of potential distributions, the double-exponential distribution and distributions with tails heavier than double-exponential (but still with all exponential moments finite), the recent paper [GKM06] makes substantial progress towards completing the geometric picture: Almost surely, the contribution coming from the complement of a random number of relevant islands is negligible compared to the mass coming from these islands, asymptotically as $t\to\infty$. In the double-exponential case, the radius of the islands stays bounded, in the heavier case the islands are single sites, and in Sznitman's case the radius tends to infinity.

Questions about the number of relevant islands remained open in all these cases. Both in [GKM06] and [Sz98] it is shown that an upper bound on the number of relevant islands is $t^{o(1)}$, but this is certainly not always best possible. In particular, the questions whether a *bounded number* of islands already carry the bulk of the mass, or when *just one* island is sufficient, are unanswered. These questions are difficult, since there are many local regions that are good candidates for being a relevant island, and the known criteria that identify relevant islands do not seem to be optimal.

In the present paper, we study the parabolic Anderson model with potential distributions that do not have any finite exponential moment. For such distributions one expects the intermittency effect to be even more pronounced than in the cases discussed above, with a very small number of relevant islands, which are just single sites. Note that in this case intermittency cannot be studied in terms of the moments $\langle U(t)^p\rangle$, which are not finite.

The main result of this paper is that, in the case of Pareto-distributed potential variables, there is only a *single* relevant island, which consists of a single site. In other words, at any large time $t$, with high probability, the total mass $U(t)$ is concentrated in a single lattice point $Z_t \in \mathbb{Z}^d$. This extreme form of intermittency is called *complete localisation*. It has been observed so far only for quite simple mean field models, see [FM90, FG92], and the present paper is the first instance where it has been found in the parabolic Anderson model or, indeed, any comparable lattice-based model. We also study the asymptotics of the location $Z_t$ of the point where the mass concentrates: We show that $Z_t$ goes to infinity like $(t/\log t)^{\alpha/(\alpha-d)}$, where $\alpha > d$ is the parameter of the Pareto distribution. The location of the relevant island is further described in terms of a weak limit theorem for the scaled quantity $(t/\log t)^{\alpha/(d-\alpha)} Z_t$ with an explicit limiting density. Precise statements are formulated in the next section.

### 1.2 The parabolic Anderson model with Pareto-distributed potential

We assume that the potential variables $\xi(z)$ at all sites $z$ are independently *Pareto-distributed* with parameter $\alpha > d$, i.e., the distribution function is

$$F(x) = \mathrm{Prob}(\xi(z) \le x) = 1 - x^{-\alpha}, \qquad x \ge 1. \tag{1.4}$$

In particular, we have $\xi(z) \ge 1$ for all $z \in \mathbb{Z}^d$, almost surely. Note from [GM90, Theorem 2.1] that the restriction to parameters $\alpha > d$ is necessary and sufficient for (1.1) to possess a unique nonnegative solution $u\colon (0,\infty) \times \mathbb{Z}^d \to [0,\infty)$. Recall that $U(t) = \sum_{z\in\mathbb{Z}^d} u(t,z)$ is the total mass of the solution at time $t > 0$. We introduce

$$q = \frac{d}{\alpha - d} \qquad \text{and} \qquad \mu = \frac{(\alpha-d)^d 2^d B(\alpha-d,d)}{d^d (d-1)!}, \tag{1.5}$$

where $B(\cdot,\cdot)$ denotes the Beta function. Throughout the paper we use $|x|$ to denote the $\ell^1$-norm of a vector $x \in \mathbb{R}^d$.

Our first main result shows the complete localisation of the solution $u(t,\,\cdot\,)$ in a single lattice point $Z_t$, as $t \to \infty$.

**Theorem 1** (Concentration in one site)**.** *There exists a process $(Z_t\colon t > 0)$ with values in $\mathbb{Z}^d$ such that*

$$\lim_{t\to\infty} \frac{u(t,Z_t)}{U(t)} = 1 \qquad \textit{in probability.}$$

Remark 1. Our statement is formulated in terms of a convergence *in probability*. The convergence does *not hold* in the almost sure sense. Indeed, suppose that $t > 0$ is sufficiently large to ensure $u(t,Z_t) \ge \frac{2}{3}U(t)$ and that $t$ is a jumping time, i.e. that $Z_{t-} \ne Z_{t+}$. Then, by continuity, we have $u(t,Z_{t-}) + u(t,Z_{t+}) \ge \frac{4}{3}U(t)$, which contradicts the nonnegativity of the solution. ◇

Remark 2. The asymptotic behaviour of $\log U(t)$ for the Anderson model with heavy-tailed potential variables is analysed in detail in [HMS06]. In the case of a Pareto-distributed potential it turns out that already the leading term in the asymptotic expansion of $\log U(t)$ is random. This is in sharp contract to potentials with exponential moments, where the leading two terms in the expansion are always deterministic. More precisely, in [HMS06, Theorem 1.2] the following limit law for $\log U(t)$ is proved,

$$\frac{(\log t)^q}{t^{q+1}} \log U(t) \Longrightarrow Y, \qquad \text{where} \quad \mathbb{P}(Y \le y) = \exp\{-\mu y^{d-\alpha}\} \text{ for } y > 0. \tag{1.6}$$

Note that the upper tails of $Y$ have the same asymptotic order as the Pareto distribution with parameter $\alpha - d$, i.e., $\mathbb{P}(Y > y) \asymp y^{d-\alpha}$ as $y \to \infty$. A careful inspection of the proof of [HMS06, Theorem 1.2] shows that also

$$\frac{(\log t)^q}{t^{q+1}} \log u(t, Z_t) \Longrightarrow Y. \tag{1.7}$$

Note, however, that a combination of (1.6) with (1.7) does not yield the concentration property in Theorem 1. Much more precise techniques are necessary. ◇

Our second main result is a limit law for the concentration site $Z_t$ in Theorem 1. Recall the definition of $q$ and $\mu$ from (1.5). As usual, we denote weak convergence by $\Rightarrow$.

**Theorem 2** (Limit law for the concentration site)**.** *As $t \to \infty$,*

$$Z_t \left(\frac{\log t}{t}\right)^{q+1} \Longrightarrow X, \tag{1.8}$$

*where $X$ is an $\mathbb{R}^d$-valued random variable with Lebesgue density*

$$p(x) = \alpha \int_0^\infty \frac{\exp\{-\mu y^{d-\alpha}\}}{(y + q|x|)^{\alpha+1}} \, \mathrm{d}y\,.$$

Remark 3. Note that $X$ is isotropic in the $\ell^1$-norm. ◇

Remark 4. The density $p$ is a probability density. Indeed,

$$\int_{\mathbb{R}^d} p(x)\,\mathrm{d}x = \alpha \int_0^\infty \mathrm{d}y\, e^{-\mu y^{d-\alpha}} \int_{\mathbb{R}^d} \mathrm{d}x\, (y + q|x|)^{-(\alpha+1)},$$

and, by [HMS06, Lemma 3.9], the inner integral equals $\frac{2^d q^{-d}}{(d-1)!} B(\alpha+1-d, d)\, y^{-\alpha+d-1}$. Using a change of variable and the definition of $\mu$ in (1.5), this simplifies to

$$\int_{\mathbb{R}^d} p(x)\,\mathrm{d}x = \frac{B(\alpha+1-d, d)}{B(\alpha-d, d)} \frac{\alpha}{\alpha - d}\, \mu \int_0^\infty e^{-\mu t}\,\mathrm{d}t.$$

The integral equals $1/\mu$, and the remaining product equals one because of the functional equation $(x+y)\,B(x+1, y) = x\,B(x, y)$ for $x, y > 1$, which is satisfied by the Beta function. Moreover, the proof of Theorem 2 shows that the two limit laws in (1.7) and in (1.8) hold jointly, and the joint density of $(X, Y)$ is the map

$$(x, y) \mapsto \alpha \frac{\exp\left\{-\mu y^{d-\alpha}\right\}}{(y + q|x|)^{\alpha+1}}.$$

This explains the structure of the density $p(x)$. ◇

### 1.3 Overview: The strategy behind the proofs

As shown in [GM90, Theorem 2.1], under the assumption $\alpha > d$, the unique nonnegative solution $u\colon (0,\infty) \times \mathbb{Z}^d \to [0,\infty)$ of (1.1) has a *Feynman-Kac representation*

$$u(t, z) = \mathbb{E}_0\Big[\mathbb{1}\{X_t = z\} \exp\Big\{ \int_0^t \xi(X_s)\,\mathrm{d}s\Big\}\Big], \qquad t > 0,\ z \in \mathbb{Z}^d,$$

where $(X_s\colon s \geq 0)$ under $\mathbb{P}_0$ (with expectation $\mathbb{E}_0$) is a continuous-time simple random walk on $\mathbb{Z}^d$ with generator $\Delta$ started in the origin. Hence, the total mass of the solution is given by

$$U(t) = \mathbb{E}_0\Big[\exp\Big\{ \int_0^t \xi(X_s)\,\mathrm{d}s\Big\}\Big].$$

Heuristically, for a fixed, large time $t > 0$, the walks $(X_s \colon 0 \le s \le t)$ that have the greatest impact on the average $U(t)$ move quickly to a remote site $z$ which,

- has a large potential value $\xi(z)$,
- and can be reached quickly, i.e. is sufficiently close to the origin.

Once this site is reached, the walk remains there until time $t$. As the probability of moving to a site $z$ within $t$ time units is approximately

$$\mathbb{P}_0(X_t = z) = \exp\Big\{ - |z| \log\big(\tfrac{|z|}{2det}\big)\,(1 + o(1))\Big\},$$

it is plausible that the optimal site $z$ at time $t$ is the maximiser $Z_t$ of the random functional

$$\Psi_t(z) = \xi(z) - \frac{|z|}{t} \log \frac{|z|}{2det}\,,$$

with the understanding that $\Psi_t(0) = \xi(0)$. This is indeed the definition of the process $(Z_t \colon t \ge 0)$, which is underlying our two main theorems.

In Section 2 we study the asymptotic behaviour of $(Z_t \colon t \ge 0)$ using techniques from extremal value theory. We prove Theorem 2 and also provide auxiliary results that compare the largest and second-largest value in the set $\{\Psi_t(z) \colon z \in \mathbb{Z}^d\}$, as needed in the proof of Theorem 1 in Section 3. Note that the arguments in this section are based entirely on the definition of $(Z_t \colon t \ge 0)$ in terms of $\Psi_t$, and not on its rôle in the parabolic Anderson problem.

Section 3 is devoted to the proof of Theorem 1. In this proof we build on techniques developed in [GKM06]. We split $u$ into three terms, which correspond to the contributions to the Feynman-Kac formula coming from paths that (1) by time $t$ have left a centred box with a certain large, $t$-dependent, random radius, (2) stay inside this box for $t$ time units but do not visit $Z_t$, and (3) stay inside this box and do visit $Z_t$. It will turn out that the total mass of the first two contributions is negligible, and that the total mass of the last one is concentrated on $Z_t$. To be more precise, we denote the three parts in the decomposition by

$$u(t, z) = u_1^{(t)}(t, z) + u_2^{(t)}(t, z) + u_3^{(t)}(t, z)\,.$$

The radius of the box will be chosen large enough that $u_1^{(t)}$ has small total mass relative to $U(t)$, since it is expensive to reach the complement of the large box.

In order to deal with $u_2^{(t)}$, we use the gap between the value of $\Psi_t$ in its maximum $Z_t$, and the maximum of $\Psi_t(z)$ over all other points $z \in \mathbb{Z}^d \setminus \{Z_t\}$, i.e. the auxiliary result provided in Section 2. From this we infer that the total mass of $u_2^{(t)}$ is small, as the site $Z_t$, which maximises $\Psi_t$, is ruled out from the exponential.

Finally, for the estimate of $u_3^{(t)}$ it is crucial that the radius of the box is chosen in such a way that $Z_t$ is also a maximiser of the field $\xi$ over the box. The main ingredient is a spectral analytical device, which is used in a similar manner as in [GKM06]: We show that $u_3^{(t)}$ can be controlled in terms of the principal eigenfunction of the Anderson Hamiltonian, $\Delta + \xi$, in the box with zero boundary conditions. This eigenfunction turns out to be exponentially concentrated in the maximal potential point in the box, which is $Z_t$. Hence the total mass of $u$ is concentrated in $Z_t$. This argument is the key step in the proof of Theorem 1.

## 2. Proof of Theorem 2: The concentration site $Z_t$

In this section, we study the top two values in the order statistics of the random variables $(\Psi_t(z) \colon z \in \mathbb{Z}^d)$. We first prove that, for any $t > 0$, the set $\{\Psi_t(z) \colon z \in \mathbb{Z}^d\}$ is almost surely bounded. Thus, by continuity of the distribution function $F$, the set $\{\Psi_t(z) \colon z \in \mathbb{Z}^d\}$ has a unique maximum $Z_t = Z_t^{(1)} \in \mathbb{Z}^d$. Moreover, the set $\{\Psi_t(z) \colon z \in \mathbb{Z}^d \setminus \{Z_t^{(1)}\}\}$ also has a unique maximum, which we denote by $Z_t^{(2)} \in \mathbb{Z}^d$. Note that $Z_t^{(1)} > Z_t^{(2)}$.

**Lemma 1.** *For any $t > 0$, $\Psi_t$ is bounded almost surely.*

**Proof.** For any $r > 0$, let $\xi_r^{(1)} = \max_{z\in\mathbb{Z}^d\colon |z|\le r} \xi(z)$ denote the maximum of the potential in the box with radius $r$. Denote $\varphi(x) = -\log(1 - F(x))$. By [GM90, Lemma 4.2], which holds for any distribution function, we have, almost surely,

$$\varphi(\xi_r^{(1)}) = d \log r(1 + o(1)), \qquad \text{as } r \to \infty.$$

Fix $\varepsilon \in (0, 1 - \frac{d}{\alpha})$. Since $\varphi(x) = \alpha \log x$, there exists a random radius $\rho_1 > 0$ such that, almost surely,

$$\xi_r^{(1)} \le r^{\frac{d}{\alpha}+\varepsilon}, \qquad \text{for all } r > \rho_1. \tag{2.1}$$

Now fix $t > 0$. Since $\frac{d}{\alpha} + \varepsilon < 1$, there exists $\rho_2(t) > 0$ such that $r^{\frac{d}{\alpha}+\varepsilon} < \frac{r}{t} \log \frac{r}{2det}$ for all $r > \rho_2(t)$. With $\rho(t) = \max\{\rho_1, \rho_2(t)\}$, we obtain

$$\sup_{|z|>\rho(t)} \Psi_t(z) \le \sup_{|z|>\rho(t)} \Big[\xi_{|z|}^{(1)} - \frac{|z|}{t} \log \frac{|z|}{2det}\Big] \le \sup_{r>\rho(t)} \Big[r^{\frac{d}{\alpha}+\varepsilon} - \frac{r}{t} \log \frac{r}{2det}\Big] \le 0.$$

Hence, the function $\Psi_t$ is positive only for finitely many $z$ and thus attains its maximum. $\square$

We define two scaling functions

$$r_t = \Big(\frac{t}{\log t}\Big)^{q+1} \qquad \text{and} \qquad a_t = \Big(\frac{t}{\log t}\Big)^{q}. \tag{2.2}$$

A limit law for $\Psi_t(Z_t)$ is given in [HMS06, Prop. 3.8]. From its proof it follows that

$$\text{Prob}\,(\Psi_t(Z_t) \le a_t y) = e^{-\mu y^{d-\alpha}} + \eta_y(t), \tag{2.3}$$

where $\lim_{t\to\infty} \sup_{y\ge\rho} \eta_y(t) = 0$ for any $\rho > 0$. The proof of [HMS06, Prop. 3.8] also contains the idea about the right scaling for $Z_t$. Here we identify the limiting law, stated in Theorem 2.

**Lemma 2.** *As $t \to \infty$, the variable $Z_t/r_t$ converges weakly towards a random variable $X$ with Lebesgue density $p$, which was defined in Theorem 2.*

**Proof.** Let $A \subset \mathbb{R}^d$ be measurable with $\text{Leb}(\partial A) = 0$. It suffices to show that

$$\lim_{t\to\infty} \text{Prob}(Z_t/r_t \in A) = \int_A p(x)\, \mathrm{d}x\,.$$

Let $\varepsilon > 0$ and recall that $d - \alpha < 0$. Pick $\rho > 0$ so small that $\exp\{-\mu\rho^{d-\alpha}\} < \varepsilon/4$ and

$$\int_0^\rho \int_A \frac{\alpha \exp\{-\mu y^{d-\alpha}\}}{(y + q|x|)^{\alpha+1}}\, \mathrm{d}x\, \mathrm{d}y < \varepsilon/4,$$

which is possible since for $\rho = \infty$ the left hand side is equal to $\int_A p(x)\, \mathrm{d}x \le 1$, by Remark 4. Let $\eta_y(t)$ be as in (2.3). Further, choose $T$ such that for all $t > T$ one has $\eta_y(t) < \varepsilon/4$ for all $y \in [\rho, \infty)$ and, moreover,

$$\sup_{y\ge\rho} |\eta_y(t)| \int_0^\infty \int_A \frac{\alpha\, \mathrm{d}x\, \mathrm{d}y}{(y + q|x|)^{\alpha+1}} < \varepsilon/4,$$

which is possible as the integral is finite and $\eta_y(t) \to 0$ uniformly for $y \ge \rho$. We have

$$\text{Prob}\Big(\frac{Z_t}{r_t} \in A\Big) \le \text{Prob}\,(\Psi_t(Z_t) < a_t\rho) + \text{Prob}\,(Z_t \in r_t A, \Psi_t(Z_t) \ge a_t\rho)\,. \tag{2.4}$$

For any $t > T$, the first probability can be bounded, with the help of (2.3), by

$$\text{Prob}\,(\Psi_t(Z_t) < a_t\rho) = \exp\big\{-\mu\rho^{d-\alpha}\big\} + \eta_\rho(t) < \varepsilon/2. \tag{2.5}$$

Further, for any $t > T$, we compute the second probability as follows:

$$\begin{aligned}
\mathrm{Prob}\,(Z_t \in r_t A, \Psi_t(Z_t) \geq a_t\rho) &= \sum_{z\in r_tA\cap\mathbb{Z}^d} \int_\rho^\infty \mathrm{Prob}\,\big(Z_t = z, a_t^{-1}\Psi_t(Z_t) \in \,\mathrm{d}y\big) \\
&= \sum_{z\in r_tA\cap\mathbb{Z}^d} \int_\rho^\infty \mathrm{Prob}\,\big(\Psi_t(\widetilde{z}) < a_t y \ \forall \widetilde{z} \neq z, a_t^{-1}\Psi_t(z) \in \,\mathrm{d}y\big) \\
&= \sum_{z\in r_tA\cap\mathbb{Z}^d} \int_\rho^\infty \Big[ \prod_{\widetilde{z}\in\mathbb{Z}^d\setminus\{z\}} \mathrm{Prob}\Big(\xi(\widetilde{z}) < a_t y + \frac{|\widetilde{z}|}{t}\log\frac{|\widetilde{z}|}{2det}\Big)\Big] \\
&\qquad \times \mathrm{Prob}\Big(a_t^{-1}\Big[\xi(z) - \frac{|z|}{t}\log\frac{|z|}{2det}\Big] \in \,\mathrm{d}y\Big) \\
&= \int_\rho^\infty \mathrm{d}y \Big[\prod_{\widetilde{z}\in\mathbb{Z}^d} F\Big(a_t y + \frac{|\widetilde{z}|}{t}\log\frac{|\widetilde{z}|}{2det}\Big)\Big] \sum_{z\in r_tA\cap\mathbb{Z}^d} a_t \frac{F'\Big(a_t y + \frac{|z|}{t}\log\frac{|z|}{2det}\Big)}{F\Big(a_t y + \frac{|z|}{t}\log\frac{|z|}{2det}\Big)},
\end{aligned}$$

since $\frac{1}{a}(\xi(0) - K)$ has the density $y \mapsto aF'(ya + K)$.

Recall (2.3) and $F(x) = 1 - x^{-\alpha}$ and therefore $a_t \frac{F'(a_t v)}{F(a_t v)} = \frac{\alpha}{v(a_t^\alpha v^\alpha - 1)}$. Hence, with

$$v_t(y, z) = y + \frac{|z|}{a_t t}\log\frac{|z|}{2det},$$

we obtain

$$\begin{aligned}
&\mathrm{Prob}\,(Z_t \in r_t A, \Psi_t(Z_t) \geq a_t\rho) \\
&\quad= \int_\rho^\infty \mathrm{d}y \Big[\mathrm{e}^{-\mu y^{d-\alpha}} + \eta_y(t)\Big] \sum_{z\in r_tA\cap\mathbb{Z}^d} \frac{\alpha}{v_t(y,z)(a_t^\alpha v_t(y,z)^\alpha - 1)}.
\end{aligned}$$

Using that $a\log\frac{a}{2de} \geq -2d$ for any $a \geq 0$, we have $v_t(y,z) \geq y - \frac{2d}{a_t} = y(1 + o(1))$, uniformly for $y \geq \rho$. Hence,

$$\begin{aligned}
&\mathrm{Prob}\,(Z_t \in r_t A, \Psi_t(Z_t) \geq a_t\rho) \\
&\quad= (1 + o(1)) \int_\rho^\infty \mathrm{d}y \Big[\mathrm{e}^{-\mu y^{d-\alpha}} + \eta_y(t)\Big] \sum_{z\in r_tA\cap\mathbb{Z}^d} \frac{\alpha}{a_t^\alpha v_t(y,z)^{\alpha+1}}.
\end{aligned} \tag{2.6}$$

Fix some small $\delta > 0$ and put $f_t = (\log t)^{-\delta}$ and $g_t = (\log t)^\delta$. We divide the sum over $z \in r_tA\cap\mathbb{Z}^d$ on the right hand side of (2.6) into the three parts where $|z| < r_t f_t$, $r_t f_t \leq |z| \leq r_t g_t$ and $r_t g_t < |z|$. Hence, using an obvious notation in (2.6),

$$\mathrm{Prob}\,(Z_t \in r_t A, \Psi_t(Z_t) \geq a_t\rho) = I_t + II_t + III_t. \tag{2.7}$$

We show that $I_t$ and $III_t$ vanish and that $|II_t - \int_A p(x)\,\mathrm{d}x| \leq \varepsilon/2 + o(1)$ as $t \to \infty$. Combining this with (2.4) and (2.5), the convergence of $Z_t/r_t$ to the distribution with density $p$ follows.

We start with the estimate for $I_t$. From (2.6) and $v_t(y,z) \geq y(1 + o(1))$, we have

$$I_t \leq \int_\rho^\infty \mathrm{d}y \Big[\mathrm{e}^{-\mu y^{d-\alpha}} + \eta_y(t)\Big] \sum_{z\in\mathbb{Z}^d, |z|\leq r_t f_t} \frac{O(1)}{a_t^\alpha y^{\alpha+1}} \leq \int_\rho^\infty \mathrm{d}y \Big[\mathrm{e}^{-\mu y^{d-\alpha}} + \eta_y(t)\Big] \frac{O(1) f_t^d}{y^{\alpha+1}},$$

where $O(1)$ does not depend on $y$ nor on $z$, and we have used that $r_t^d = a_t^\alpha$. Since $\lim_{t\to\infty} f_t^d = 0$, we see that $\lim_{t\to\infty} I_t = 0$.

Now we turn to $II_t$. Recall that $q = \frac{d}{\alpha-d}$. For $r_t f_t \le |z| \le r_t g_t$ we have

$$\log \frac{|z|}{2det} = q\,(1+o(1))\,\log t. \tag{2.8}$$

Using (2.8) and the relations $ta_t = r_t \log t$ and $r_t^d = a_t^\alpha$, we obtain, uniformly for $z \in r_t A \cap \mathbb{Z}^d$ satisfying $r_t f_t \le |z| \le r_t g_t$, and uniformly for $y \in (\rho, \infty)$,

$$\begin{aligned}\frac{\alpha}{a_t^\alpha v_t(y,z)^{\alpha+1}} &= (\alpha+o(1))a_t^{-\alpha}\Big(y + \frac{|z|}{r_t \log t} q\,(1+o(1))\,\log t\Big)^{-\alpha-1}\\ &= (\alpha+o(1))r_t^{-d}\Big(y+\frac{|z|}{r_t}q\Big)^{-\alpha-1}.\end{aligned}$$

Substituting this into (2.6), using (2.3), $\mathrm{Leb}(\partial A) = 0$, and interchanging the integrals gives

$$\begin{aligned}II_t &= (1+o(1)) \int_\rho^\infty \Big(e^{-\mu y^{d-\alpha}} + \eta_y(t)\Big) \int_A \mathbb{1}\{f_t \le |x| \le g_t\} \frac{\alpha}{(y+q|x|)^{\alpha+1}}\,\mathrm{d}x\,\mathrm{d}y\\ &= \int_A p(x)\,\mathrm{d}x - \int_0^\rho \int_A \frac{\alpha e^{-\mu y^{d-\alpha}}\,\mathrm{d}x\,\mathrm{d}y}{(y+q|x|)^{\alpha+1}} + \int_\rho^\infty \int_A \frac{\alpha \eta_y(t)\,\mathrm{d}x\,\mathrm{d}y}{(y+q|x|)^{\alpha+1}} + o(1).\end{aligned} \tag{2.9}$$

Hence, by our choice of $\rho$, we have that $|II_t - \int_A p(x)\,\mathrm{d}x| \le \varepsilon/2 + o(1)$.

Finally, we estimate $III_t$. For $|z| \ge r_t g_t$, we estimate $\log \frac{|z|}{2det} \ge \log \frac{r_t g_t}{2det} = (q+o(1))\log t$ and use the monotonicity to estimate, in the same way as for the term $II_t$,

$$\begin{aligned}III_t &\le O(1) \int_\rho^\infty \Big(e^{-\mu y^{d-\alpha}} + \eta_y(t)\Big) \int_{\mathbb{R}^d} \mathbb{1}\{|x| \ge g_t\} \frac{1}{(y+q|x|)^{\alpha+1}}\,\mathrm{d}x\,\mathrm{d}y\\ &\le O(1) \int_{\mathbb{R}^d} \mathbb{1}\{|x| \ge g_t\}\, p(x)\,\mathrm{d}x.\end{aligned}$$

Since $p$ is integrable over $\mathbb{R}^d$ and $\lim_{t\to\infty} g_t = \infty$, we also have that $III_t$ vanishes as $t \to \infty$. This finishes the proof. □

We now quantify the difference between the largest and the second-largest value of $\Psi_t$ in terms of their joint limit law. Recall the definition of $Z_t^{(1)}$ and $Z_t^{(2)}$ from the beginning of this section, and also that $Z_t$ is identical to $Z_t^{(1)}$.

**Lemma 3.** *$a_t{}^{-1}(\Psi_t(Z_t^{(1)}), \Psi_t(Z_t^{(2)})) \Longrightarrow (Y_1, Y_2)$ weakly as $t \to \infty$, where $(Y_1, Y_2)$ is a $(0,\infty)\times(0,\infty)$-valued random variable with distribution function*

$$P(Y_1 \le y_1, Y_2 \le y_2) = \begin{cases} \exp\Big\{-\mu y_2^{d-\alpha}\Big\}\Big[1+\mu\Big(y_2^{d-\alpha} - y_1^{d-\alpha}\Big)\Big] & \text{if } 0 < y_2 \le y_1,\\ \exp\Big\{-\mu y_1^{d-\alpha}\Big\} & \text{if } 0 < y_1 < y_2.\end{cases}$$

**Proof.** First we argue that $\Psi_t(Z_t^{(1)}) \ge \Psi_t(Z_t^{(2)}) \ge 0$ almost surely, for all sufficiently large $t \ge 0$. Indeed, since $\Psi_t(Z_t^{(2)})$ is the second-largest of the values $\xi(z) - \frac{|z|}{t}\log\frac{|z|}{2det}$ with $z \in \mathbb{Z}^d$, it may be bounded from below against the minimum of any two of these. Picking $z = 0$ and $z = z_0$ equal to a neighbour of the origin, we obtain the lower bound

$$\Psi_t(Z_t^{(2)}) \ge \min\left\{\xi(0), \xi(z_0) - \tfrac{1}{t}\log\tfrac{1}{2det}\right\},$$

which is nonnegative for all sufficiently large $t$. Hence it is sufficient to consider $y_1, y_2 > 0$.

First, consider the case $0 < y_1 < y_2$. Using that $\Psi_t(Z_t^{(2)}) \le \Psi_t(Z_t^{(1)})$ and (2.3), we obtain

$$\mathrm{Prob}\left(\Psi_t(Z_t^{(1)}) \le a_t y_1, \Psi_t(Z_t^{(2)}) \le a_t y_2\right) = \mathrm{Prob}\left(\Psi_t(Z_t^{(1)}) \le a_t y_1\right) \to \exp\big\{-\mu y_1^{d-\alpha}\big\}.$$

Second, assume $0 < y_2 \le y_1$. Using that $\Psi_t(Z_t^{(2)}) \le \Psi_t(Z_t^{(1)})$, we obtain

$$\begin{aligned} &\mathrm{Prob}\left(\Psi_t(Z_t^{(1)}) \le a_t y_1, \Psi_t(Z_t^{(2)}) \le a_t y_2\right) \\ &\quad = \mathrm{Prob}\left(\Psi_t(Z_t^{(1)}) \le a_t y_2\right) + \mathrm{Prob}\left(a_t y_2 < \Psi_t(Z_t^{(1)}) \le a_t y_1, \Psi_t(Z_t^{(2)}) \le a_t y_2\right). \end{aligned} \tag{2.10}$$

Because of (2.3), it suffices to study the second term on the right. Taking into account the independence of the random variables $(\Psi_t(z)\colon z \in \mathbb{Z}^d)$, we compute we obtain

$$\begin{aligned} &\mathrm{Prob}\left(a_t y_2 < \Psi_t(Z_t^{(1)}) \le a_t y_1, \Psi_t(Z_t^{(2)}) \le a_t y_2\right) \\ &\quad = \sum_{z \in \mathbb{Z}^d} \mathrm{Prob}\left(a_t y_2 < \Psi_t(z) \le a_t y_1, \Psi_t(\widetilde{z}) \le a_t y_2 \ \forall \widetilde{z} \ne z\right) \\ &\quad = \sum_{z \in \mathbb{Z}^d} \frac{\mathrm{Prob}\big(a_t y_2 < \Psi_t(z) \le a_t y_1\big)}{\mathrm{Prob}\big(\Psi_t(z) \le a_t y_2\big)} \prod_{\widetilde{z} \in \mathbb{Z}^d} \mathrm{Prob}\big(\Psi_t(\widetilde{z}) \le a_t y_2\big) \\ &\quad = \mathrm{Prob}\big(\Psi_t(Z_t^{(1)}) \le a_t y_2\big) \sum_{z \in \mathbb{Z}^d} \frac{\mathrm{Prob}\big(a_t y_2 < \xi(z) - \frac{|z|}{t} \log \frac{|z|}{2det} \le a_t y_1\big)}{\mathrm{Prob}\big(\xi(z) \le a_t y_2 + \frac{|z|}{t} \log \frac{|z|}{2det}\big)} \\ &\quad = \mathrm{Prob}\left(\Psi_t(Z_t^{(1)}) \le a_t y_2\right) \sum_{z \in \mathbb{Z}^d} \frac{\overline{F}\big(a_t y_2 + \frac{|z|}{t} \log \frac{|z|}{2det}\big) - \overline{F}\big(a_t y_1 + \frac{|z|}{t} \log \frac{|z|}{2det}\big)}{F\big(a_t y_2 + \frac{|z|}{t} \log \frac{|z|}{2det}\big)}, \end{aligned} \tag{2.11}$$

where $\overline{F}(x) = 1 - F(x) = x^{-\alpha}$ is the tail of the distribution. Note that all the denominators are positive for all sufficiently large $t$.

Since $a \log \frac{a}{2de} \ge -2d$ for any $a \ge 0$, we have $F(a_t y_2 + \frac{|z|}{t} \log \frac{|z|}{2det}) \ge F(a_t y_2 - 2d) = 1 + o(1)$ uniformly in $z$. To calculate the numerator, fix some small $\delta > 0$ and denote $f_t = (\log t)^{-\delta}$ and $g_t = (\log t)^{\delta}$. We split the sum into the three parts, as to where $|z|/r_t$ is smaller than $f_t$, between $f_t$ and $g_t$ and larger than $g_t$. We show next that the two boundary contributions vanish, while the middle one has a nontrivial limit.

First, consider the domain where $|z| < r_t f_t$. We have

$$\frac{|z|}{t} \log \frac{|z|}{2det} \le \frac{r_t f_t}{t} \log \frac{r_t f_t}{2det} = \frac{q f_t r_t \log t (1 + o(1))}{t} = q f_t a_t (1 + o(1)) = o(a_t),$$

which, together with $r_t^d = a_t^\alpha$, implies, for any $y > 0$,

$$\sum_{|z| < r_t f_t} \overline{F}\Big(a_t y + \frac{|z|}{t} \log \frac{|z|}{2det}\Big) = \sum_{|z| < r_t f_t} \overline{F}\left(a_t y (1 + o(1))\right) = O\big((r_t f_t)^d\big)(a_t y)^{-\alpha} = o(1). \tag{2.12}$$

Second, consider the domain where $r_t f_t \le |z| \le r_t g_t$. In this case $\log \frac{|z|}{2det} = q \log t (1 + o(1))$ uniformly in $z$. Hence, using $\overline{F}(x) = x^{-\alpha}$, $r_t \log t = t a_t$ and $a_t^\alpha = r_t^d$, we obtain

$$\overline{F}\Big(a_t y + \frac{|z|}{t} \log \frac{|z|}{2det}\Big) = \overline{F}\Big(a_t\big(y + q\frac{|z|}{r_t}(1 + o(1))\big)\Big) = (1 + o(1))\, r_t^{-d} \Big(y + q\frac{|z|}{r_t}\Big)^{-\alpha}. \tag{2.13}$$

Summing over $r_t f_t \le |z| \le r_t g_t$, and turning the sum into an integral, we obtain

$$\begin{aligned} \sum_{f_t r_t \le |z| \le g_t r_t} \overline{F}\Big(a_t y + \frac{|z|}{t} \log \frac{|z|}{2det}\Big) &= (1 + o(1)) \int_{\mathbb{R}^d} \mathbb{1}\{f_t \le |x| \le g_t\} \big(y + q|x|\big)^{-\alpha} \, \mathrm{d}x \\ &= (1 + o(1))\, \mu\, y^{d-\alpha}, \end{aligned} \tag{2.14}$$

where we use [HMS06, Lemma 3.9] to evaluate the integral and recall the definition of $\mu$ from (1.5).

Finally, consider $|z| > r_t g_t$. Since $\log \frac{|z|}{2det} \geq q \log t(1+o(1))$ uniformly in $z$, we have '$\leq$' instead of the first equality in (2.13). By the same procedure as in the case $r_t f_t \leq |z| \leq r_t g_t$,

$$\sum_{|z|>r_t g_t} \overline{F}\Big(a_t y + \frac{|z|}{t} \log \frac{|z|}{2det}\Big) \leq (1+o(1)) \int_{\mathbb{R}^d} 1\!\!1\{|x| \geq g_t\}\big(y + q|x|\big)^{-\alpha}\, \mathrm{d}x = o(1). \qquad (2.15)$$

Using (2.12), (2.14), and (2.15), we obtain, for any $y > 0$,

$$\sum_z \overline{F}\Big(a_t y + \frac{|z|}{t} \log \frac{|z|}{2det}\Big) = \mu y^{d-\alpha} + o(1).$$

Using this and (2.3) in (2.11) and substituting this and again (2.3) in (2.10), we obtain

$$\mathrm{Prob}\,\big(\Psi_t(Z_t^{(1)}) \leq a_t y_1, \Psi_t(Z_t^{(2)}) \leq a_t y_2\big) = \exp\big\{-\mu y_2^{d-\alpha}\big\}\,\Big[1 + \mu\big(y_2^{d-\alpha} - y_1^{d-\alpha}\big)\Big] + o(1),$$

which completes the proof. □

## 3. Proof of Theorem 1: Complete localisation

In this section, we prove Theorem 1. Section 3.1 presents the details of the decomposition of $u$ into three parts, which is announced informally in Section 1.3. Subject to the two main propositions, whose proofs are deferred to Section 3.2 and Section 3.3, we finish the proof of Theorem 1 in this section. Proposition 1 is proved in Section 3.2, where we show that the total mass of the first two contributions is negligible, using extreme value theory and certain limit laws. Proposition 2 is proved in Section 3.3, where we show that the third contribution is asymptotically concentrated in $Z_t$.

### 3.1 Decomposing $u$.

Let $(X_s : s \in [0,\infty))$ be the continuous-time simple random walk on $\mathbb{Z}^d$ with generator $\Delta$. By $\mathbb{P}_z$ and $\mathbb{E}_z$ we denote the probability measure and the expectation with respect to the walk starting at $z \in \mathbb{Z}^d$. According to [GM90, Theorem 2.1], the unique nonnegative solution of (1.1) can be expressed in terms of the Feynman-Kac formula as

$$u(t,z) = \mathbb{E}_0\Big[\exp\Big\{\int_0^t \xi(X_s)\,\mathrm{d}s\Big\} 1\!\!1\{X_t = z\}\Big], \qquad t > 0,\ z \in \mathbb{Z}^d, \qquad (3.1)$$

where we also used the time-reversal property of the random walk. We denote the entrance time into a set $A \subset \mathbb{Z}^d$ by $\tau_A = \inf\{t \geq 0 : X_t \in A\}$ and abbreviate $\tau_z = \tau_{\{z\}}$. By $B_R = \{z \in \mathbb{Z}^d : |z| \leq R\}$ we denote the box in $\mathbb{Z}^d$ with radius $R > 0$. Let $h : (0,\infty) \to (0,\infty)$ be such that

$$\lim_{t\to\infty} h_t = 0 \qquad \text{and} \qquad \lim_{t\to\infty} h_t \frac{\sqrt{\log t}}{\log\log t} = \infty, \qquad (3.2)$$

and define the *random* radius

$$R_t = |Z_t|(1 + h_t). \qquad (3.3)$$

We write $u(\theta,z) = u_1^{(t)}(\theta,z) + u_2^{(t)}(\theta,z) + u_3^{(t)}(\theta,z)$, where

$$u_1^{(t)}(\theta,z) = \mathbb{E}_0\Big[\exp\Big\{\int_0^\theta \xi(X_s)\,\mathrm{d}s\Big\} 1\!\!1\{X_\theta = z\}\, 1\!\!1\{\tau_{B_{R_t}^c} \leq \theta\}\Big]$$

$$u_2^{(t)}(\theta,z) = \mathbb{E}_0\Big[\exp\Big\{\int_0^\theta \xi(X_s)\,\mathrm{d}s\Big\} 1\!\!1\{X_\theta = z\}\, 1\!\!1\{\tau_{B_{R_t}^c} > \theta\} 1\!\!1\{\tau_{Z_t} > \theta\}\Big]$$

$$u_3^{(t)}(\theta,z) = \mathbb{E}_0\Big[\exp\Big\{\int_0^\theta \xi(X_s)\,\mathrm{d}s\Big\} 1\!\!1\{X_\theta = z\}\, 1\!\!1\{\tau_{B_{R_t}^c} > \theta\} 1\!\!1\{\tau_{Z_t} \leq \theta\}\Big],$$

for $(\theta,z) \in (0,\infty) \times \mathbb{Z}^d$ and $t > 0$. We are mainly interested in this decomposition for $\theta = t$.

**Proposition 1** (Estimating $u_1^{(t)}$ and $u_2^{(t)}$)**.**

$$\lim_{t\to\infty}\frac{\sum_{z\in\mathbb{Z}^d}u_1^{(t)}(t,z)}{U(t)}=0 \qquad \textit{and} \qquad \lim_{t\to\infty}\frac{\sum_{z\in\mathbb{Z}^d}u_2^{(t)}(t,z)}{U(t)}=0 \qquad \textit{in probability.}$$

**Proposition 2** (Estimating $u_3^{(t)}$)**.**

$$\lim_{t\to\infty}\frac{\sum_{z\in\mathbb{Z}^d\setminus\{Z_t\}}u_3^{(t)}(t,z)}{U(t)}=0 \qquad \textit{in probability.}$$

These two propositions will be proved in the next two sections. Using them, we can easily finish the proof of our first main result:

**Proof of Theorem 1.** Recall that $u=u_1^{(t)}+u_2^{(t)}+u_3^{(t)}$. Since $u_1^{(t)}$ and $u_2^{(t)}$ are nonnegative,

$$\frac{\sum_{z\in\mathbb{Z}^d\setminus\{Z_t\}}u(t,z)}{U(t)}\le\frac{\sum_{z\in\mathbb{Z}^d}u_1^{(t)}(t,z)}{U(t)}+\frac{\sum_{z\in\mathbb{Z}^d}u_2^{(t)}(t,z)}{U(t)}+\frac{\sum_{z\in\mathbb{Z}^d\setminus\{Z_t\}}u_3^{(t)}(t,z)}{U(t)},$$

and the right hand side vanishes in probability as $t\to\infty$, by Propositions 1 and 2. □

### 3.2 Proof of Proposition 1: Estimating $u_1^{(t)}$ and $u_2^{(t)}$.

In this section we prove Proposition 1, i.e., we show that the contributions coming from $u_1^{(t)}$ and $u_2^{(t)}$ are negligible. To prepare this, we first show that $Z_t=Z_t^{(1)}$, the maximal point of $\Psi_t$, is also maximal for the potential $\xi$ in the smallest centred box that contains it. Then we show that, by our choice of $R_t$ in (3.3), $Z_t$ is also maximal for $\xi$ in the box with radius $R_t$. Finally, the difference to the second-largest value of $\xi$ in this box diverges.

In order to formulate these statements, we define the two upper order statistics for the potential $\xi$ by

$$\xi_r^{(1)}=\max\{\xi(z)\colon |z|\le r\} \qquad \text{and} \qquad \xi_r^{(2)}=\max\left\{\xi(z)\colon |z|\le r,\xi(z)\ne\xi_r^{(1)}\right\}.$$

It follows from the continuity of distribution of $\xi(0)$ that, for any $r>0$, each of the sets $\left\{x\in\mathbb{Z}^d\colon |x|\le r,\xi(x)=\xi_r^{(i)}\right\}$, $i=1,2$, contains exactly one point, almost surely.

**Lemma 4.**

(i) $\lim_{t\to\infty}\mathrm{Prob}\big(\xi(Z_t)=\xi^{(1)}_{|Z_t|}\big)=1$;

(ii) $\lim_{t\to\infty}\mathrm{Prob}\big(\xi(Z_t)=\xi^{(1)}_{R_t}\big)=1$;

(iii) $\lim_{t\to\infty}\mathrm{Prob}\big(t\xi(Z_t)>|Z_t|\big)=1$.

(iv) *There exists* $b\colon(0,\infty)\to(0,\infty)$ *such that* $\lim_{t\to\infty}b_t=\infty$ *and*

$$\lim_{t\to\infty}\mathrm{Prob}\big(\xi^{(1)}_{R_t}-\xi^{(2)}_{R_t}\ge b_t\big)=1.$$

**Proof.** (i) Set $f_t=(\log t)^{-\delta}$ for some small $\delta>0$. By Lemma 2, $\lim\mathrm{Prob}\,(|Z_t|\ge f_tr_t)=1$. It thus suffices to show that $\xi(Z_t)=\xi^{(1)}_{|Z_t|}$ on the set $\{|Z_t|\ge f_tr_t\}$ for all large $t$.

Suppose for contradiction that $|Z_t|\ge f_tr_t$, but there exists $z\ne Z_t$ such that $|z|\le|Z_t|$ and $\xi(z)>\xi(Z_t)$. Since $f_tr_t/t\to\infty$, we may assume that $t$ is large enough to satisfy $\frac{f_tr_t}{t}\log(\frac{f_tr_t}{2det})>0$. Using that $r\mapsto\frac{r}{t}\log\frac{r}{2det}$ is increasing for $r>2det$ and nonpositive otherwise, we get

$$\Psi_t(z)=\xi(z)-\frac{|z|}{t}\log\frac{|z|}{2det}>\xi(Z_t)-\frac{|Z_t|}{t}\log\frac{|Z_t|}{2det}=\Psi_t(Z_t),$$

which contradicts the fact that $Z_t$ is the maximum of $\Psi_t$. Hence (i) is proved.

(ii) As $R_t \ge |Z_t|$ we clearly have $\xi^{(1)}_{R_t} \ge \xi^{(1)}_{|Z_t|}$. Let $f, g\colon (0,\infty) \to (0,\infty)$ be such that $f_t \to 0$, $g_t \to \infty$ and $h_t \log(g_t/f_t) \to 0$. Then we obtain

$$\begin{aligned}
&\mathrm{Prob}\big(\xi^{(1)}_{R_t} > \xi^{(1)}_{|Z_t|}, r_t f_t \le |Z_t| \le r_t g_t, \xi(Z_t) = \xi^{(1)}_{|Z_t|}\big)\\
&\quad= \sum_{r=\lceil r_t f_t\rceil}^{\lfloor r_t g_t\rfloor} \mathrm{Prob}\big(\xi^{(1)}_{R_t} > \xi^{(1)}_{|Z_t|}, |Z_t| = r, \xi(Z_t) = \xi^{(1)}_{|Z_t|}\big)\\
&\quad\le \sum_{r=\lceil r_t f_t\rceil}^{\lfloor r_t g_t\rfloor} \mathrm{Prob}\big(\xi^{(1)}_{r(1+h_t)} > \xi^{(1)}_{r} > \xi^{(1)}_{r-1}\big)\\
&\quad= \sum_{r=\lceil r_t f_t\rceil}^{\lfloor r_t g_t\rfloor} \big[\mathrm{Prob}\big(\xi^{(1)}_{r} > \xi^{(1)}_{r-1}\big) - \mathrm{Prob}\big(\xi^{(1)}_{r(1+h_t)} = \xi^{(1)}_{r} > \xi^{(1)}_{r-1}\big)\big].
\end{aligned} \tag{3.4}$$

Observe that, for any two finite non-empty subsets $A \subset B$ of $\mathbb{Z}^d$, we have $\mathrm{Prob}(\max_{z\in B}\xi(z) = \max_{z\in A}\xi(z)) = |A|/|B|$, since all the values $\xi(z)$ with $z \in B$ are different, and the index of the maximal value is uniformly distributed over $B$. Also, observe that

$$\big\{\xi^{(1)}_{r} > \xi^{(1)}_{r-1}\big\} = \big\{\max_{z\in B_r}\xi(z) = \max_{z\in\partial B_r}\xi(z)\big\}$$

and

$$\big\{\xi^{(1)}_{r(1+h_t)} = \xi^{(1)}_{r} > \xi^{(1)}_{r-1}\big\} = \big\{\max_{z\in B_{r(1+h_t)}}\xi(z) = \max_{z\in\partial B_r}\xi(z)\big\},$$

where the *inner boundary* of $B_r$ is defined by

$$\partial B_r = \big\{x \in B_r\colon \text{ there is } y \notin B_r \text{ with } |y-x| = 1\big\}.$$

Denoting $\sigma_d = \lim_{r\to\infty}|\partial B_r| r^{1-d} > 0$ and $\kappa_d = \lim_{r\to\infty}|B_r| r^{-d} > 0$, we therefore obtain from (3.4) that

$$\begin{aligned}
&\mathrm{Prob}\Big(\xi^{(1)}_{R_t} > \xi^{(1)}_{|Z_t|}, r_t f_t \le |Z_t| \le r_t g_t, \xi(Z_t) = \xi^{(1)}_{|Z_t|}\Big)\\
&\quad\le (1+o(1)) \sum_{r=\lceil r_t f_t\rceil}^{\lfloor r_t g_t\rfloor} \Big[\frac{\sigma_d r^{d-1}}{\kappa_d r^d} - \frac{\sigma_d r^{d-1}}{\kappa_d r^d (1+h_t)^d}\Big]\\
&\quad= (1+o(1))\,\frac{\sigma_d}{\kappa_d}\Big[1 - \frac{1}{(1+h_t)^d}\Big] \sum_{r=\lceil r_t f_t\rceil}^{\lfloor r_t g_t\rfloor} \frac{1}{r} = O(1)\, d\, h_t \log\frac{g_t}{f_t},
\end{aligned} \tag{3.5}$$

and this vanishes as $t \to \infty$ because of our assumption on $f_t$ and $g_t$. Since we know from Lemma 2, respectively from (i), that the probabilities of the events $\{r_t f_t \le |Z_t| \le r_t g_t\}$ and $\{\xi(Z_t) = \xi^{(1)}_{|Z_t|}\}$ tend to one as $t \to \infty$, the assertion (ii) is proved.

(iii) Let $f_t = 1/\log t$. Using that $t\xi(Z_t) = \Psi_t(Z_t) + |Z_t| \log\frac{|Z_t|}{2det}$ and $\Psi_t(Z_t) > 0$, we obtain

$$\begin{aligned}
\mathrm{Prob}\big(t\xi(Z_t) \le |Z_t|, |Z_t| \ge r_t f_t\big) &= \mathrm{Prob}\big(t\Psi_t(Z_t) + |Z_t| \log\tfrac{|Z_t|}{2det} \le |Z_t|, |Z_t| \ge r_t f_t\big)\\
&\le \mathrm{Prob}\big(\log\tfrac{|Z_t|}{2det} \le 1, |Z_t| \ge r_t f_t\big) \le \mathrm{Prob}\big(\log\tfrac{r_t f_t}{2det} \le 1\big).
\end{aligned}$$

The right hand side is equal to zero if $t$ is sufficiently large, as $r_t f_t/2det \to \infty$. By Lemma 2, the probability of the event $\{|Z_t| \ge r_t f_t\}$ tends to one, and this ends the proof of (iii).

(iv) There exists a scale function $\overline{b}\colon (0,\infty)\to(0,\infty)$ such that $\overline{b}_r\to\infty$ and $\mathrm{Prob}(\xi_r^{(1)}-\xi_r^{(2)}\geq \overline{b}_r)\to 1$ as $r\to\infty$. Indeed, this follows from the fact that the top two values of the order statistics satisfy the limit law

$$r^{-\frac{d}{\alpha}}\left(\xi_r^{(1)},\xi_r^{(2)}\right)\Longrightarrow(\Xi_1,\Xi_2)\qquad\text{as } r\to\infty, \tag{3.6}$$

where $\Xi_1$ and $\Xi_2$ are two continuous $(0,\infty)$-valued random variables that satisfy $\Xi_1>\Xi_2$ almost surely, see [EKM97, Th. 4.2.8] for the general limit assertion and [EKM97, p. 153] for the discussion of the Pareto case. From this limit law, it is easy to construct the desired scale function $\overline{b}$. Note that (iv), which we now prove, does not follow immediately from this, because the radius $R_t$ is chosen randomly. Define

$$p_r=\mathrm{Prob}\left(\xi_r^{(1)}-\xi_r^{(2)}<\overline{b}_r\right).$$

and choose $\overline{f}\colon(0,\infty)\to(0,\infty)$ in such a way that $\overline{f}_t\to 0$ and $r_t\overline{f_t}\to\infty$. As $p_r\to 0$ this implies $\overline{p}_t=\sup_{r>r_t\overline{f_t}}p_r\to 0$. Now we can choose $f\colon(0,\infty)\to(0,\infty)$ so that $f_t\to 0$, $f_t>\overline{f}_t$ and $\overline{p}_t\log f_t\to 0$. Finally, we choose $g\colon(0,\infty)\to(0,\infty)$ such that $g_t\to\infty$ and $\overline{p}_t\log g_t\to 0$. This gives

$$\sup_{r\geq r_tf_t}p_{r(1+h_t)}\log\frac{g_t}{f_t}\leq\sup_{r\geq r_t\overline{f}_t}p_{r(1+h_t)}\log\frac{g_t}{f_t}\leq\overline{p}_t\log\frac{g_t}{f_t}\to 0. \tag{3.7}$$

Define

$$b_t=\inf_{r_tf_t\leq r\leq r_tg_t}\overline{b}_{r(1+h_t)}$$

and note that $b_t\to\infty$ since $\overline{b}_r\to\infty$ and $r_tf_t\to\infty$. Using the spatial homogeneity of the family $(\xi(z)\colon z\in\mathbb{Z}^d)$ and (3.7), we obtain

$$\begin{aligned}
&\mathrm{Prob}\left(\xi_{R_t}^{(1)}-\xi_{R_t}^{(2)}<b_t,r_tf_t\leq|Z_t|\leq r_tg_t,\xi(Z_t)=\xi_{|Z_t|}^{(1)}=\xi_{R_t}^{(1)}\right)\\
&\quad=\sum_{r_tf_t\leq|z|\leq r_tg_t}\mathrm{Prob}\left(\xi_{R_t}^{(1)}-\xi_{R_t}^{(2)}<b_t,Z_t=z,\xi(Z_t)=\xi_{|Z_t|}^{(1)}=\xi_{R_t}^{(1)}\right)\\
&\quad\leq\sum_{r_tf_t\leq|z|\leq r_tg_t}\mathrm{Prob}\left(\xi_{|z|(1+h_t)}^{(1)}-\xi_{|z|(1+h_t)}^{(2)}<b_t,\xi(z)=\xi_{|z|(1+h_t)}^{(1)}\right).
\end{aligned} \tag{3.8}$$

Observe that the top two values in the order statistics are independent of the indices at which they are realised, i.e., the two events on the right hand side are independent, and that the probability of the second event is $1/|B_{|z|(1+h_t)}|$. As before, we denote $\sigma_d=\lim_{r\to\infty}|\partial B_r|r^{1-d}$ and $\kappa_d=\lim_{r\to\infty}|B_r|r^{-d}$. Using $b_t\leq\overline{b}_{|z|(1+h_t)}$ and the definition of $p_{r(1+h_t)}$, we therefore obtain

$$\begin{aligned}
&\mathrm{Prob}\Big(\xi_{R_t}^{(1)}-\xi_{R_t}^{(2)}<b_t,r_tf_t\leq|Z_t|\leq r_tg_t,\xi(Z_t)=\xi_{|Z_t|}^{(1)}=\xi_{R_t}^{(1)}\Big)\\
&\quad\leq\sum_{r_tf_t\leq|z|\leq r_tg_t}\mathrm{Prob}\Big(\xi_{|z|(1+h_t)}^{(1)}-\xi_{|z|(1+h_t)}^{(2)}<b_t\Big)\,\mathrm{Prob}\Big(\xi(z)=\xi_{|z|(1+h_t)}^{(1)}\Big)\\
&\quad\leq(1+o(1))\sum_{r_tf_t\leq|z|\leq r_tg_t}\mathrm{Prob}\Big(\xi_{|z|(1+h_t)}^{(1)}-\xi_{|z|(1+h_t)}^{(2)}<\overline{b}_{|z|(1+h_t)}\Big)\frac{1}{\kappa_d|z|^d(1+h_t)^d}\\
&\quad\leq O(1)\sum_{r=\lceil r_tf_t\rceil}^{\lfloor r_tg_t\rfloor}\frac{1}{r}\sup_{r_tf_t\leq r\leq r_tg_t}p_{r(1+h_t)}\leq O(1)\,\log\frac{g_t}{f_t}\sup_{r\geq r_tf_t}p_{r(1+h_t)},
\end{aligned} \tag{3.9}$$

where we changed the sum over $z$ into a sum over $r$, which turns the term $|z|^{-d}$ into $\frac{1}{r}$. By our assumptions, the right hand side vanishes as $t\to\infty$. Since we know from Lemma 2 and (i) and (ii) that the probabilities of the events $\{r_t f_t \le |Z_t| \le r_t g_t\}$ and $\{\xi(Z_t) = \xi^{(1)}_{|Z_t|} = \xi^{(1)}_{R_t}\}$ tend to one, the proof of (iv) is finished. $\square$

Now we give a lower bound for the total mass $U(t)$ in terms of the maximal point $Z_t$ of $\Psi_t$. Recall that $O(t)$ denotes some deterministic function $(0,\infty) \to (0,\infty)$ that is at most linear in $t$ at infinity.

**Lemma 5** (Lower bound for $U(t)$)**.**

$$\lim_{t\to\infty} \mathrm{Prob}\big(\log U(t) \ge t\xi(Z_t) - |Z_t| \log \xi(Z_t) + O(t)\big) = 1.$$

**Proof.** Fix $\varepsilon \in (0, 1 - \frac{d}{\alpha})$ and note from (2.1) that $\xi^{(1)}_r$ is asymptotically sublinear in $r$, almost surely. Hence, by [HMS06, Lemma 2.2] there exists a random time $T$ such that, for all $t > T$,

$$\log U(t) \ge t \max_{0<\rho<1} \max_{z\in\mathbb{Z}^d} \left[(1-\rho)\xi(z) - \frac{|z|}{t}\log\frac{|z|}{e\rho t}\right] + O(t). \tag{3.10}$$

On the event $\{t\xi(Z_t) > |Z_t|\}$, we substitute $\rho = \frac{|Z_t|}{t\xi(Z_t)} \in (0,1)$ and $z = Z_t$ in (3.10) and obtain

$$\log U(t) \ge \big(t - \tfrac{|Z_t|}{\xi(Z_t)}\big)\xi(Z_t) - |Z_t| \log \tfrac{|Z_t|\xi(Z_t)}{e|Z_t|} + O(t) = t\xi(Z_t) - |Z_t|\log\xi(Z_t) + O(t).$$

By Lemma 4(iii) the probability of $\{t\xi(Z_t) > |Z_t|\}$ tends to one, which implies the claim. $\square$

Now we derive upper bounds for the total mass in terms of the sites $Z_t = Z^{(1)}_t$ and $Z^{(2)}_t$.

**Lemma 6** (Upper bounds for $u^{(t)}_1$ and $u^{(t)}_2$)**.**

(i)

$$\lim_{t\to\infty} \mathrm{Prob}\Big(\log \sum_{z\in\mathbb{Z}^d} u^{(t)}_2(t,z) \le t\Psi_t(Z^{(2)}_t) + O(t)\Big) = 1.$$

(ii)

$$\lim_{t\to\infty} \mathrm{Prob}\Big(\log \sum_{z\in\mathbb{Z}^d} u^{(t)}_1(t,z) \le \max\Big\{t\Psi_t(Z^{(2)}_t), t\xi(Z^{(1)}_t) - R_t \log\frac{R_t}{2det}\Big\} + O(t)\Big) = 1.$$

**Proof.** (i) Note that

$$\sum_{z\in\mathbb{Z}^d} u^{(t)}_2(t,z) \le \mathbb{E}_0\left[\exp\left\{\int_0^t \xi(X_s)\,\mathrm{d}s\right\} \mathbb{1}\{\tau_{Z^{(1)}_t} > t\}\right].$$

Denote

$$\zeta_r = \max_{z\in B_r\setminus Z^{(1)}_t} \xi(z).$$

Denote by $J_t$ the number of jumps of the random walk $(X_s : s \ge 0)$ before time $t$. Note that $J_t$ has a Poisson distribution with parameter $2dt$, and that the path stays inside the box $B_{J_t}$ up to time $t$. Therefore, on the event $\{\tau_{Z^{(1)}_t} > t\}$, we can estimate $\xi(X_s) \le \zeta_{J_t}$ for $s \in [0,t]$. Summing over all values of $J_t$, we obtain

$$\sum_{z\in\mathbb{Z}^d} u^{(t)}_2(t,z) \le \sum_{r=0}^\infty \mathbb{E}_0\left[\exp\{t\zeta_{J_t}\}\,\mathbb{1}\{\tau_{Z^{(1)}_t} > t\}\mathbb{1}\{J_t = r\}\right] \le \sum_{r=0}^\infty e^{t\zeta_r - 2dt}\frac{(2dt)^r}{r!}. \tag{3.11}$$

We now give an upper bound for the tail of the series on the right. Fix some $\theta > 1$, $\varepsilon > 0$, $1 > \eta > d/\alpha$ and let $\beta = (1-\eta)^{-1}(1+\varepsilon)$. Using Stirling's formula,

$$r! = \sqrt{2\pi r}\,\left(\tfrac{r}{e}\right)^r e^{\delta(r)}, \qquad \text{with } \lim_{r\uparrow\infty} \delta(r) = 0\,,$$

and the bound $\zeta_r \le \xi_r^{(1)} \le r^\eta$ for all large $r$, we obtain, for all $r > t^\beta$ and large $t$, that

$$\begin{aligned} &t\,\zeta_r - 2dt + r\log(2dt) - \log(r!) \le tr^\eta - r\log\tfrac{r}{2\mathrm{d}et} - \delta(r) \\ &\le tr^\eta\left(1 - \tfrac{r^{1-\eta}}{t}\log\tfrac{r}{2\mathrm{d}et} - \tfrac{\delta(r)}{tr^\eta}\right) \le tr^\eta\left(1 - t^\varepsilon\log\tfrac{t^{\beta-1}}{2de} - \tfrac{\delta(r)}{tr^\eta}\right) \le -\theta\,\log r\,. \end{aligned}$$

Splitting the sum on the right of (3.11) at $r = \lceil t^\beta\rceil$ and noting that $\sum_{r>\lceil t^\beta\rceil} r^{-\theta} = o(1)$, we obtain

$$\begin{aligned} \log\sum_{z\in\mathbb{Z}^d} u_2^{(t)}(t,z) &\le t \max_{0\le r\le t^\beta}\left[\zeta_r - \tfrac{r}{t}\log\tfrac{r}{2\mathrm{d}et} - \tfrac{1}{t}\log\sqrt{2\pi r} - \tfrac{\delta(r)}{t}\right] - 2dt + o(t) \\ &\le t\max_{r\in\mathbb{N}_0}\left[\zeta_r - \tfrac{r}{t}\log\tfrac{r}{2det}\right] + O(t). \end{aligned} \tag{3.12}$$

Our goal is to show that the maximum on the right hand side is not larger than $\Psi_t(Z_t^{(2)})$, the second-largest value of $\Psi_t$, with probability tending to one.

Denote by $\rho_t$ the value at which this maximum is attained and let $z_t \in B_{\rho_t} \setminus \{Z_t^{(1)}\}$ be the maximal point in the definition of $\zeta_{\rho_t}$. Both $\rho_t$ and $z_t$ are unique by the continuity of the potential distribution. For any $z \in \mathbb{Z}^d \setminus \{Z_t^{(1)}\}$, we have, using the definition of $\Psi_t$, then the definition of $\zeta_{|z|}$ and $z \ne Z_t^{(1)}$, the definition of $\rho_t$ and finally the definition of $z_t$,

$$\begin{aligned} \Psi_t(z) &= \xi(z) - \frac{|z|}{t}\log\frac{|z|}{2det} \le \zeta_{|z|} - \frac{|z|}{t}\log\frac{|z|}{2det} \le \zeta_{\rho_t} - \frac{\rho_t}{t}\log\frac{\rho_t}{2det} \\ &= \xi(z_t) - \frac{\rho_t}{t}\log\frac{\rho_t}{2det}. \end{aligned} \tag{3.13}$$

On the event $\{\rho_t \ge 2det\}$, one can estimate $-\frac{\rho_t}{t}\log\frac{\rho_t}{2det} \le -\frac{|z_t|}{t}\log\frac{|z_t|}{2det}$, since $|z_t| \le \rho_t$, and since the map $r \mapsto \frac{r}{t}\log\frac{r}{2det}$ is increasing on $[2dt,\infty)$ and positive precisely on $(2det,\infty)$. Then (3.13) implies that $\Psi_t(z) \le \Psi_t(z_t)$. Hence, $\Psi_t(z_t)$ turns out to be the second-largest value of $\Psi_t$, and it follows that $z_t = Z_t^{(2)}$. Note that the last two terms of (3.13) are equal to the maximum on the right hand side of (3.12), which therefore is not smaller than $\Psi_t(Z_t^{(2)})$. Summarising, on the event $\{\rho_t \ge 2det\}$, we have the desired estimate. Hence, it remains to show that the probability of this event tends to one.

Recall that $q = d/(\alpha-d)$ and pick $\varepsilon_1 \in (0,q)$. It suffices to show that the probability of the event $\{\rho_t < t^{q+1-\varepsilon_1}\}$ vanishes. For this purpose, pick $0 < \varepsilon_2 < \varepsilon_3 < d/\alpha\varepsilon_1$ and $\varepsilon_4 > 0$ such that $\varepsilon_4(q+1-\varepsilon_1) < d/\alpha\varepsilon_1 - \varepsilon_3$, and observe that

$$\begin{aligned} \mathrm{Prob}\left(\rho_t < t^{q+1-\varepsilon_1}\right) &\le \mathrm{Prob}\left(\Psi_t(Z_t^{(2)}) < t^{q-\varepsilon_2}\right) + \mathrm{Prob}\left(\xi^{(1)}_{t^{q+1-\varepsilon_1}} > t^{q-\varepsilon_3}\right) \\ &\quad + \mathrm{Prob}\left(\rho_t < t^{q+1-\varepsilon_1}, \Psi_t(Z_t^{(2)}) \ge t^{q-\varepsilon_2}, \xi^{(1)}_{t^{q+1-\varepsilon_1}} \le t^{q-\varepsilon_3}\right). \end{aligned} \tag{3.14}$$

The first term on the right hand side vanishes since, by Lemma 3, $\Psi_t(Z_t^{(2)})$ is of order $a_t = (t/\log t)^q$. The second term vanishes by (2.1) applied to $t^{q+1-\varepsilon_1}$, because, almost surely, for any sufficiently large $t$,

$$\xi^{(1)}_{t^{q+1-\varepsilon_1}} \le \left(t^{q+1-\varepsilon_1}\right)^{\frac{d}{\alpha}+\varepsilon_4} \le t^{q-\varepsilon_3}.$$

Finally, we show that the third term is equal to zero for any sufficiently large $t$. Indeed, first estimate

$$\begin{aligned}\Psi_t(Z_t^{(2)}) &= \xi(Z_t^{(2)}) - \tfrac{|Z_t^{(2)}|}{t}\log\tfrac{|Z_t^{(2)}|}{2det} \le \zeta_{|Z_t^{(2)}|} - \tfrac{|Z_t^{(2)}|}{t}\log\tfrac{|Z_t^{(2)}|}{2det} \\ &\le \max_{r\in\mathbb{N}_0}\left[\zeta_r - \tfrac{r}{t}\log\tfrac{r}{2det}\right] = \zeta_{\rho_t} - \tfrac{\rho_t}{t}\log\tfrac{\rho_t}{2det} \le \xi_{\rho_t}^{(1)} + 2d,\end{aligned} \tag{3.15}$$

since $\frac{r}{t}\log\frac{r}{2det} \ge -2d$ for any $r \ge 0$. Hence, on the event

$$\left\{\rho_t < t^{q+1-\varepsilon_1},\ \Psi_t(Z_t^{(2)}) \ge t^{q-\varepsilon_2},\ \xi^{(1)}_{t^{q+1-\varepsilon_1}} \le t^{q-\varepsilon_3}\right\},$$

we have the estimate

$$t^{q-\varepsilon_2} \le \Psi_t(Z_t^{(2)}) \le \xi^{(1)}_{\rho_t} + 2d \le \xi^{(1)}_{t^{q+1-\varepsilon_1}} + 2d \le t^{q-\varepsilon_3} + 2d,$$

which is impossible for any sufficiently large $t$ since $\varepsilon_2 < \varepsilon_3$. This finishes the proof of (i).

(ii) Note that

$$\sum_{z\in\mathbb{Z}^d} u_1^{(t)}(t,z) = \mathbb{E}_0\Big[\exp\Big\{\int_0^t \xi(X_s)\,\mathrm{d}s\Big\}\mathbb{1}\{\tau_{B^c_{R_t}} \le t\}\Big].$$

Again we denote by $J_t$ the number of jumps of the random walk $(X_s\colon s\ge 0)$ before time $t$. As in the proof of (3.12), using that $J_t \ge R_t$, we obtain

$$\log\sum_{z\in\mathbb{Z}^d} u_1^{(t)}(t,z) \le \log\Big[\sum_{r\ge R_t} e^{t\xi_r^{(1)}-2dt}\frac{(2dt)^r}{r!}\Big] \le t\max_{r\ge R_t}\Big[\xi_r^{(1)} - \frac{r}{t}\log\frac{r}{2det}\Big] + O(t). \tag{3.16}$$

Denote by $\overline{\rho}_t$ the radius in $\mathbb{N}\cap[R_t,\infty)$ at which the maximum on the right hand side is attained, and by $\overline{z}_t$ the maximum point of $\xi$ in the box $B_{\overline{\rho}_t}$, i.e., the point satisfying $|\overline{z}_t| \le \overline{\rho}_t$ and $\xi(\overline{z}_t) = \xi^{(1)}_{\overline{\rho}_t}$.

Let $f_t = (\log t)^{-\delta}$ and consider the event $\{|Z_t^{(1)}| \ge r_t f_t\}\cap\{\xi(Z_t^{(1)}) = \xi^{(1)}_{R_t}\}$. By Lemma 2 and Lemma 4 the probability of this converges to one, and so it is sufficient to prove the desired estimate on this set. For large $t$ we have $\overline{\rho}_t \ge R_t \ge |Z_t^{(1)}| \ge r_t f_t \ge 2dt$. Supposing for the moment that $|\overline{z}_t| < \overline{\rho}_t$, we obtain, using that $r \mapsto \frac{r}{t}\log\frac{r}{2det}$ is positive and strictly increasing on the interval $(2dt,\infty)$,

$$\xi^{(1)}_{|\overline{z}_t|} - \frac{|\overline{z}_t|}{t}\log\frac{|\overline{z}_t|}{2det} = \xi^{(1)}_{\overline{\rho}_t} - \frac{|\overline{z}_t|}{t}\log\frac{|\overline{z}_t|}{2det} > \xi^{(1)}_{\overline{\rho}_t} - \frac{\overline{\rho}_t}{t}\log\frac{\overline{\rho}_t}{2det},$$

which implies $|\overline{z}_t| < R_t$ by definition of $\overline{\rho}_t$. Hence either $|\overline{z}_t| = \overline{\rho}_t$ holds, or $|\overline{z}_t| < R_t$.

In the case $|\overline{z}_t| = \overline{\rho}_t$ we have

$$\xi^{(1)}_{\overline{\rho}_t} - \frac{\overline{\rho}_t}{t}\log\frac{\overline{\rho}_t}{2det} = \xi(\overline{z}_t) - \frac{|\overline{z}_t|}{t}\log\frac{|\overline{z}_t|}{2det} = \Psi_t(\overline{z}_t) \le \Psi_t(Z_t^{(2)}), \tag{3.17}$$

where the last inequality follows from the fact that $|\overline{z}_t| = \overline{\rho_t} \ge R_t > |Z_t^{(1)}|$ and so $\overline{z}_t \ne Z_t^{(1)}$.

In the case $|\overline{z}_t| < R_t$ we use the condition $\xi(Z_t^{(1)}) = \xi^{(1)}_{R_t}$ and get

$$\xi^{(1)}_{\overline{\rho}_t} - \frac{\overline{\rho}_t}{t}\log\frac{\overline{\rho}_t}{2det} = \xi(\overline{z}_t) - \frac{\overline{\rho}_t}{t}\log\frac{\overline{\rho}_t}{2det} \le \xi^{(1)}_{R_t} - \frac{R_t}{t}\log\frac{R_t}{2det} = \xi(Z_t^{(1)}) - \frac{R_t}{t}\log\frac{R_t}{2det}. \tag{3.18}$$

Combining (3.16), (3.17), (3.18) we obtain, on the event $\{|Z_t^{(1)}| \ge r_t f_t\}\cap\{\xi(Z_t^{(1)}) = \xi^{(1)}_{R_t}\}$,

$$\log\sum_{z\in\mathbb{Z}^d} u_1^{(t)}(t,z) \le \max\left\{t\Psi_t(Z_t^{(2)}), t\xi(Z_t^{(1)}) - R_t\log\frac{R_t}{2det}\right\} + O(t).$$

This completes the proof. □

**Proof of Proposition 1.** Recall the random variables $Y_1 \ge Y_2$ from Lemma 3. Since their joint distribution is continuous, we have $\mathbb{P}(Y_1 = Y_2) = 0$. Fix some function $t \mapsto \eta_t$ tending to 0 as $t \to \infty$ (to be determined later), then we have

$$\lim_{t\to\infty} \mathrm{Prob}\left(\Psi_t(Z_t^{(1)}) - \Psi_t(Z_t^{(2)}) \ge a_t\eta_t\right) = 1. \tag{3.19}$$

Fix $0 < \delta < 1/4$ and put $f_t = (\log t)^{-\delta}$ and $g_t = (\log t)^{\delta}$. Recall $q = d/(\alpha - d)$ and the scale functions $r_t = (t/\log t)^{q+1}$ and $a_t = (t/\log t)^q$. Consider the event

$$\begin{aligned}
\Lambda_t = &\Big\{\log \sum_{z\in\mathbb{Z}^d} u_2^{(t)}(t,z) \le t\Psi_t(Z_t^{(2)}) + O(t)\Big\}\\
&\cap \Big\{\log \sum_{z\in\mathbb{Z}^d} u_1^{(t)}(t,z) \le \max\{t\Psi_t(Z_t^{(2)}), t\xi(Z_t^{(1)}) - R_t \log \tfrac{R_t}{2det}\} + O(1)\Big\}\\
&\cap \Big\{\log U(t) \ge t\xi(Z_t^{(1)}) - |Z_t^{(1)}| \log \xi(Z_t^{(1)}) + O(t)\Big\} \cap \Big\{r_t f_t \le |Z_t^{(1)}| \le r_t g_t\Big\}\\
&\cap \Big\{\Psi_t(Z_t^{(1)}) \le a_t g_t\Big\} \cap \Big\{\Psi_t(Z_t^{(1)}) - \Psi_t(Z_t^{(2)}) \ge a_t\eta_t\Big\}
\end{aligned}$$

Then $\lim_{t\to\infty} \mathrm{Prob}(\Lambda_t) = 1$, according to Lemmas 6, 5, 2 and (2.3) and (3.19), respectively. On the set $\Lambda_t$ we have the following estimates. *First*,

$$\begin{aligned}
\log \frac{\sum_x u_2^{(t)}(t,x)}{U(t)} &\le t\Psi_t(Z_t^{(2)}) - t\xi(Z_t^{(1)}) + |Z_t^{(1)}| \log \xi(Z_t^{(1)}) + O(t)\\
&= -t\left(\Psi_t(Z_t^{(1)}) - \Psi_t(Z_t^{(2)})\right) + |Z_t^{(1)}| \log \left[2de \tfrac{t\xi(Z_t^{(1)})}{|Z_t^{(1)}|}\right] + O(t).
\end{aligned} \tag{3.20}$$

For the first of the two terms we get,

$$-t\left(\Psi_t(Z_t^{(1)}) - \Psi_t(Z_t^{(2)})\right) \le -\eta_t t a_t = -\frac{\eta_t t^{q+1}}{(\log t)^q},$$

and for the second,

$$\begin{aligned}
|Z_t^{(1)}| \log \left[2de \tfrac{t\xi(Z_t^{(1)})}{|Z_t^{(1)}|}\right] + O(t) &\le |Z_t^{(1)}| \log \left[\tfrac{t\Psi_t(Z_t^{(1)})}{|Z_t^{(1)}|} + \log \tfrac{|Z_t^{(1)}|}{2det}\right] + O(|Z_t^{(1)}|)\\
&\le r_t g_t \log \left[\tfrac{t a_t g_t}{r_t f_t} + \log \tfrac{r_t g_t}{2det}\right] + O(r_t g_t) \le \tfrac{t^{q+1} \log\log t}{(\log t)^{q+1-\delta}} (1 + 2\delta)(1 + o(1)).
\end{aligned} \tag{3.21}$$

Now it is clear that we may pick $\eta_t \downarrow 0$ such that the right hand side of (3.20) diverges to $-\infty$.

*Second*,

$$\begin{aligned}
\log \frac{\sum_z u_1^{(t)}(t,z)}{U(t)} \le \max &\left\{t\Psi_t(Z_t^{(2)}), t\xi(Z_t^{(1)}) - R_t \log \frac{R_t}{2det}\right\}\\
&- t\xi(Z_t^{(1)}) + |Z_t^{(1)}| \log \xi(Z_t^{(1)}) + O(t).
\end{aligned} \tag{3.22}$$

We have already shown in (3.20) that the expression produced by the first option in the maximum converges to $-\infty$ on $\Lambda_t$. It remains to show that the same is true for the second option, i.e., for

$$-R_t \log \frac{R_t}{2det} + |Z_t^{(1)}| \log \xi(Z_t^{(1)}) + O(t)\,.$$

Recalling from (3.2) and (3.3) that $R_t = |Z_t|(1 + h_t) > |Z_t^{(1)}|$, we obtain an upper bound of

$$-h_t |Z_t^{(1)}| \log \frac{R_t}{2det} + |Z_t^{(1)}| \log \left[2de \tfrac{t\xi(Z_t^{(1)})}{|Z_t^{(1)}|}\right] + O(t).$$

The first term is estimated by

$$-h_t|Z_t^{(1)}|\log\frac{R_t}{2det}\le -h_tr_tf_t\log\frac{r_tf_t}{2det}=-\frac{qh_tt^{q+1}}{(\log t)^{q+\delta}}(1+o(1)),$$

while the second is estimated in (3.21). One observes that the sum of these two upper bounds diverges to $-\infty$, provided that $\frac{(\log t)^{1-2\delta}h_t}{\log\log t}\to\infty$. This follows from our assumption $\delta<1/4$ and the definition of $h_t$ in (3.2). Hence the right hand side of (3.22) goes to $-\infty$. □

### 3.3 Proof of Proposition 2: Estimating $u_3^{(t)}$.

In this section we prove Proposition 2, i.e., we show that the total mass of $u_3$ is concentrated on $Z_t$. Denote by $\lambda_t$ and $v_t$ the principal eigenvalue and the corresponding positive eigenfunction of $\Delta+\xi$ in the box $B_{R_t}$ with zero boundary condition. We assume that $v_t$ is normalised to $v_t(Z_t)=1$ and not, as more common, in the $\ell^2$-sense. Then we have the following probabilistic representation of $v_t$,

$$v_t(z)=\mathbb{E}_z\Big[\exp\Big\{\int_0^{\tau_{Z_t}}(\xi(X_s)-\lambda_t)\ \mathrm{d}s\Big\}1\!\!1\{\tau_{Z_t}<\tau_{B^{\rm c}_{R_t}}\}\Big],\qquad\text{for } z\in B^{\rm c}_{R_t}.\tag{3.23}$$

**Lemma 7.**

(i) *For any $t>0$ and all $(\theta,z)\in(0,\infty)\times\mathbb{Z}^d$, we have*

$$u_3^{(t)}(\theta,z)\le u_3^{(t)}(\theta,Z_t)\,||v_t||_2^2\,v_t(z).$$

(ii) *The eigenfunction $v_t$ is localised around $Z_t$ so that*

$$||v_t||_2^2\sum_{z\neq Z_t}v_t(z)\to 0\qquad\textit{in probability.}$$

**Proof.** (i) The first estimate is a special case of [GKM06, Th. 4.1] (with $\Gamma=\{Z_t\}$ in their notation) but we repeat the proof here for the sake of completeness. By time reversal,

$$u_3^{(t)}(\theta,z)=\mathbb{E}_z\Big[\exp\Big\{\int_0^\theta\xi(X_s)\ \mathrm{d}s\Big\}\,1\!\!1\{X_\theta=0\}\,1\!\!1\{\tau_{B^{\rm c}_{R_t}}>\theta\}1\!\!1\{\tau_{Z_t}\le\theta\}\Big].$$

We obtain a lower bound for $u_3^{(t)}(\theta,Z_t)$ by requiring that the random walk is in $Z_t$ at time $u\in(0,\theta)$. Using the Markov property at time $u$, we obtain

$$\begin{aligned}u_3^{(t)}(\theta,Z_t)\ge\ &\mathbb{E}_{Z_t}\Big[\exp\Big\{\int_0^u\xi(X_s)\ \mathrm{d}s\Big\}\,1\!\!1\{Z_t=X_u\}\,1\!\!1\{\tau_{B^{\rm c}_{R_t}}>u\}\Big]\\&\times\mathbb{E}_{Z_t}\Big[\exp\Big\{\int_0^{\theta-u}\xi(X_s)\,\mathrm{d}s\Big\}\,1\!\!1\{X_{\theta-u}=0\}\,1\!\!1\{\tau_{B^{\rm c}_{R_t}}>\theta-u\}\Big].\end{aligned}\tag{3.24}$$

Using an eigenvalue expansion for the parabolic problem in $B_{R_t}$ represented by the first factor in the formula above, we obtain the bound

$$\mathbb{E}_{Z_t}\Big[\exp\Big\{\int_0^u\xi(X_s)\ \mathrm{d}s\Big\}\,1\!\!1\{Z_t=X_u\}\,1\!\!1\{\tau_{B^{\rm c}_{R_t}}>u\}\Big]\ge \mathrm{e}^{\lambda_tu}\,\frac{v_t(Z_t)^2}{||v_t||_2^2}=\mathrm{e}^{\lambda_tu}\,||v_t||_2^{-2}.$$

Substituting this into (3.24), for $0<u<\theta$,

$$\mathbb{E}_{Z_t}\Big[\exp\Big\{\int_0^{\theta-u}\xi(X_s)\ \mathrm{d}s\Big\}\,1\!\!1\{X_{\theta-u}=0\}\,1\!\!1\{\tau_{B^{\rm c}_{R_t}}>\theta-u\}\Big]\le\mathrm{e}^{-\lambda_tu}\,||v_t||_2^2\,u_3^{(t)}(\theta,Z_t).\tag{3.25}$$

Since the claimed estimate is obvious for $z = Z_t$ due to the norming of $v_t$, we may assume that $z \in B_{R_t} \setminus \{Z_t\}$. Using the strong Markov property at time $\tau_{Z_t}$ and (3.25) with $u = \tau_{Z_t}$ we obtain

$$\begin{aligned}
u_3^{(t)}(\theta, z) &= \mathbb{E}_z \Big[ \exp\Big\{ \int_0^{\tau_{Z_t}} \xi(X_s)\, \mathrm{d}s \Big\} \mathbb{1}\{\tau_{B^c_{R_t}} > \tau_{Z_t}\} \mathbb{1}\{\tau_{Z_t} \le \theta\} \\
&\qquad \times \mathbb{E}_{Z_t} \Big[ \exp\Big\{ \int_0^{\theta-u} \xi(X_s)\, \mathrm{d}s \Big\} \mathbb{1}\{X_{\theta-u} = 0\} \mathbb{1}\{\tau_{B^c_{R_t}} > \theta - u\} \Big]_{u=\tau_{Z_t}} \Big] \\
&\le u_3^{(t)}(\theta, Z_t)\, ||v_t||_2^2\, \mathbb{E}_z \Big[ \exp\Big\{ \int_0^{\tau_{Z_t}} (\xi(X_s) - \lambda_t)\, \mathrm{d}s \Big\} \mathbb{1}\{\tau_{Z_t} < \tau_{B^c_{R_t}}\} \Big] \\
&= u_3^{(t)}(\theta, Z_t)\, ||v_t||_2^2\, v_t(z).
\end{aligned}$$

(ii) To prove the localisation of $v_t$ around $Z_t$, first note that by the Rayleigh-Ritz formula

$$\begin{aligned}
\lambda_t &= \sup \Big\{ \langle (\Delta + \xi) f, f \rangle \colon f \in \ell^2(\mathbb{Z}^d), \mathrm{supp}(f) \subset B_{R_t}, ||f||_2 = 1 \Big\} \\
&\ge \sup \{ \langle (\Delta + \xi)\delta_z, \delta_z \rangle \colon z \in B_{R_t} \} = \sup \{ \xi(z) - 2d \colon z \in B_{R_t} \} \\
&= \xi^{(1)}_{R_t} - 2d.
\end{aligned} \tag{3.26}$$

Recall the diverging function $b \colon (0,\infty) \to (0,\infty)$ from Lemma 4(iv) and consider the event

$$\Lambda_t = \big\{ \xi^{(1)}_{R_t} - \xi^{(2)}_{R_t} \ge b_t \big\} \cap \big\{ \xi(Z_t) = \xi^{(1)}_{R_t} \big\}.$$

By Lemma 4(ii) and (iv), its probability converges to 1. It follows from (3.26) that, on $\Lambda_t$,

$$\xi^{(2)}_{R_t} - \lambda_t \le \xi^{(1)}_{R_t} - b_t - \lambda_t \le 2d - b_t.$$

Since the paths of the random walk $(X_s \colon s \ge 0)$ in (3.23) do not leave $B_{R_t}$ and avoid the point $Z_t$ where the maximum $\xi^{(1)}_{R_t}$ is achieved, we can estimate the integrand in terms of the second-largest value of $\xi$ in $B_{R_t}$. Hence, we obtain

$$v_t(z) \le \mathbb{E}_z \big[ \exp \big\{ \tau_{Z_t} \big( \xi^{(2)}_{R_t} - \lambda_t \big) \big\} \big] \le \mathbb{E}_z \big[ \exp \big\{ \tau_{Z_t} \, (2d - b_t) \big\} \big]. \tag{3.27}$$

Under $\mathbb{P}_z$ the random variable $\tau_{Z_t}$ is stochastically bounded from below by a sum of $|z - Z_t|$ independent exponentially distributed random times with parameter $2d$. If $\tau$ denotes such a random time, we therefore have

$$v_t(z) \le \mathbb{E}_z \Big[ \exp \big\{ \tau_{Z_t} (2d - b_t) \big\} \Big] \le \Big( \mathbb{E} \big[ e^{-[b_t - 2d]\tau} \big] \Big)^{|z - Z_t|} = \Big( \frac{2d}{b_t} \Big)^{|z - Z_t|}.$$

From this, it is easy to see that the assertion holds. □

**Proof of Proposition 2.** Lemma 7(i) yields that

$$\frac{\sum_{z \ne Z_t} u_3^{(t)}(t, z)}{U(t)} \le \frac{\sum_{z \ne Z_t} u_3^{(t)}(t, z)}{u_3^{(t)}(t, Z_t)} \le ||v_t||_2^2 \sum_{z \ne Z_t} v_t(z),$$

and the right hand side vanishes as $t \to \infty$ in probability, by Lemma 7(ii). □

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## References


[An58] P.W. Anderson, Absence of diffusion in certain random lattices. *Phys. Rev.* 109, 1492-1505 (1958)

[BMR05] G. Ben Arous, S. Molchanov, and A.F. Ramirez, Transition from the annealed to the quenched asymptotics for a random walk on random obstacles. *Ann. Probab.* 33, 2149-2187 (2005)

[CM94] R. Carmona, S. Molchanov, Parabolic Anderson problem and intermittency. *Mem. Amer. Math. Soc.* 108, no. 518, (1994).

[EKM97] P. Embrechts, C. Klüppelberg and T. Mikosch, *Modelling Extremal Events*, Springer, 1997.

[FG92] K. Fleischmann and A. Greven, Localization and selection in a mean field branching random walk in a random environment. *Ann. Probab.* 20, 2141-2163 (1992).

[FM90] K. Fleischmann and S. Molchanov, Exact asymptotics in a mean field model with random potential. *Probab. Th. Rel. Fields* 86, 239-251 (1990).

[GK05] J. Gärtner and W. König, The parabolic Anderson model. in: J.-D. Deuschel and A. Greven (Eds.), *Interacting Stochastic Systems*, pp. 153-179, Springer, 2005.

[GKM06] J. Gärtner, W. König and S. Molchanov, Geometric characterisation of intermittency in the parabolic Anderson model, to appear in *Ann. Probab.* (2006).

[GH06] J. Gärtner and F. den Hollander, Intermittency in a catalytic random medium, to appear in *Ann. Probab.* (2006).

[GM90] J. Gärtner and S. Molchanov, Parabolic problems for the Anderson model. I. Intermittency and related topics. *Commun. Math. Phys.* 132, 613-655, (1990).

[GM98] J. Gärtner and S. Molchanov, Parabolic problems for the Anderson model. II. Second-order asymptotics and structure of high peaks. *Probab. Theory Relat. Fields* 111, 17-55, (1998).

[HKM06] R. van der Hofstad, W. König and P. Mörters, The universality classes in the parabolic Anderson model. to appear in *Comm. Math. Phys.* (2006).

[HMS06] R. van der Hofstad, P. Mörters and N. Sidorova, Weak and almost sure limits for the parabolic Anderson model with heavy tailed potentials, Preprint 2006.

[Mo94] S. Molchanov, Lectures on random media. In: D. Bakry, R.D. Gill, and S. Molchanov, Ecole d'Eté de Probabilités de Saint-Flour XXII-1992, LNM 1581, pp. 242-411. Berlin, Springer, 1994.

[Sz98] A.-S. Sznitman, *Brownian motion, Obstacles and Random Media.* Springer, Berlin, 1998.

[ZM+87] Ya. Zel'dovitch, S. Molchanov, S. Ruzmaikin, and D. Sokolov, Intermittency in random media. *Sov. Phys. Uzpekhi.* 30:5, 353-369 (1987)



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