Weierstrass Institute for
Applied Analysis and Stochastics

Leibniz
Gemeinschaft

The parabolic Anderson model

Based on joint works with Marek Biskup (České Budějovice and Los Angeles), Jürgen Gärtner (Berlin), Remco van der Hofstad (Eindhoven), Stanislav Molchanov (Charlotte), Peter Mörters (Bath) and Nadia Sidorova (London)

Wolfgang König

WIAS Berlin and TU Berlin

Random Motions in Random Media

Important models for a variety of situations and real-world applications. Examples:

- random walk in random environment
- random walk in random scenery
- random walk among random conductances

Random Motions in Random Media

Important models for a variety of situations and real-world applications. Examples:

- random walk in random environment
- random walk in random scenery
- random walk among random conductances

But we will be concerned with

- random motions in random potential,

which are closely connected with

- spectra of random operators.

The operators that we consider have a kinetic part and a random potential. More precisely, they are random Schrödinger operators.
We look at the time dependent problem and study long-time properties. This is closely connected with spectral theory, in particular, Anderson localisation properties, but only close to the top of the spectrum of the random operator.

Warning: We use probabilistic sign convention.

We consider the Cauchy problem for the heat equation with random coefficients and localised initial datum:

$$\frac{\partial}{\partial t}u(t,z) = \Delta^{\mathrm{d}}u(t,z)+\xi(z)u(t,z), \qquad \text{for } (t,z)\in(0,\infty)\times\mathbb{Z}^d, \tag{1}$$

$$u(0,z) = \delta_0(z), \qquad \text{for } z\in\mathbb{Z}^d. \tag{2}$$

The Parabolic Anderson Model

We consider the Cauchy problem for the heat equation with random coefficients and localised initial datum:

$$\frac{\partial}{\partial t} u(t,z) = \Delta^{\mathrm{d}} u(t,z) + \xi(z) u(t,z), \qquad \text{for } (t,z) \in (0,\infty) \times \mathbb{Z}^d, \tag{1}$$

$$u(0,z) = \delta_0(z), \qquad \text{for } z \in \mathbb{Z}^d. \tag{2}$$

- $\xi = (\xi(z) \colon z \in \mathbb{Z}^d)$ i.i.d. random potential, $[-\infty,\infty)$-valued.
- $\Delta^{\mathrm{d}} f(z) = \sum_{y \sim z} \big[f(y) - f(z) \big]$ discrete Laplacian
- $\Delta^{\mathrm{d}} + \xi$ Anderson Hamiltonian (a celebrated random Schrödinger operator)

The Parabolic Anderson Model

We consider the Cauchy problem for the heat equation with random coefficients and localised initial datum:

$$\frac{\partial}{\partial t}u(t,z) = \Delta^{\mathrm{d}}u(t,z) + \xi(z)u(t,z), \qquad \text{for } (t,z) \in (0,\infty)\times\mathbb{Z}^d, \tag{1}$$

$$u(0,z) = \delta_0(z), \qquad \text{for } z \in \mathbb{Z}^d. \tag{2}$$

- $\xi = (\xi(z) \colon z \in \mathbb{Z}^d)$ i.i.d. random potential, $[-\infty,\infty)$-valued.
- $\Delta^{\mathrm{d}} f(z) = \sum_{y\sim z}\big[f(y) - f(z)\big]$ discrete Laplacian
- $\Delta^{\mathrm{d}} + \xi$ Anderson Hamiltonian (a celebrated random Schrödinger operator)

The solution $u(t,\cdot)$ is a random time-dependent shift-invariant field.

Its a.s. existence is guaranteed under a mild moment condition on the potential.

It has all moments finite if all positive exponential moments of $\xi(0)$ are finite.

Interpretations / Motivations:

- Random mass transport through a random field of sinks and sources.
- Expected particle number in a branching random walk model in a field of random branching and killing rates.
- Anderson Hamiltonian $\Delta^d + \xi$ describes conductance properties of alloys of metals, or optical properties of glasses with impurities. Many open questions about delocalised versus extended states.

Interpretations / Motivations:

- Random mass transport through a random field of sinks and sources.
- Expected particle number in a branching random walk model in a field of random branching and killing rates.
- Anderson Hamiltonian $\Delta^d + \xi$ describes conductance properties of alloys of metals, or optical properties of glasses with impurities. Many open questions about delocalised versus extended states.

Comments:

- In the special case $\xi(z) \in \{-\infty, 0\}$, we call sites z with $\xi(z) = -\infty$ a (hard) trap. Then $u(t,x)$ is equal to the survival probability up to time t in x.
- The spatially continuous version (Brownian motion instead of random walk) is also highly interesting.

Background literature and surveys: [MOLCHANOV 1994], [CARMONA/MOLCHANOV 1994], [SZNITMAN 1998], [GÄRTNER/K. 2005].

Main tools

Feynman-Kac formula

$$u(t,z) = \mathbb{E}_0\Big[\exp\Big\{\int_0^t \xi(X(s))\,\mathrm{d}s\Big\}\mathbb{1}\{X(t)=z\}\Big], \qquad z\in\mathbb{Z}^d, t>0,$$

where $(X(s))_{s\in[0,\infty)}$ is the simple random walk on $\mathbb{Z}^d$ with generator Δ^{d}, starting from z under $\mathbb{P}_z$.

Main tools

Feynman-Kac formula

$$u(t,z) = \mathbb{E}_0\Big[\exp\Big\{\int_0^t \xi(X(s))\,\mathrm{d}s\Big\}\mathbb{1}\{X(t)=z\}\Big], \qquad z\in\mathbb{Z}^d, t>0,$$

where $(X(s))_{s\in[0,\infty)}$ is the simple random walk on $\mathbb{Z}^d$ with generator Δ^{d}, starting from z under $\mathbb{P}_z$.

Eigenvalue expansion

$$u(t,z) \approx \mathbb{E}_0\Big[\exp\Big\{\int_0^t \xi(X(s))\,\mathrm{d}s\Big\}\mathbb{1}\{X(t)=z\}\mathbb{1}\{X_{[0,t]}\subset B^{(2)}(t)\}\Big]$$
$$= \sum_k \mathrm{e}^{t\lambda_k(\xi,B^{(2)}(t))}\varphi_k(0)\varphi_k(z),$$

where $(\lambda_k(\xi,B^{(2)}(t)),\varphi_k)_k$ is a sequence of eigenvalues $\lambda_1>\lambda_2\geq\lambda_3\geq\dots$ and L^2-orthonormal eigenfunctions $\varphi_1,\varphi_2,\varphi_3,\dots$ of $\Delta+\xi$ in some box $B^{(2)}(t)=t\log^2 t\times[-1,1]^d$ with zero boundary condition.

Questions and Heuristics I

MAIN GOAL: Describe the large-t behavior of the solution $u(t,\cdot)$.

In particular: Where does the main bulk of the total mass stem from?

Total mass of the solution: $$U(t) = \sum_{z\in\mathbb{Z}^d} u(t,z), \qquad \text{for } t>0.$$

Questions and Heuristics I

MAIN GOAL: Describe the large-t behavior of the solution $u(t,\cdot)$.

In particular: Where does the main bulk of the total mass stem from?

Total mass of the solution: $$U(t) = \sum_{z\in\mathbb{Z}^d} u(t,z), \qquad \text{for } t > 0.$$

Much work is devoted to a thorough understanding of the effect of

Intermittency:

Asymptotically as $t \to \infty$, the main contribution to $U(t)$ comes from few small remote islands.

Questions and Heuristics I

MAIN GOAL: Describe the large-t behavior of the solution $u(t,\cdot)$.

In particular: Where does the main bulk of the total mass stem from?

Total mass of the solution: $$U(t) = \sum_{z\in\mathbb{Z}^d} u(t,z), \qquad \text{for } t>0.$$

Much work is devoted to a thorough understanding of the effect of

Intermittency:

Asymptotically as $t \to \infty$, the main contribution to $U(t)$ comes from few small remote islands.

- These islands are randomly located, t-dependent, not too far from the origin.

Questions and Heuristics I

MAIN GOAL: Describe the large-t behavior of the solution $u(t,\cdot)$.

In particular: Where does the main bulk of the total mass stem from?

Total mass of the solution: $$U(t) = \sum_{z\in\mathbb{Z}^d} u(t,z), \qquad \text{for } t > 0.$$

Much work is devoted to a thorough understanding of the effect of

Intermittency:

Asymptotically as $t \to \infty$, the main contribution to $U(t)$ comes from few small remote islands.

- These islands are randomly located, t-dependent, not too far from the origin.
- The solution $u(t,\cdot)$ and the potential $\xi(\cdot)$ are extremely large in these islands.

MAIN GOAL: Describe the large-t behavior of the solution $u(t,\cdot)$.

In particular: Where does the main bulk of the total mass stem from?

Total mass of the solution: $$U(t) = \sum_{z\in\mathbb{Z}^d} u(t,z), \qquad \text{for } t>0.$$

Much work is devoted to a thorough understanding of the effect of

Intermittency:

Asymptotically as $t\to\infty$, the main contribution to $U(t)$ comes from few small remote islands.

- These islands are randomly located, t-dependent, not too far from the origin.
- The solution $u(t,\cdot)$ and the potential $\xi(\cdot)$ are extremely large in these islands.
- The large-t behavior is determined by the largest eigenvalue of the Anderson Hamiltonian $\Delta^{\mathrm{d}}+\xi$ (i.e., by the bottom of the spectrum of $-\Delta^{\mathrm{d}}-\xi$) in large t-dependent boxes.

Questions and Heuristics I

MAIN GOAL: Describe the large-t behavior of the solution $u(t,\cdot)$.

In particular: Where does the main bulk of the total mass stem from?

Total mass of the solution: $$U(t) = \sum_{z\in\mathbb{Z}^d} u(t,z), \qquad \text{for } t>0.$$

Much work is devoted to a thorough understanding of the effect of

Intermittency:

Asymptotically as $t\to\infty$, the main contribution to $U(t)$ comes from few small remote islands.

- These islands are randomly located, t-dependent, not too far from the origin.
- The solution $u(t,\cdot)$ and the potential $\xi(\cdot)$ are extremely large in these islands.
- The large-t behavior is determined by the largest eigenvalue of the Anderson Hamiltonian $\Delta^{\rm d}+\xi$ (i.e., by the bottom of the spectrum of $-\Delta^{\rm d}-\xi$) in large t-dependent boxes.
- This in turn is determined by the extreme values of the potential ξ.

MAIN GOAL: Describe the large-t behavior of the solution $u(t,\cdot)$.

In particular: Where does the main bulk of the total mass stem from?

Total mass of the solution: $$U(t) = \sum_{z\in\mathbb{Z}^d} u(t,z), \qquad \text{for } t > 0.$$

Much work is devoted to a thorough understanding of the effect of

Intermittency:

Asymptotically as $t \to \infty$, the main contribution to $U(t)$ comes from few small remote islands.

- These islands are randomly located, t-dependent, not too far from the origin.
- The solution $u(t,\cdot)$ and the potential $\xi(\cdot)$ are extremely large in these islands.
- The large-t behavior is determined by the largest eigenvalue of the Anderson Hamiltonian $\Delta^{\mathrm{d}} + \xi$ (i.e., by the bottom of the spectrum of $-\Delta^{\mathrm{d}} - \xi$) in large t-dependent boxes.
- This in turn is determined by the extreme values of the potential ξ.
- Hence, only the upper tails of $\xi(0)$ matter.

Questions and Heuristics II

If all the moments $\langle U(t)^p \rangle$ of $U(t)$ are finite for any $p,t > 0$, then intermittency can be characterised by the requirement

$$0 < p < q \qquad \Longrightarrow \qquad \limsup_{t\to\infty} \frac{1}{t} \log \frac{\langle U(t)^p \rangle^{1/p}}{\langle U(t)^q \rangle^{1/q}} < 0.$$

Questions and Heuristics II

If all the moments $\langle U(t)^p\rangle$ of $U(t)$ are finite for any $p,t>0$, then intermittency can be characterised by the requirement

$$0<p<q \qquad \Longrightarrow \qquad \limsup_{t\to\infty}\frac{1}{t}\log\frac{\langle U(t)^p\rangle^{1/p}}{\langle U(t)^q\rangle^{1/q}}<0.$$

Presence of intermittency

[GÄRTNER/MOLCHANOV 1990]: In this sense, intermittency holds as soon as the potential is not a.s. constant.

Questions and Heuristics II

If all the moments $\langle U(t)^p \rangle$ of $U(t)$ are finite for any $p,t > 0$, then intermittency can be characterised by the requirement

$$0 < p < q \qquad \Longrightarrow \qquad \limsup_{t\to\infty} \frac{1}{t} \log \frac{\langle U(t)^p\rangle^{1/p}}{\langle U(t)^q\rangle^{1/q}} < 0.$$

Presence of intermittency

[GÄRTNER/MOLCHANOV 1990]: In this sense, intermittency holds as soon as the potential is not a.s. constant.

How many random potentials are interesting?

Under some mild regularity assumption, the case of finite positive exponential moments has been classified in four universality classes ([VAN DER HOFSTAD/K./MÖRTERS 2006]): the double-exponential distribution, a boundary case, bounded potentials and almost bounded ones.

How large are the islands? How large are the potential and the solution there? What do their shapes look like there?

- Strong answer in two main special cases ([GÄRTNER/K./MOLCHANOV 2007], [SZNITMAN 1998]): the complement of certain islands is negligible.
  In the islands, the potential and the solution approach the minimizer φ of a characteristic formula and the eigenfunction of $\Delta + \varphi$, respectively.
- Weakly answered in many cases ([GÄRTNER/MOLCHANOV 1998], [BISKUP/K. 2001], [VAN DER HOFSTAD/K./MÖRTERS 2006]): Identification of moment asymptotics and the almost sure asymptotics of $U(t)$.

Questions and Heuristics III

How large are the islands? How large are the potential and the solution there? What do their shapes look like there?

- Strong answer in two main special cases ([GÄRTNER/K./MOLCHANOV 2007], [SZNITMAN 1998]): the complement of certain islands is negligible.
  In the islands, the potential and the solution approach the minimizer φ of a characteristic formula and the eigenfunction of $\Delta + \varphi$, respectively.
- Weakly answered in many cases ([GÄRTNER/MOLCHANOV 1998], [BISKUP/K. 2001], [VAN DER HOFSTAD/K./MÖRTERS 2006]):
  Identification of moment asymptotics and the almost sure asymptotics of $U(t)$.

How many islands have to be taken into account?

- Rough bounds: $t^{o(1)}$ ([SZNITMAN 1998], [GÄRTNER/K./MOLCHANOV 2007]).
- Conjecture: $O(1)$.
  Open for finite positive exponential moments (ongoing work [BISKUP/K.]).
  Proved for heavy-tailed potentials ([K./LACOIN/MÖRTERS/SIDOROVA 2009]).

Heuristics using Fourier expansion

Let $\lambda_k(\xi)$ be the eigenvalues of $\Delta^{\mathrm{d}} + \xi$ with zero boundary condition in the cube $Q_t = [-t,t]^d \cap \mathbb{Z}^d$ and corresponding eigenfunctions φ_k, forming an ONS.

Heuristics using Fourier expansion

Let $\lambda_k(\xi)$ be the eigenvalues of $\Delta^{\mathrm{d}} + \xi$ with zero boundary condition in the cube $Q_t = [-t,t]^d \cap \mathbb{Z}^d$ and corresponding eigenfunctions φ_k, forming an ONS.
Then, with some suitable $H(t) \gg t$,

$$\langle U(t)\rangle \approx \Big\langle \sum_k \mathrm{e}^{t\lambda_k(\xi)} \varphi_k(0) \langle \varphi_k, 1\!\!1 \rangle \Big\rangle$$

Heuristics using Fourier expansion

Let $\lambda_k(\xi)$ be the eigenvalues of $\Delta^{\mathrm{d}} + \xi$ with zero boundary condition in the cube $Q_t = [-t,t]^d \cap \mathbb{Z}^d$ and corresponding eigenfunctions φ_k, forming an ONS.
Then, with some suitable $H(t) \gg t$,

$$\langle U(t) \rangle \approx \Big\langle \sum_k \mathrm{e}^{t\lambda_k(\xi)} \varphi_k(0) \langle \varphi_k, \mathbb{1} \rangle \Big\rangle \approx \mathrm{e}^{H(t)} \big\langle \mathrm{e}^{t\lambda_1(\xi - H(t)/t)} \big\rangle$$

Heuristics using Fourier expansion

Let $\lambda_k(\xi)$ be the eigenvalues of $\Delta^{\rm d}+\xi$ with zero boundary condition in the cube $Q_t=[-t,t]^d\cap\mathbb{Z}^d$ and corresponding eigenfunctions φ_k, forming an ONS.
Then, with some suitable $H(t)\gg t$,

$$\langle U(t)\rangle \approx \Big\langle \sum_k \mathrm{e}^{t\lambda_k(\xi)}\varphi_k(0)\langle\varphi_k,1\!\!1\rangle\Big\rangle \approx \mathrm{e}^{H(t)}\big\langle \mathrm{e}^{t\lambda_1(\xi-H(t)/t)}\big\rangle$$
$$\approx \mathrm{e}^{H(t)}\exp\Big\{t\sup_{\varphi}\big(\lambda_1(\varphi)-\mathscr{L}(\varphi)\big)\Big\},$$

Heuristics using Fourier expansion

Let $\lambda_k(\xi)$ be the eigenvalues of $\Delta^{\rm d}+\xi$ with zero boundary condition in the cube $Q_t=[-t,t]^d\cap\mathbb{Z}^d$ and corresponding eigenfunctions φ_k, forming an ONS.
Then, with some suitable $H(t)\gg t$,

$$\langle U(t)\rangle \approx \Big\langle \sum_k \mathrm{e}^{t\lambda_k(\xi)}\varphi_k(0)\langle\varphi_k,\mathbb{1}\rangle\Big\rangle \approx \mathrm{e}^{H(t)}\big\langle \mathrm{e}^{t\lambda_1(\xi-H(t)/t)}\big\rangle$$
$$\approx \mathrm{e}^{H(t)}\exp\Big\{t\sup_{\varphi}\big(\lambda_1(\varphi)-\mathscr{L}(\varphi)\big)\Big\},$$

where a large-deviation rate function is given by

$$\mathscr{L}(\varphi)=-\lim_{t\to\infty}\frac{1}{t}\log\mathrm{Prob}\big(\xi(\cdot)-H(t)/t\approx\varphi(\cdot)\big).$$

Heuristics using Fourier expansion

Let $\lambda_k(\xi)$ be the eigenvalues of $\Delta^{\mathrm{d}} + \xi$ with zero boundary condition in the cube $Q_t = [-t,t]^d \cap \mathbb{Z}^d$ and corresponding eigenfunctions φ_k, forming an ONS.
Then, with some suitable $H(t) \gg t$,

$$\langle U(t)\rangle \approx \Big\langle \sum_k \mathrm{e}^{t\lambda_k(\xi)} \varphi_k(0)\langle\varphi_k, \mathbb{1}\rangle \Big\rangle \approx \mathrm{e}^{H(t)} \big\langle \mathrm{e}^{t\lambda_1(\xi - H(t)/t)} \big\rangle$$
$$\approx \mathrm{e}^{H(t)} \exp\Big\{ t \sup_{\varphi} \big(\lambda_1(\varphi) - \mathscr{L}(\varphi)\big)\Big\},$$

where a large-deviation rate function is given by

$$\mathscr{L}(\varphi) = -\lim_{t\to\infty} \frac{1}{t} \log \mathrm{Prob}\big(\xi(\cdot) - H(t)/t \approx \varphi(\cdot)\big).$$

Hence, there is a large-deviation principle at work:
On the event $\{\xi(\cdot) \approx H(t)/t + \varphi(\cdot)\}$,
the exponential contribution to the Feynman-Kac formula is $H(t) + t\lambda_1(\varphi)$,
and the probabilistic price is $\exp\{-t\mathscr{L}(\varphi)\}$.

Finally, optimise over all shapes φ.

Heuristics using local times

Introducing the walker's local times $\ell_t(z) = \int_0^t \mathrm{d}s\, \delta_{X_s}(z)$ and the cumulant generating function $H(t) = \log\langle \mathrm{e}^{t\xi(0)}\rangle$,

Heuristics using local times

Introducing the walker's local times $\ell_t(z) = \int_0^t \mathrm{d}s\, \delta_{X_s}(z)$ and the cumulant generating function $H(t) = \log\langle \mathrm{e}^{t\xi(0)}\rangle$, we have

$$\langle U(t)\rangle = \Big\langle \mathbb{E}_0\Big[\exp\Big\{\sum_{z\in\mathbb{Z}^d} \xi(z)\ell_t(z)\Big\}\Big]\Big\rangle$$

Heuristics using local times

Introducing the walker's local times $\ell_t(z) = \int_0^t \mathrm{d}s\, \delta_{X_s}(z)$ and the cumulant generating function $H(t) = \log\langle \mathrm{e}^{t\xi(0)}\rangle$, we have

$$\langle U(t)\rangle = \Big\langle \mathbb{E}_0\Big[\exp\Big\{\sum_{z\in\mathbb{Z}^d} \xi(z)\ell_t(z)\Big\}\Big]\Big\rangle = \mathbb{E}_0\Big\langle \exp\Big\{\sum_{z\in\mathbb{Z}^d} \xi(z)\ell_t(z)\Big\}\Big\rangle$$

Heuristics using local times

Introducing the walker's local times $\ell_t(z) = \int_0^t \mathrm{d}s\, \delta_{X_s}(z)$ and the cumulant generating function $H(t) = \log\langle \mathrm{e}^{t\xi(0)}\rangle$, we have

$$\langle U(t)\rangle = \Big\langle \mathbb{E}_0\Big[\exp\Big\{\sum_{z\in\mathbb{Z}^d} \xi(z)\ell_t(z)\Big\}\Big]\Big\rangle = \mathbb{E}_0\Big\langle \exp\Big\{\sum_{z\in\mathbb{Z}^d} \xi(z)\ell_t(z)\Big\}\Big\rangle$$
$$= \mathbb{E}_0 \exp\Big\{\sum_{z\in\mathbb{Z}^d} H(\ell_t(z))\Big\}$$

Heuristics using local times

Introducing the walker's local times $\ell_t(z) = \int_0^t \mathrm{d}s\, \delta_{X_s}(z)$ and the cumulant generating function $H(t) = \log\langle \mathrm{e}^{t\xi(0)}\rangle$, we have

$$\begin{aligned}\langle U(t)\rangle &= \Big\langle \mathbb{E}_0\Big[\exp\Big\{\sum_{z\in\mathbb{Z}^d} \xi(z)\ell_t(z)\Big\}\Big]\Big\rangle = \mathbb{E}_0\Big\langle \exp\Big\{\sum_{z\in\mathbb{Z}^d} \xi(z)\ell_t(z)\Big\}\Big\rangle \\ &= \mathbb{E}_0 \exp\Big\{\sum_{z\in\mathbb{Z}^d} H(\ell_t(z))\Big\} = \mathrm{e}^{H(t)}\mathbb{E}_0 \exp\Big\{t\Phi_t\big(\tfrac{1}{t}\ell_t\big)\Big\},\end{aligned}$$

with

$$\Phi_t(\psi) = \sum_{z\in\mathbb{Z}^d} \frac{H(t\psi(z)) - \psi(z)H(t)}{t}.$$

Heuristics using local times

Introducing the walker's local times $\ell_t(z) = \int_0^t \mathrm{d}s\, \delta_{X_s}(z)$ and the cumulant generating function $H(t) = \log\langle \mathrm{e}^{t\xi(0)}\rangle$, we have

$$\langle U(t)\rangle = \Big\langle \mathbb{E}_0\Big[\exp\Big\{\sum_{z\in\mathbb{Z}^d} \xi(z)\ell_t(z)\Big\}\Big]\Big\rangle = \mathbb{E}_0\Big\langle \exp\Big\{\sum_{z\in\mathbb{Z}^d} \xi(z)\ell_t(z)\Big\}\Big\rangle$$

$$= \mathbb{E}_0 \exp\Big\{\sum_{z\in\mathbb{Z}^d} H(\ell_t(z))\Big\} = \mathrm{e}^{H(t)}\mathbb{E}_0 \exp\Big\{t\Phi_t(\tfrac{1}{t}\ell_t)\Big\},$$

with

$$\Phi_t(\psi) = \sum_{z\in\mathbb{Z}^d} \frac{H(t\psi(z)) - \psi(z)H(t)}{t}.$$

Now use that $\Phi_t(\psi) \to \Phi(\psi) = \rho \sum_z \psi(z)\log\psi(z)$,

Heuristics using local times

Introducing the walker's local times $\ell_t(z) = \int_0^t \mathrm{d}s\, \delta_{X_s}(z)$ and the cumulant generating function $H(t) = \log\langle \mathrm{e}^{t\xi(0)}\rangle$, we have

$$\langle U(t)\rangle = \Big\langle \mathbb{E}_0\Big[\exp\Big\{\sum_{z\in\mathbb{Z}^d} \xi(z)\ell_t(z)\Big\}\Big]\Big\rangle = \mathbb{E}_0\Big\langle \exp\Big\{\sum_{z\in\mathbb{Z}^d} \xi(z)\ell_t(z)\Big\}\Big\rangle$$
$$= \mathbb{E}_0 \exp\Big\{\sum_{z\in\mathbb{Z}^d} H(\ell_t(z))\Big\} = \mathrm{e}^{H(t)}\mathbb{E}_0 \exp\Big\{t\Phi_t\big(\tfrac{1}{t}\ell_t\big)\Big\},$$

with

$$\Phi_t(\psi) = \sum_{z\in\mathbb{Z}^d} \frac{H(t\psi(z)) - \psi(z)H(t)}{t}.$$

Now use that $\Phi_t(\psi) \to \Phi(\psi) = \rho \sum_z \psi(z)\log\psi(z)$, and the large-deviations principle [DONSKER/VARADHAN $\approx$ '75], [GÄRTNER $\approx$ '75]

$$\mathbb{P}_0\big(\tfrac{1}{t}\ell_t \approx \psi\big) \approx \exp\{-tI(\psi)\}, \qquad \text{with } I(\psi) = \textstyle\sum_{x\sim y}\Big(\psi(x) - \psi(y)\Big)^2,$$

Introducing the walker's local times $\ell_t(z) = \int_0^t \mathrm{d}s\, \delta_{X_s}(z)$ and the cumulant generating function $H(t) = \log\langle \mathrm{e}^{t\xi(0)}\rangle$, we have

$$\langle U(t)\rangle = \Big\langle \mathbb{E}_0\Big[\exp\Big\{\sum_{z\in\mathbb{Z}^d}\xi(z)\ell_t(z)\Big\}\Big]\Big\rangle = \mathbb{E}_0\Big\langle \exp\Big\{\sum_{z\in\mathbb{Z}^d}\xi(z)\ell_t(z)\Big\}\Big\rangle$$

$$= \mathbb{E}_0 \exp\Big\{\sum_{z\in\mathbb{Z}^d} H(\ell_t(z))\Big\} = \mathrm{e}^{H(t)}\mathbb{E}_0 \exp\Big\{t\Phi_t\big(\tfrac{1}{t}\ell_t\big)\Big\},$$

with

$$\Phi_t(\psi) = \sum_{z\in\mathbb{Z}^d} \frac{H(t\psi(z)) - \psi(z)H(t)}{t}.$$

Now use that $\Phi_t(\psi) \to \Phi(\psi) = \rho\sum_z \psi(z)\log\psi(z)$, and the large-deviations principle [DONSKER/VARADHAN $\approx$ '75], [GÄRTNER $\approx$ '75]

$$\mathbb{P}_0\big(\tfrac{1}{t}\ell_t \approx \psi\big) \approx \exp\{-tI(\psi)\}, \qquad \text{with } I(\psi) = \textstyle\sum_{x\sim y}\Big(\psi(x) - \psi(y)\Big)^2,$$

to get that

$$\langle U(t)\rangle \approx \mathrm{e}^{H(t)} \exp\Big\{t \sup_{\|\psi\|_1=1}\big(\Phi(\psi) - I(\psi)\big)\Big\}.$$

The double-exponential distribution I

With $\rho \in (0,\infty)$, consider the upper tails

$$\mathrm{Prob}(\xi(0) > r) \approx \exp\Big\{-\mathrm{e}^{r/\rho}\Big\}, \qquad r \to \infty.$$

The double-exponential distribution I

With $\rho \in (0,\infty)$, consider the upper tails

$$\mathrm{Prob}(\xi(0) > r) \approx \exp\Big\{ -\mathrm{e}^{r/\rho} \Big\}, \qquad r \to \infty.$$

Then $H(t) = \log\langle \mathrm{e}^{t\xi(0)}\rangle$ turns out to be $H(t) = \rho t \log(\rho t) - \rho t + o(t)$.

The double-exponential distribution I

With $\rho \in (0,\infty)$, consider the upper tails

$$\mathrm{Prob}(\xi(0) > r) \approx \exp\Big\{-\mathrm{e}^{r/\rho}\Big\}, \qquad r \to \infty.$$

Then $H(t) = \log\langle \mathrm{e}^{t\xi(0)}\rangle$ turns out to be $H(t) = \rho t \log(\rho t) - \rho t + o(t)$.

Moment asymptotics.

[GÄRTNER/MOLCHANOV 1998]: For any $p \in \mathbb{N}$, as $t \to \infty$,

$$\langle U(t)^p\rangle = \mathrm{e}^{H(tp)}\mathrm{e}^{-tp(\chi+o(1))}, \quad \text{where} \quad \chi = \inf_{\varphi\colon \mathbb{Z}^d \to \mathbb{R}} \Big[\mathscr{L}(\varphi) - \lambda(\varphi)\Big],$$

and $\mathscr{L}(\varphi) = \frac{\rho}{\mathrm{e}}\sum_{z\in\mathbb{Z}^d} \mathrm{e}^{\varphi(z)/\rho}$, and $\lambda(\varphi)$ is the top of the spectrum of $\Delta^{\mathrm{d}} + \varphi$ in $\mathbb{Z}^d$.

The double-exponential distribution I

With $\rho \in (0,\infty)$, consider the upper tails

$$\mathrm{Prob}(\xi(0) > r) \approx \exp\Big\{-\mathrm{e}^{r/\rho}\Big\}, \qquad r \to \infty.$$

Then $H(t) = \log\langle \mathrm{e}^{t\xi(0)}\rangle$ turns out to be $H(t) = \rho t \log(\rho t) - \rho t + o(t)$.

Moment asymptotics.

[GÄRTNER/MOLCHANOV 1998]: For any $p \in \mathbb{N}$, as $t \to \infty$,

$$\langle U(t)^p\rangle = \mathrm{e}^{H(tp)}\mathrm{e}^{-tp(\chi+o(1))}, \quad \text{where} \quad \chi = \inf_{\varphi\colon \mathbb{Z}^d\to\mathbb{R}} \Big[\mathscr{L}(\varphi) - \lambda(\varphi)\Big],$$

and $\mathscr{L}(\varphi) = \frac{\rho}{\mathrm{e}}\sum_{z\in\mathbb{Z}^d} \mathrm{e}^{\varphi(z)/\rho}$, and $\lambda(\varphi)$ is the top of the spectrum of $\Delta^{\mathrm{d}} + \varphi$ in $\mathbb{Z}^d$.

- Minimiser(s) exist. They are unique for ρ sufficiently large and rather inexplicit.
- $\mathscr{L}(\varphi)$ is a large-deviation rate function for the shifted potential $\xi_t = \xi - H(t)/t$. On the event $\{\xi_t \approx \varphi\}$, the contribution to the Feynman-Kac formula is quantified by $\lambda(\varphi)$. The optimal profile describes the total expected mass.
- The structure of the asymptotic optimal profile is discrete; no spatial scaling is involved.

The double-exponential distribution II

Almost sure asymptotics.

[GÄRTNER/MOLCHANOV 1998]: As $t \to \infty$,

$$\frac{1}{t} \log U(t) = \frac{H(\log t)}{\log t} - \widetilde{\chi} + o(1), \quad \text{with} \quad -\widetilde{\chi} = \sup\Big\{\lambda(\varphi) \Big| \varphi : \mathbb{Z}^d \to \mathbb{R}, \mathscr{L}(\varphi) \le d\Big\}.$$

Almost sure asymptotics.

[GÄRTNER/MOLCHANOV 1998]: As $t \to \infty$,

$$\frac{1}{t}\log U(t) = \frac{H(\log t)}{\log t} - \widetilde{\chi} + o(1), \quad \text{with} \quad -\widetilde{\chi} = \sup\Big\{\lambda(\varphi)\Big|\varphi\colon \mathbb{Z}^d \to \mathbb{R}, \mathscr{L}(\varphi) \le d\Big\}.$$

- The maximiser(s) of χ and $\widetilde{\chi}$ are identical.
- A Borel-Cantelli argument shows that, with probability one, for all large t, every potential shape φ satisfying $\mathscr{L}(\varphi) \le d$ appears on some island in the cube $Q_t = [-t,t]^d \cap \mathbb{Z}^d$ in the potential $\xi_{\log t} = \xi - H(\log t)/\log t$. The contribution to the Feynman-Kac formula coming from those paths that go quickly there and spend most of the time there is quantified by $\lambda(\varphi)$. The optimal such profile φ describes the total contribution.
- Every such island with φ an optimal profile is potentially one of the intermittent islands we mentioned above. The corresponding eigenfunction is localised.

The double-exponential distribution II

Almost sure asymptotics.

[GÄRTNER/MOLCHANOV 1998]: As $t \to \infty$,

$$\frac{1}{t}\log U(t) = \frac{H(\log t)}{\log t} - \widetilde{\chi} + o(1), \quad \text{with} \quad -\widetilde{\chi} = \sup\Big\{\lambda(\varphi)\Big|\varphi \colon \mathbb{Z}^d \to \mathbb{R}, \mathscr{L}(\varphi) \le d\Big\}.$$

- The maximiser(s) of χ and $\widetilde{\chi}$ are identical.
- A Borel-Cantelli argument shows that, with probability one, for all large t, every potential shape φ satisfying $\mathscr{L}(\varphi) \le d$ appears on some island in the cube $Q_t = [-t,t]^d \cap \mathbb{Z}^d$ in the potential $\xi_{\log t} = \xi - H(\log t)/\log t$. The contribution to the Feynman-Kac formula coming from those paths that go quickly there and spend most of the time there is quantified by $\lambda(\varphi)$. The optimal such profile φ describes the total contribution.
- Every such island with φ an optimal profile is potentially one of the intermittent islands we mentioned above. The corresponding eigenfunction is localised.

This is one of the universality classes. The second one is the boundary case $\rho = \infty$. The third one is introduced on the next slide, the fourth one is a kind of continuous version of the first one.

Potentials bounded from above: Moments

Assume $\operatorname{esssup}(\xi(0)) = 0$. With $\gamma \in [0,1)$, consider the upper-tail behavior

$$\mathrm{Prob}(\xi(0) > -x) \approx \exp\Big\{ -\mathrm{const.}\, x^{-\gamma/(1-\gamma)} \Big\}, \qquad x \downarrow 0.$$

Potentials bounded from above: Moments

Assume $\mathrm{esssup}(\xi(0)) = 0$. With $\gamma \in [0,1)$, consider the upper-tail behavior

$$\mathrm{Prob}(\xi(0) > -x) \approx \exp\Big\{ -\mathrm{const.}\, x^{-\gamma/(1-\gamma)} \Big\}, \qquad x \downarrow 0.$$

- The case $\gamma = 0$ contains the case of i.i.d. Bernoulli traps, where $\xi(0) \in \{-\infty, 0\}$.
- $H(t) = \log\langle e^{t\xi(0)}\rangle$ is roughly $H(t) \approx -\mathrm{const.}\, t^{\gamma}$.

Potentials bounded from above: Moments

Assume $\operatorname{esssup}(\xi(0))=0$. With $\gamma\in[0,1)$, consider the upper-tail behavior

$$\operatorname{Prob}(\xi(0)>-x)\approx\exp\Big\{-\text{const.}\,x^{-\gamma/(1-\gamma)}\Big\},\qquad x\downarrow 0.$$

- The case $\gamma=0$ contains the case of i.i.d. Bernoulli traps, where $\xi(0)\in\{-\infty,0\}$.
- $H(t)=\log\langle e^{t\xi(0)}\rangle$ is roughly $H(t)\approx-\text{const.}\,t^{\gamma}$.

The relevant islands have diameter of the order

$$\alpha(t)=t^{\nu},\qquad \text{where}\qquad \nu=\frac{1-\gamma}{d+2-d\gamma}\in\big(0,\tfrac{1}{d+2}\big].$$

Moment asymptotics.

[BISKUP/K. 2001]: For any $p\in(0,\infty)$, as $t\to\infty$,

$$\frac{1}{tp}\log\langle U(t)^p\rangle=-\frac{\chi+o(1)}{\alpha(pt)^2},\qquad \text{where}\qquad \chi=\inf_{\varphi\in\mathscr{C}(\mathbb{R}^d\to[-\infty,0])}\Big[\mathscr{L}(\varphi)-\lambda(\varphi)\Big],$$

and $\mathscr{L}(\varphi)=\text{const.}\int_{\mathbb{R}^d}|\varphi(x)|^{-\gamma/(1-\gamma)}\,\mathrm{d}x$,
and $\lambda(\varphi)$ is the top of the spectrum of $\Delta+\varphi$ in $L^2(\mathbb{R}^d)$.

Potentials bounded from above: Moments

Assume $\operatorname{esssup}(\xi(0)) = 0$. With $\gamma \in [0,1)$, consider the upper-tail behavior

$$\mathrm{Prob}(\xi(0) > -x) \approx \exp\Big\{ -\mathrm{const.}\, x^{-\gamma/(1-\gamma)} \Big\}, \qquad x \downarrow 0.$$

- The case $\gamma = 0$ contains the case of i.i.d. Bernoulli traps, where $\xi(0) \in \{-\infty, 0\}$.
- $H(t) = \log\langle \mathrm{e}^{t\xi(0)}\rangle$ is roughly $H(t) \approx -\mathrm{const.}\, t^{\gamma}$.

The relevant islands have diameter of the order

$$\alpha(t) = t^{\nu}, \qquad \text{where} \qquad \nu = \frac{1-\gamma}{d+2-d\gamma} \in \big(0, \tfrac{1}{d+2}\big].$$

Moment asymptotics.

[BISKUP/K. 2001]: For any $p \in (0,\infty)$, as $t \to \infty$,

$$\frac{1}{tp} \log\langle U(t)^p\rangle = -\frac{\chi + o(1)}{\alpha(pt)^2}, \qquad \text{where} \qquad \chi = \inf_{\varphi \in \mathscr{C}(\mathbb{R}^d \to [-\infty,0])} \Big[\mathscr{L}(\varphi) - \lambda(\varphi) \Big],$$

and $\mathscr{L}(\varphi) = \mathrm{const.} \int_{\mathbb{R}^d} |\varphi(x)|^{-\gamma/(1-\gamma)}\, \mathrm{d}x$,
and $\lambda(\varphi)$ is the top of the spectrum of $\Delta + \varphi$ in $L^2(\mathbb{R}^d)$.

- $\mathscr{L}$ is a LD rate function for the rescaled potential $\xi_t(\cdot) = \alpha(t)^2 \xi(\cdot\, \alpha(t))$.

Potentials bounded from above: Lifshitz tails

- $\nu_R = \frac{1}{\#Q_R}\sum_k \delta_{-\lambda_{R,k}}$ spectral measure of $-\Delta^{\mathrm{d}} - \xi$ in the box Q_R.
- $N(E) = \lim_{R\to\infty} \nu_R([0,E])$ integrated density of states (IDS).

Potentials bounded from above: Lifshitz tails

- $\nu_R = \frac{1}{\# Q_R} \sum_k \delta_{-\lambda_{R,k}}$ spectral measure of $-\Delta^{\mathrm{d}} - \xi$ in the box Q_R.
- $N(E) = \lim_{R\to\infty} \nu_R([0,E])$ integrated density of states (IDS).

Lifshitz tails.

[BISKUP/K. 2001]: $$\lim_{E\downarrow 0} \frac{\log|\log N(E)|}{\log E} = -\frac{1-2\nu}{2\nu} \in \Big(-\infty, -\frac{d}{2}\Big].$$

- $\nu_R = \frac{1}{\#Q_R} \sum_k \delta_{-\lambda_{R,k}}$ spectral measure of $-\Delta^{\mathrm{d}} - \xi$ in the box Q_R.
- $N(E) = \lim_{R\to\infty} \nu_R([0,E])$ integrated density of states (IDS).

Lifshitz tails.

[BISKUP/K. 2001]: $$\lim_{E\downarrow 0} \frac{\log|\log N(E)|}{\log E} = -\frac{1-2\nu}{2\nu} \in \Big(-\infty, -\frac{d}{2}\Big].$$

Proof idea: Laplace transform

$$\mathscr{L}(\nu_R, t) = \int_0^\infty \mathrm{e}^{-Et}\,\nu_R(\mathrm{d}E) = \frac{1}{\#Q_R}\sum_k \mathrm{e}^{t\lambda_{R,k}}$$

Potentials bounded from above: Lifshitz tails

- $\nu_R = \frac{1}{\#Q_R} \sum_k \delta_{-\lambda_{R,k}}$ spectral measure of $-\Delta^{\mathrm{d}} - \xi$ in the box Q_R.
- $N(E) = \lim_{R\to\infty} \nu_R([0,E])$ integrated density of states (IDS).

Lifshitz tails.

[BISKUP/K. 2001]: $$\lim_{E\downarrow 0} \frac{\log|\log N(E)|}{\log E} = -\frac{1-2\nu}{2\nu} \in \Big(-\infty, -\frac{d}{2}\Big].$$

Proof idea: Laplace transform

$$\mathscr{L}(\nu_R, t) = \int_0^\infty \mathrm{e}^{-Et}\, \nu_R(\mathrm{d}E) = \frac{1}{\#Q_R} \sum_k \mathrm{e}^{t\lambda_{R,k}}$$

$$= \frac{1}{\#Q_R} \sum_{z\in\mathbb{Z}^d} \mathbb{E}_z\Big[\exp\Big\{\int_0^t \xi(X_s)\,\mathrm{d}s\Big\} \mathbb{1}\{X_{[0,t]} \subset Q_R\} \mathbb{1}\{X_t = z\}\Big]$$

Potentials bounded from above: Lifshitz tails

- $\nu_R = \frac{1}{\#Q_R}\sum_k \delta_{-\lambda_{R,k}}$ spectral measure of $-\Delta^{\rm d} - \xi$ in the box Q_R.
- $N(E) = \lim_{R\to\infty} \nu_R([0,E])$ integrated density of states (IDS).

Lifshitz tails.

[Biskup/K. 2001]:
$$\lim_{E\downarrow 0} \frac{\log|\log N(E)|}{\log E} = -\frac{1-2\nu}{2\nu} \in \Big(-\infty, -\frac{d}{2}\Big].$$

Proof idea: Laplace transform

$$\mathscr{L}(\nu_R, t) = \int_0^\infty \mathrm{e}^{-Et}\,\nu_R(\mathrm{d}E) = \frac{1}{\#Q_R}\sum_k \mathrm{e}^{t\lambda_{R,k}}$$
$$= \frac{1}{\#Q_R}\sum_{z\in\mathbb{Z}^d} \mathbb{E}_z\Big[\exp\Big\{\int_0^t \xi(X_s)\,\mathrm{d}s\Big\}\mathbb{1}\{X_{[0,t]}\subset Q_R\}\mathbb{1}\{X_t = z\}\Big]$$
$$\overset{R\to\infty}{\longrightarrow} \Big\langle \mathbb{E}_0\Big[\exp\Big\{\int_0^t \xi(X_s)\,\mathrm{d}s\Big\}\mathbb{1}\{X_t = 0\}\Big]\Big\rangle \qquad \text{(ergodic theorem)}$$

- $\nu_R = \frac{1}{\#Q_R}\sum_k \delta_{-\lambda_{R,k}}$ spectral measure of $-\Delta^{\mathrm{d}} - \xi$ in the box Q_R.
- $N(E) = \lim_{R\to\infty} \nu_R([0,E])$ integrated density of states (IDS).

Lifshitz tails.

[BISKUP/K. 2001]: $$\lim_{E\downarrow 0} \frac{\log|\log N(E)|}{\log E} = -\frac{1-2\nu}{2\nu} \in \Big(-\infty, -\frac{d}{2}\Big].$$

Proof idea: Laplace transform

$$\begin{aligned}
\mathscr{L}(\nu_R,t) &= \int_0^\infty \mathrm{e}^{-Et}\,\nu_R(\mathrm{d}E) = \frac{1}{\#Q_R}\sum_k \mathrm{e}^{t\lambda_{R,k}} \\
&= \frac{1}{\#Q_R}\sum_{z\in\mathbb{Z}^d} \mathbb{E}_z\Big[\exp\Big\{\int_0^t \xi(X_s)\,\mathrm{d}s\Big\}\mathbb{1}\{X_{[0,t]}\subset Q_R\}\mathbb{1}\{X_t=z\}\Big] \\
&\overset{R\to\infty}{\to} \Big\langle \mathbb{E}_0\Big[\exp\Big\{\int_0^t \xi(X_s)\,\mathrm{d}s\Big\}\mathbb{1}\{X_t=0\}\Big]\Big\rangle \qquad \text{(ergodic theorem)} \\
&\overset{t\to\infty}{\approx} \exp\Big\{-\frac{t}{\alpha(t)^2}\chi\Big\},
\end{aligned}$$

according to the moment asymptotics. Now invert the Laplace transform.

Heavy-tailed potentials I

Another potential class, the Pareto-distribution:

$$\mathrm{Prob}(\xi(0) > r) = r^{-\alpha}, \qquad r \in [1,\infty), \qquad \text{(Parameter } \alpha > d\text{)}.$$

Then the parabolic Anderson model possesses a.s. a solution $u(t,\cdot)$, but $U(t)$ has no moments.

Heavy-tailed potentials I

Another potential class, the Pareto-distribution:

$$\mathrm{Prob}(\xi(0) > r) = r^{-\alpha}, \qquad r \in [1,\infty), \qquad (\text{Parameter } \alpha > d).$$

Then the parabolic Anderson model possesses a.s. a solution $u(t,\cdot)$, but $U(t)$ has no moments.

Weak Asymptotics for Pareto-Distributed Potentials

[VAN DER HOFSTAD/MÖRTERS/SIDOROVA 2008]. For $x \in \mathbb{R}$,

$$\lim_{t\to\infty} \mathrm{Prob}\Big(\Big(\frac{t}{\log t}\Big)^{-\frac{d}{\alpha-d}} \frac{1}{t} \log U(t) \le x\Big) = \exp\big(-\mu x^{d-\alpha}\big),$$

where $\mu \in (0,\infty)$ is some suitable, explicit constant.

Heavy-tailed potentials I

Another potential class, the Pareto-distribution:

$$\mathrm{Prob}(\xi(0) > r) = r^{-\alpha}, \qquad r \in [1,\infty), \qquad (\text{Parameter } \alpha > d).$$

Then the parabolic Anderson model possesses a.s. a solution $u(t,\cdot)$, but $U(t)$ has no moments.

Weak Asymptotics for Pareto-Distributed Potentials

[VAN DER HOFSTAD/MÖRTERS/SIDOROVA 2008]. For $x \in \mathbb{R}$,

$$\lim_{t\to\infty} \mathrm{Prob}\Big(\Big(\frac{t}{\log t}\Big)^{-\frac{d}{\alpha-d}} \frac{1}{t} \log U(t) \le x\Big) = \exp\big(-\mu x^{d-\alpha}\big),$$

where $\mu \in (0,\infty)$ is some suitable, explicit constant.

- $\frac{1}{t}\log U(t)$ has the same weak asymptotics as the maximum of t^d independent Pareto $(\alpha-d)$-distributed random variables.
- Apparently, the potential's random fluctuations dominate the smoothing effect of Δ^{d}.

Heavy-tailed potentials II

Well-known: Large values of a sum of i.i.d. heavy tailed random variables are most easily realised by having just one of the values extremely large (and the others of moderate size).

Heavy-tailed potentials II

Well-known: Large values of a sum of i.i.d. heavy tailed random variables are most easily realised by having just one of the values extremely large (and the others of moderate size).

This principle can be translated to the solution of the parabolic Anderson model:

Complete Localisation for Pareto-Distributed Potentials

[K./LACOIN/MÖRTERS/SIDOROVA 2009]. There is a process $(Z_t)_{t>0}$ in $\mathbb{Z}^d$ such that

$$\lim_{t\to\infty} \frac{u(t,Z_t)}{U(t)} = 1 \qquad \text{in probability.}$$

Furthermore, $Z_t(\log t/t)^{\alpha/(\alpha-d)}$ converges in distribution to some non-degenerate random variable.

Heavy-tailed potentials II

Well-known: Large values of a sum of i.i.d. heavy tailed random variables are most easily realised by having just one of the values extremely large (and the others of moderate size).

This principle can be translated to the solution of the parabolic Anderson model:

Complete Localisation for Pareto-Distributed Potentials

[K./LACOIN/MÖRTERS/SIDOROVA 2009]. There is a process $(Z_t)_{t>0}$ in $\mathbb{Z}^d$ such that

$$\lim_{t\to\infty} \frac{u(t,Z_t)}{U(t)} = 1 \qquad \text{in probability.}$$

Furthermore, $Z_t(\log t/t)^{\alpha/(\alpha-d)}$ converges in distribution to some non-degenerate random variable.

- Hence, there is precisely one intermittent island, a singleton.
- The localisation statement is not true in almost sure sense, since the process $(Z_t)_{t>0}$ jumps.

Open problems and further questions

- Is just one island enough for general potentials? What is the contribution from the others? Make the connection to Anderson localisation mathematically rigorous. (ongoing work with MAREK BISKUP, see the next slides)
- Describe transition from localised solution ($u(0,\cdot) = \delta_0(\cdot)$) to homogeneous solution ($u(0,\cdot) \equiv 1$) (see works by [BEN AROUS, MOLCHANOV, RAMIREZ]).
- Replace Δ^{d} by $\sigma\Delta^{\mathrm{d}}$ and let $\sigma = \sigma(t)$ depend on time (recent thesis by SYLVIA SCHMIDT).
- Replace Δ^{d} by $\sigma\Delta^{\mathrm{d}}$ and let $\sigma = \sigma(x)$ depend on time and a new randomness (randomly perturbed Laplace operator). Ongoing thesis.
- Ageing properties: asymptotics for time-correlation

$$\frac{\langle U(t)U(t+s(t))\rangle}{\sqrt{\langle U(t)^2\rangle\langle U(t+s(t))^2\rangle}}$$

and interpretation ([GÄRTNER/SCHNITZLER 2010]).

Mass Concentration and Eigenvalue Order Statistics

From now: joint work in progress with M. Biskup (České Budějovice and Los Angeles).

- Mass concentration: The total mass $U(t) = \sum_{z \in \mathbb{Z}^d} u(t,z)$ comes in probability from just one island (strong form of intermittency).
- Eigenvalue order statistics: The top eigenvalues and the concentration centres of the corresponding eigenfunctions (after rescaling and shifting) form a Poisson point process.

Mass Concentration and Eigenvalue Order Statistics

From now: joint work in progress with M. Biskup (České Budějovice and Los Angeles).

- Mass concentration: The total mass $U(t) = \sum_{z\in\mathbb{Z}^d} u(t,z)$ comes in probability from just one island (strong form of intermittency).
- Eigenvalue order statistics: The top eigenvalues and the concentration centres of the corresponding eigenfunctions (after rescaling and shifting) form a Poisson point process.

Explanation

- The top eigenvalues satisfy an order statistics in some box $B^{(1)}(t) \subset B^{(2)}(t)$. In particular, we have control on their differences, i.e., the spectral gaps.
- The corresponding eigenfunctions are exponentially localised in islands $B_{r_t}(z_k)$ whose locations z_k form a Poisson point process.
- The main contribution to $U(t)$ inside $B_t^{(1)}$ comes from precisely that summand k which maximises $\mathrm{e}^{t\lambda_k(\xi,B_t^{(1)})}|\varphi_k(0)|$.
- The contribution to $U(t)$ from the outside of the *a priori* box $B^{(2)}(t)$ is negligible.
- The contribution to $U(t)$ from $B^{(2)}(t)\setminus B^{(1)}(t)$ is negligible since the values of $\mathrm{e}^{t\lambda_k(\xi,B^{(2)}(t))}\varphi_k(0)\langle\varphi_k,1\rangle$ are substantially worse.

Earlier results

- [SZNITMAN 98] (Brownian motion among Poisson obstacles) and [GÄRTNER/K./MOLCHANOV 07] (double-exponential distribution): mass concentration a.s. in $t^{o(1)}$ islands.
- [K./LACOIN/MÖRTERS/SIDOROVA 09] (Pareto distribution): mass concentration in one site in probability, and in two sites a.s.
- [KILLIP/NAKANO 07], [GERMINET/KLOPP 10] (bounded distributions with smooth density):
  Poisson process convergence for rescaled eigenvalues and localisation centers of eigenfunctions in large boxes in the localised regime
- [ASTRAUSKAS 08] (all heavy-tailed potentials, '$\rho = \infty$'):
  Eigenfunction localisation and eigenvalue order statistics at the top of the spectrum.

Earlier results

- [SZNITMAN 98] (Brownian motion among Poisson obstacles) and [GÄRTNER/K./MOLCHANOV 07] (double-exponential distribution): mass concentration a.s. in $t^{o(1)}$ islands.
- [K./LACOIN/MÖRTERS/SIDOROVA 09] (Pareto distribution): mass concentration in one site in probability, and in two sites a.s.
- [KILLIP/NAKANO 07], [GERMINET/KLOPP 10] (bounded distributions with smooth density):
  Poisson process convergence for rescaled eigenvalues and localisation centers of eigenfunctions in large boxes in the localised regime
- [ASTRAUSKAS 08] (all heavy-tailed potentials, '$\rho = \infty$'):
  Eigenfunction localisation and eigenvalue order statistics at the top of the spectrum.

We are working here for ξ double-exponentially distributed, i.e., for some $\rho \in (0,\infty)$,

$$\mathrm{Prob}(\xi(0) > r) = \exp\left\{-\mathrm{e}^{r/\rho}\right\}, \qquad r \in \mathbb{R}.$$

Earlier papers:

[GÄRTNER/MOLCHANOV 98], [GÄRTNER/DEN HOLLANDER 99],

[GÄRTNER/K./MOLCHANOV 07].

The potential is unbounded to $+\infty$. The islands are of bounded size. The potential and of the solution approach (after shifting and normalization) certain shapes, whic

First result: Eigenvalue order statistics

Abbreviate $B_L = L \times [-\frac{1}{2}, \frac{1}{2}]^d$.

Theorem 1

There is a number $\chi = \chi_\rho \in (0, 2d)$ and a sequence $(a_L)_{L\in\mathbb{N}}$ with $a_L = \rho \log\log|B_L| - \chi + o(1)$ as $L \to \infty$ and, for any $L \in \mathbb{N}$, a sequence $(X_k^{(L)})_k$ in B_L such that, in probability,

$$\lim_{L\to\infty} \sum_{z\colon |z-X_k^{(L)}|\le \log L} \varphi_k(z)^2 = 1, \qquad k \in \mathbb{N},$$

and the law of

$$\sum_{k\in\mathbb{N}} \delta_{\left(\frac{X_k^{(L)}}{L}, (\lambda_k(\xi, B_L) - a_L)\log L\right)}$$

converges weakly to a Poisson process on $B_1 \times \mathbb{R}$ with intensity measure $\mathrm{d}x \otimes \mathrm{e}^{-\lambda}\,\mathrm{d}\lambda$.

Hence, the top eigenvalues in B_L are of order $\log\log L$, leave gaps of order $1/\log L$, are in the max-domain of attraction of the Gumbel distribution, and the localisation centres are separated by a distance of order L and are uniformly distributed.

Theorem 2

Put $r_L = L \log L \log\log\log L$ and

$$\Psi_{L,t}(z,\lambda) = \frac{t}{r_L}(\lambda - a_L)\log L - \frac{|z|}{L},$$

and pick k such that $\Psi_{L,t}(X_k^{(L)}, \lambda_k(\xi, B_L))$ is maximal, and put $Z_{L,t} = X_k^{(L)}/L$. Then, with L_t defined by $r_{L_t} = t$,

$$\lim_{t\to\infty} \frac{1}{U(t)} \sum_{z\colon\, |z - L_t Z_{L_t,t}| \le R_t} u(t,z) = 1 \qquad \text{in probability,}$$

for any $R_t \gg \log t$.

Hence, the total mass essentially comes from a single $\gg \log t$-island in the centred box $B^{(1)}(t)$ with radius $\approx t/(\log t \log\log\log t)$

Ageing

A system is said to age if its significant changes come after longer and longer (or shorter and shorter) time lags, such that one can see from the frequency of changes how much time has elapsed.

Ageing properties of the PAM can now be studied in terms of the time lags between jumps of the concentration site.

These ones, in turn, may be described as follows.

Theorem 3: Scaling limit of concentration location

As $L \to \infty$, the process $(Z_{L,tr_L})_{t\in[0,\infty)}$ converges in distribution to the process of maximizers of $z \mapsto t\lambda - |z|$ over the points (z,λ) of a Poisson process on $[-\frac{1}{2},\frac{1}{2}]^d \times \mathbb{R}$ with intensity measure $\mathrm{d}x \otimes \mathrm{e}^{-\lambda}\,\mathrm{d}\lambda$.

Comments

- The rescaled and shifted eigenvalues $(\lambda_k(\xi, B_L) - a_L)\log L$ are asymptotically independent and lie in the max-domain of attraction of the Gumbel distribution.

Comments

- The rescaled and shifted eigenvalues $(\lambda_k(\xi, B_L) - a_L)\log L$ are asymptotically independent and lie in the max-domain of attraction of the Gumbel distribution.
- The gaps between subsequent eigenvalues in B_L at the edge of the spectrum are of order $1/\log L$, rather than $1/|B_L|$ as in the bulk of the spectrum.

Comments

- The rescaled and shifted eigenvalues $(\lambda_k(\xi, B_L) - a_L)\log L$ are asymptotically independent and lie in the max-domain of attraction of the Gumbel distribution.
- The gaps between subsequent eigenvalues in B_L at the edge of the spectrum are of order $1/\log L$, rather than $1/|B_L|$ as in the bulk of the spectrum.
- We have in particular Anderson localisation at the edge of the spectrum.

Comments

- The rescaled and shifted eigenvalues $(\lambda_k(\xi, B_L) - a_L)\log L$ are asymptotically independent and lie in the max-domain of attraction of the Gumbel distribution.
- The gaps between subsequent eigenvalues in B_L at the edge of the spectrum are of order $1/\log L$, rather than $1/|B_L|$ as in the bulk of the spectrum.
- We have in particular Anderson localisation at the edge of the spectrum.
- The control from Poisson process convergence holds only in the box $B^{(1)}(t)$ of radius $\asymp t/\log t \log\log\log t$, and the contribution from the outside of the box $B^{(2)}(t)$ of radius $t\log^2 t$ is easily seen to be negligible. The treatment of the region $B^{(2)}(t) \setminus B^{(1)}(t)$ is delicate and requires a comparison of the two eigenvalue expansions.

Comments

- The rescaled and shifted eigenvalues $(\lambda_k(\xi, B_L) - a_L)\log L$ are asymptotically independent and lie in the max-domain of attraction of the Gumbel distribution.
- The gaps between subsequent eigenvalues in B_L at the edge of the spectrum are of order $1/\log L$, rather than $1/|B_L|$ as in the bulk of the spectrum.
- We have in particular Anderson localisation at the edge of the spectrum.
- The control from Poisson process convergence holds only in the box $B^{(1)}(t)$ of radius $\asymp t/\log t \log\log\log t$, and the contribution from the outside of the box $B^{(2)}(t)$ of radius $t\log^2 t$ is easily seen to be negligible. The treatment of the region $B^{(2)}(t) \setminus B^{(1)}(t)$ is delicate and requires a comparison of the two eigenvalue expansions.
- The two terms in the optimized functional $\Psi_{L,t}$ come from the eigenvalue and the probabilistic cost for the random walk in the Feynman-Kac formula to reach the island. The latter term can also be seen as coming from the decay of the eigenfunction term $\varphi_k(0)$.

The top eigenvalues in $B = B_L$ remain the top eigenvalues after discarding potential values significantly less than the eigenvalues. Put $\varepsilon_R = 2d\left(1+\frac{A}{2d}\right)^{1-2R}$.

Proposition 1

Fix $A > 0$ and $R \in \mathbb{N}$ and put $U = \bigcup_{z\in B:\ \xi(z)\geq\lambda_1(\xi,B)} B_R(z)$. Then

$$\lambda_k(\xi,B) \geq \lambda_1(\xi,B) - A/2 \qquad \Longrightarrow \qquad |\lambda_k(\xi,B) - \lambda_k(\xi,U)| \leq \varepsilon_R.$$

- Any ℓ^2-normalized eigenvector $v = v_{k,\xi}$ with eigenvalue $\lambda = \lambda_k(\xi,B) \geq \lambda_1 - A/2$ decays rapidly away from U.
- Proof uses the martingale $(v(Y_n)\prod_{k=0}^{n-1}\frac{2d}{2d+\lambda-\xi(Y_k)})_{n\in\mathbb{N}}$ (with $(Y_n)_n$ an SRW).

Some elements of the proof of Theorem 1 (I)

The top eigenvalues in $B = B_L$ remain the top eigenvalues after discarding potential values significantly less than the eigenvalues. Put $\varepsilon_R = 2d\left(1+\frac{A}{2d}\right)^{1-2R}$.

Proposition 1

Fix $A > 0$ and $R \in \mathbb{N}$ and put $U = \bigcup_{z\in B:\ \xi(z)\geq\lambda_1(\xi,B)} B_R(z)$. Then

$$\lambda_k(\xi,B) \geq \lambda_1(\xi,B) - A/2 \qquad \Longrightarrow \qquad |\lambda_k(\xi,B) - \lambda_k(\xi,U)| \leq \varepsilon_R.$$

- Any ℓ^2-normalized eigenvector $v = v_{k,\xi}$ with eigenvalue $\lambda = \lambda_k(\xi,B) \geq \lambda_1 - A/2$ decays rapidly away from U.
- Proof uses the martingale $(v(Y_n)\prod_{k=0}^{n-1}\frac{2d}{2d+\lambda-\xi(Y_k)})_{n\in\mathbb{N}}$ (with $(Y_n)_n$ an SRW).
- Furthermore, we use that $\partial_{\xi(z)}\lambda_k(\xi,B) = v(z)^2$.
- Introduce $\xi_s = \xi - s\mathbb{1}_{B\setminus U}$ for $s \in [0,\infty]$. Then

$$|\partial_s\lambda_k(\xi_s,B)| = \sum_{z\in B\setminus U} v_{k,\xi_s}(z)^2,$$

which is very small. Integrating over $s \in [0,\infty]$ gives the estimate.

Some elements of the proof of Theorem 1 (II)

The top eigenvalues are the principal eigenvalues in local regions, and the corresponding eigenfunctions are exponentially localised.

A bit more precisely, with the help of the variational characterisation of the asymptotics of the PAM [GÄRTNER/K./MOLCHANOV 07], one proves the following.

- U consists of connected components of bounded size, which are far away from each other.

The top eigenvalues are the principal eigenvalues in local regions, and the corresponding eigenfunctions are exponentially localised.

A bit more precisely, with the help of the variational characterisation of the asymptotics of the PAM [Gärtner/K./Molchanov 07], one proves the following.

- U consists of connected components of bounded size, which are far away from each other.
- For any component C, if $\lambda_1(\xi,C)$ is close to $a_L \approx \rho \log\log L$, then $\lambda_1(\xi,C)$ is bounded away from $\lambda_2(\xi,C)$.

The top eigenvalues are the principal eigenvalues in local regions, and the corresponding eigenfunctions are exponentially localised.

A bit more precisely, with the help of the variational characterisation of the asymptotics of the PAM [GÄRTNER/K./MOLCHANOV 07], one proves the following.

- U consists of connected components of bounded size, which are far away from each other.
- For any component C, if $\lambda_1(\xi,C)$ is close to $a_L \approx \rho \log\log L$, then $\lambda_1(\xi,C)$ is bounded away from $\lambda_2(\xi,C)$.
- If λ is an eigenvalue of $\Delta+\xi$ larger than $\lambda_1(\xi,B_L)-A/2$ and v a corresponding ℓ^2-normalised eigenfunction such that the distance of λ to the nearest eigenvalue (spectral gap) is larger than $3\varepsilon_R$, then v decays exponentially away from one of the components of U.

The top eigenvalues are the principal eigenvalues in local regions, and the corresponding eigenfunctions are exponentially localised.

A bit more precisely, with the help of the variational characterisation of the asymptotics of the PAM [GÄRTNER/K./MOLCHANOV 07], one proves the following.

- U consists of connected components of bounded size, which are far away from each other.
- For any component C, if $\lambda_1(\xi,C)$ is close to $a_L \approx \rho \log\log L$, then $\lambda_1(\xi,C)$ is bounded away from $\lambda_2(\xi,C)$.
- If λ is an eigenvalue of $\Delta+\xi$ larger than $\lambda_1(\xi,B_L)-A/2$ and v a corresponding ℓ^2-normalised eigenfunction such that the distance of λ to the nearest eigenvalue (spectral gap) is larger than $3\varepsilon_R$, then v decays exponentially away from one of the components of U.
- The proof uses that the path $[0,\infty] \ni s \mapsto \lambda_k(\xi_s, B_L)$ (with $\xi_s = \xi - s\mathbb{1}_{B_L\setminus U}$) does not cross other eigenvalues and therefore admits a continuous choice of corresponding eigenfunctions. The one for $s=\infty$ puts all its mass in one component, and the one for $s=0$ is uniformly close.

Some elements of the proof of Theorem 1 (III)

The scale a_L satisfies $\mathrm{Prob}(\lambda_1(\xi, B_R) > a_L) = 1/|B_L|$, hence we may expect finitely many sites in B_L where the local eigenvalue is $\approx a_L$. Then the random variable $\lambda_1(\xi, B_R)$ lies in the max-domain of a Gumbel random variable:

Proposition 2

As $L \to \infty$, for any $s \in \mathbb{R}$,

$$\mathrm{Prob}(\lambda_1(\xi, B_R) > a_L + s/\log L) = \mathrm{e}^{-s} \frac{1}{|B_L|}(1 + o(1)).$$

- The event $\{\lambda_1(\xi, B_R) > a\}$ is more or less the same as the event that some shift of the potential $\xi(\cdot)$ is larger than $a + \chi + \psi(\cdot)$ for some well-chosen function ψ.

Some elements of the proof of Theorem 1 (III)

The scale a_L satisfies $\mathrm{Prob}(\lambda_1(\xi, B_R) > a_L) = 1/|B_L|$, hence we may expect finitely many sites in B_L where the local eigenvalue is $\approx a_L$. Then the random variable $\lambda_1(\xi, B_R)$ lies in the max-domain of a Gumbel random variable:

Proposition 2

As $L \to \infty$, for any $s \in \mathbb{R}$,

$$\mathrm{Prob}(\lambda_1(\xi, B_R) > a_L + s/\log L) = \mathrm{e}^{-s} \frac{1}{|B_L|}(1 + o(1)).$$

- The event $\{\lambda_1(\xi, B_R) > a\}$ is more or less the same as the event that some shift of the potential $\xi(\cdot)$ is larger than $a + \chi + \psi(\cdot)$ for some well-chosen function ψ.
- Shifting ξ by an amount of $s/\log L$ yields an additional factor of e^{-s}, using properties of ψ and of the distribution of ξ and some information from the variational characterisation.

Some elements of the proof of Theorem 2 (I)

Consider boxes $B^{(1)}(t) = B_{L_t^{(1)}}$ and $B^{(2)}(t) = B_{L_t^{(2)}}$ with

$$L_t^{(1)} = \text{const.} \times \frac{t}{\log t \log\log\log t} \qquad \text{and} \qquad L_t^{(2)} = t \log^2 t.$$

- Inside $B^{(1)}(t)$, we have the Poisson process convergence.
- Outside $B^{(2)}(t)$, the contribution is negligible.

Why is the contribution from $B^{(2)}(t) \setminus B^{(1)}(t)$ negligible?

Our Strategy:

- Consider the eigenvalue expansion in $B^{(2)}(t)$. A version of Minami's estimate gives that each spectral gap close to the top is $\geq \varepsilon_R$, with high probability.
- This enables us to prove exponential localisation of the top eigenfunctions in $B^{(2)}(t)$. This makes the top eigenvalues in $B^{(2)}(t)$ essentially independent.

Some elements of the proof of Theorem 2 (II)

Our Strategy (continued):

- The top eigenvalues of $B^{(1)}(t)$ are also top eigenvalues in $B^{(2)}(t)$.
  But the $B^{(2)}(t)$-eigenfunctions are located much further away (if 'const' is large). Hence their contributions in the eigenvalue expansion are negligible w.r.t. the optimizer of $\Psi_{L_t^{(1)},t}$ in $B^{(1)}(t)$.
- For N large enough, the eigenvalues $\lambda_k(\xi, B^{(2)}(t))$ for $k > N$ are negligible w.r.t. the optimizer of $\Psi_{L_t^{(1)},t}$ in $B^{(1)}(t)$ and hence their contribution to the eigenvalue expansion.
- The remaining N eigenvalues can be ordered with gaps $\asymp 1/\log L_t^{(1)} \approx 1/\log t$ between them, and the optimizer is among them.

Future directions

- Presumably, the mass concentration property of the PAM also holds in almost sure sense, but the assertion must be adapted.
- Control on the process of localisation centres, $(Z_{L_t,tr_{L_t}})_{t\in[0,\infty)}$, opens up the possibility to study the time-evolution of the PAM, e.g. in terms of ageing properties.
- Replacing double-exponential distribution by bounded distributions will lead to the same max-domains of attractions for the top eigenvalues, but other rescalings of the gaps.