Weierstrass Institute for
Applied Analysis and Stochastics

Leibniz Gemeinschaft

The universality classes in the parabolic Anderson model

Based on joint works with Marek Biskup (Ceske Budejowice and Los Angeles), Jürgen Gärtner (Berlin), Remco van der Hofstad (Eindhoven), Stanislav Molchanov (Charlotte), Peter Mörters (Bath) and Nadia Sidorova (London)

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The Parabolic Anderson Model

We consider the Cauchy problem for the heat equation with random coefficients and localised initial datum:

$$\frac{\partial}{\partial t}u(t,z) = \Delta^{\mathrm{d}}u(t,z) + \xi(z)u(t,z), \qquad \text{for } (t,z) \in (0,\infty)\times\mathbb{Z}^d, \tag{1}$$

$$u(0,z) = \delta_0(z), \qquad \text{for } z \in \mathbb{Z}^d. \tag{2}$$

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- $\xi = (\xi(z)\colon z\in\mathbb{Z}^d)$ i.i.d. random potential, $[-\infty,\infty)$-valued.
- $\Delta^{\mathrm{d}} f(z) = \sum_{y\sim z}\big[f(y)-f(z)\big]$ discrete Laplacian
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The solution $u(t,\cdot)$ is a random time-dependent shift-invariant field.

Its a.s. existence is guaranteed under a mild moment condition on the potential.

It has all moments finite if all positive exponential moments of $\xi(0)$ are finite.

Motivations and Feynman-Kac formula

Interpretations / Motivations:

- Random mass transport through a random field of sinks and sources.
- Expected particle number in a branching random walk model in a field of random branching and killing rates.
- Anderson Hamiltonian $\Delta^d + \xi$ describes conductance properties of alloys of metals, or optical properties of glasses with impurities. Many open questions about delocalised versus extended states.

Background literature and surveys: [MOLCHANOV 1994], [CARMONA/MOLCHANOV 1994], [SZNITMAN 1998], [GÄRTNER/K. 2005].

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Main tool for analysis: Feynman-Kac formula

$$u(t,z) = \mathbb{E}_0\Big[\exp\Big\{\int_0^t \xi(X(s))\,\mathrm{d}s\Big\}\mathbb{1}\{X(t)=z\}\Big], \qquad z \in \mathbb{Z}^d, t > 0,$$

where $(X(s))_{s\in[0,\infty)}$ is a simple random walk on $\mathbb{Z}^d$ with generator $\Delta^{\rm d}$, starting from z under $\mathbb{P}_z$.

Questions and Heuristics I

MAIN GOAL: Describe the large-t behavior of the solution $u(t,\cdot)$.

In particular: Where does the main bulk of the total mass stem from?

Total mass of the solution: $$U(t) = \sum_{z\in\mathbb{Z}^d} u(t,z), \qquad \text{for } t > 0.$$

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- This in turn is determined by the extreme values of the potential ξ.
- Hence, only the upper tails of $\xi(0)$ matter.

Questions and Heuristics II

If all the moments $\langle U(t)^p \rangle$ of $U(t)$ are finite for any $p, t > 0$, then intermittency can be characterised by the requirement

$$0 < p < q \qquad \Longrightarrow \qquad \limsup_{t \to \infty} \frac{\langle U(t)^p \rangle^{1/p}}{\langle U(t)^q \rangle^{1/q}} = 0.$$

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How many random potentials are interesting?

Under some mild regularity assumption, the case of finite positive exponential moments has been classified in four universality classes ([VAN DER HOFSTAD/K./MÖRTERS 2006]): the double-exponential distribution, a boundary case, bounded potentials and almost bounded ones.

How large are the islands? How large are the potential and the solution there? What do their shapes look like there?

- Strong answer in two main special cases ([Gärtner/K./Molchanov 2007], [Sznitman 1998]): the complement of certain islands is negligible.
  In the islands, the potential and the solution approach the minimizer φ of a characteristic formula and the eigenfunction of $\Delta+\varphi$, respectively.
- Weakly answered in many cases ([Gärtner/Molchanov 1998], [Biskup/K. 2001], [van der Hofstad/K./Mörters 2006]):
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How many islands have to be taken into account?

- Rough bounds: $t^{o(1)}$ ([SZNITMAN 1998], [GÄRTNER/K./MOLCHANOV 2007]).
- Conjecture: $O(1)$.
  Open for finite positive exponential moments (ongoing work [BISKUP/K.]).
  Proved for heavy-tailed potentials ([K./LACOIN/MÖRTERS/SIDOROVA 2009]).

Heuristics using Fourier expansion

Let $\lambda_k(\xi)$ be the eigenvalues of $\Delta^{\rm d}+\xi$ with zero boundary condition in the cube $Q_t=[-t,t]^d\cap\mathbb{Z}^d$ and corresponding eigenfunctions φ_k, forming an ONS.

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Then, with some suitable $H(t) \gg t$,

$$\langle U(t)\rangle \approx \Big\langle \sum_k \mathrm{e}^{t\lambda_k(\xi)} \varphi_k(0) \langle \varphi_k, \mathbb{1}\rangle \Big\rangle$$

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where a large-deviation rate function is given by

$$\mathscr{L}(\varphi) = -\lim_{t\to\infty} \frac{1}{t} \log \mathrm{Prob}\big(\xi(\cdot) - H(t)/t \approx \varphi(\cdot)\big).$$

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Hence, there is a large-deviation principle at work:
On the event $\{\xi(\cdot) \approx H(t)/t + \varphi(\cdot)\}$,
the exponential contribution to the Feynman-Kac formula is $H(t) + t\lambda_1(\varphi)$,
and the probabilistic price is $\exp\{-t\mathscr{L}(\varphi)\}$.

FInally, optimise over all shapes φ.

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Introducing the walker's local times $\ell_t(z) = \int_0^t \mathrm{d}s \, \delta_{X_s}(z)$ and the cumulant generating function $H(t) = \log\langle \mathrm{e}^{t\xi(0)} \rangle$,

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to get that

$$\langle U(t)\rangle \approx \mathrm{e}^{H(t)} \exp\Big\{t \sup_{\|\psi\|_1=1} \big(\Phi(\psi) - I(\psi)\big)\Big\}.$$

The double-exponential distribution I

With $\rho \in (0,\infty)$, consider the upper tails

$$\mathrm{Prob}(\xi(0) > r) \approx \exp\Big\{-\mathrm{e}^{r/\rho}\Big\}, \qquad r \to \infty.$$

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Moment asymptotics.

[Gärtner/Molchanov 1998]: For any $p \in \mathbb{N}$, as $t \to \infty$,

$$\langle U(t)^p \rangle = \mathrm{e}^{H(tp)}\mathrm{e}^{-tp(\chi + o(1))}, \quad \text{where} \quad \chi = \inf_{\varphi\colon \mathbb{Z}^d \to \mathbb{R}} \Big[\mathscr{L}(\varphi) - \lambda(\varphi)\Big],$$

and $\mathscr{L}(\varphi) = \frac{\rho}{\mathrm{e}} \sum_{z\in\mathbb{Z}^d} \mathrm{e}^{\varphi(z)/\rho}$, and $\lambda(\varphi)$ is the top of the spectrum of $\Delta^{\mathrm{d}} + \varphi$ in $\mathbb{Z}^d$.

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- Minimiser(s) exist. They are unique for ρ sufficiently large and rather inexplicit.
- $\mathscr{L}(\varphi)$ is a large-deviation rate function for the shifted potential $\xi_t = \xi - H(t)/t$. On the event $\{\xi_t \approx \varphi\}$, the contribution to the Feynman-Kac formula is quantified by $\lambda(\varphi)$. The optimal profile describes the total expected mass.
- The structure of the asymptotic optimal profile is discrete; no spatial scaling is involved.

The double-exponential distribution II

Almost sure asymptotics.

[GÄRTNER/MOLCHANOV 1998]: As $t \to \infty$,

$$\frac{1}{t}\log U(t) = \frac{H(\log t)}{\log t} - \widetilde{\chi} + o(1), \quad \text{with} \quad -\widetilde{\chi} = \sup\Big\{\lambda(\varphi)\Big|\varphi\colon \mathbb{Z}^d \to \mathbb{R}, \mathscr{L}(\varphi) \le d\Big\}.$$

The double-exponential distribution II

Almost sure asymptotics.

[GÄRTNER/MOLCHANOV 1998]: As $t \to \infty$,

$$\frac{1}{t}\log U(t) = \frac{H(\log t)}{\log t} - \widetilde{\chi} + o(1), \quad \text{with} \quad -\widetilde{\chi} = \sup\Big\{\lambda(\varphi)\Big|\varphi\colon \mathbb{Z}^d \to \mathbb{R}, \mathscr{L}(\varphi) \le d\Big\}.$$

- The maximiser(s) of χ and $\widetilde{\chi}$ are identical.
- A Borel-Cantelli argument shows that, with probability one, for all sufficiently large t, every potential shape φ satisfying $\mathscr{L}(\varphi) \le d$ appears on some island in the cube $Q_t = [-t,t]^d \cap \mathbb{Z}^d$ in the potential $\xi_{\log t} = \xi - H(\log t)/\log t$. The contribution to the Feynman-Kac formula coming from those paths that go quickly there and spend most of the time there is quantified by $\lambda(\varphi)$. The optimal such profile φ describes the total contribution.
- Every such island with φ an optimal profile is potentially one of the intermittent islands we mentioned above. The corresponding eigenfunction is localised.

Geometric characterisation of intermittency.

[GÄRTNER/K./MOLCHANOV 2007]: Almost surely, for any sufficiently large t, there is a random set $\Gamma_t \subset Q_t$ such that $|\Gamma_t| = t^{o(1)}$ and

- $\displaystyle\min_{z,\widetilde{z}\in\Gamma_t:\, z\neq\widetilde{z}} |z-\widetilde{z}| = t^{1-o(1)}$,
- $\displaystyle\lim_{R\to\infty}\liminf_{t\to\infty}\frac{1}{U(t)}\sum_{z\in Q_R(\Gamma_t)} u(t,z) = 1$,
- For any $z\in\Gamma_t$ and any $R>0$, in the cube $Q_R(z)$, the shifted potential $\xi_{\log t}$ resembles the maximiser φ of $-\widetilde{\chi}$, and the solution $u(t,\cdot)$ resembles C_t times the principal eigenfunction of $\Delta^{\mathrm{d}}+\varphi$ for some suitable norming $C_t>0$.

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This is one of the universality classes. The second one is the boundary case $\rho=\infty$. The third one is introduced on the next slide, the fourth one is a kind of continuous version of the first one.

Potentials bounded from above

Assume $\operatorname{esssup}(\xi(0)) = 0$. With $\gamma \in [0,1)$, consider the upper-tail behavior

$$\mathrm{Prob}(\xi(0) > -x) \approx \exp\Big\{ -\mathrm{const.}\, x^{-\gamma/(1-\gamma)} \Big\}, \qquad x \downarrow 0.$$

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The relevant islands have diameter of the order

$$\alpha(t) = t^{\nu}, \qquad \text{where} \qquad \nu = \frac{1-\gamma}{d+2-d\gamma} \in \big(0, \tfrac{1}{d+2}\big].$$

Moment asymptotics.

[BISKUP/K. 2001]: For any $p \in (0,\infty)$, as $t \to \infty$,

$$\frac{1}{tp} \log\langle U(t)^p \rangle = -\frac{\chi + o(1)}{\alpha(pt)^2}, \qquad \text{where} \qquad \chi = \inf_{\varphi \in \mathscr{C}(\mathbb{R}^d \to [-\infty,0])} \Big[\mathscr{L}(\varphi) - \lambda(\varphi) \Big],$$

and $\mathscr{L}(\varphi) = \mathrm{const.} \int_{\mathbb{R}^d} |\varphi(x)|^{-\gamma/(1-\gamma)}\, \mathrm{d}x$,
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- $\mathscr{L}$ is a LD rate function for the rescaled potential $\xi_t(\cdot) = \alpha(t)^2 \xi(\cdot\, \alpha(t))$.

Lifshitz tails

- $\nu_R = \frac{1}{\#Q_R} \sum_k \delta_{-\lambda_{R,k}}$ spectral measure of $-\Delta^d - \xi$ in the box Q_R.
- $N(E) = \lim_{R\to\infty} \nu_R([0,E])$ integrated density of states (IDS).

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&\overset{t\to\infty}{\approx} \exp\Big\{-\frac{t}{\alpha(t)^2}\chi\Big\},
\end{aligned}$$

according to the moment asymptotics. Now invert the Laplace transform.

Heavy-tailed potentials I

Another potential class, the Pareto-distribution:

$$\mathrm{Prob}(\xi(0) > r) = r^{-\alpha}, \qquad r \in [1,\infty), \qquad (\text{Parameter } \alpha > d).$$

Then the parabolic Anderson model possesses a.s. a solution $u(t,\cdot)$, but $U(t)$ has no moments.

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Weak Asymptotics for Pareto-Distributed Potentials

[VAN DER HOFSTAD/MÖRTERS/SIDOROVA 2008]. For $x \in \mathbb{R}$,

$$\lim_{t\to\infty} \mathrm{Prob}\Big(\Big(\frac{t}{\log t}\Big)^{-\frac{d}{\alpha-d}} \frac{1}{t} \log U(t) \leq x\Big) = \exp\big(-\mu x^{d-\alpha}\big),$$

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where $\mu \in (0,\infty)$ is some suitable, explicit constant.

- $\frac{1}{t}\log U(t)$ has the same weak asymptotics as the maximum of t^d independent Pareto $(\alpha - d)$-distributed random variables.
- Apparently, the potential's random fluctuations dominate the smoothing effect of Δ^{d}.

Heavy-tailed potentials II

Well-known: Large values of a sum of i.i.d. heavy tailed random variables are most easily realised by having just one of the values extremely large (and the others of moderate size).

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This principle can be translated to the solution of the parabolic Anderson model:

Complete Localisation for Pareto-Distributed Potentials

[K./LACOIN/MÖRTERS/SIDOROVA 2009]. There is a process $(Z_t)_{t>0}$ in $\mathbb{Z}^d$ such that

$$\lim_{t\to\infty} \frac{u(t,Z_t)}{U(t)} = 1 \qquad \text{in probability.}$$

Furthermore, $Z_t(\log t/t)^{\alpha/(\alpha-d)}$ converges in distribution to some non-degenerate random variable.

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Furthermore, $Z_t(\log t/t)^{\alpha/(\alpha-d)}$ converges in distribution to some non-degenerate random variable.

- Hence, there is precisely one intermittent island, a singleton.
- The localisation statement is not true in almost sure sense, since the process $(Z_t)_{t>0}$ jumps.

Open problems and further questions

- Is just one island enough for general potentials? What is the contribution from the others? (ongoing work with MAREK BISKUP)

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- Ageing properties: asymptotics for time-correlation

$$\frac{\langle U(t)U(t+s(t))\rangle}{\sqrt{\langle U(t)^2\rangle\langle U(t+s(t))^2\rangle}}$$

and interpretation (ongoing work [GÄRTNER/SCHNITZLER 2010]).