Weierstrass Institute for
Applied Analysis and Stochastics

Leibniz Association

Ordered random walks

Wolfgang König
WIAS Berlin and TU Berlin

joint work with Peter Eichelsbacher (Bochum), [EJP08]

The Goal

Consider k i.i.d. random walks $X_i = (X_i(n))_{n\in\mathbb{N}_0}$ ($i = 1, \dots, k$) on $\mathbb{R}$.

Questions:

- What is the conditional version given that the walkers stay in strict order for ever?
- What is the asymptotic probability that they stay in strict order until a late time?
- What is the large-time behaviour of the k walkers given that they stay in strict order until a late time or for ever?

The Goal

Consider k i.i.d. random walks $X_i = (X_i(n))_{n\in\mathbb{N}_0}$ ($i = 1, \dots, k$) on $\mathbb{R}$.

Questions:

- What is the conditional version given that the walkers stay in strict order for ever?
- What is the asymptotic probability that they stay in strict order until a late time?
- What is the large-time behaviour of the k walkers given that they stay in strict order until a late time or for ever?

Denote $X = (X_1, \dots, X_k)$, starting from $x \in \mathbb{R}^k$ under $\mathbb{P}_x$, and

$$W = \{x \in \mathbb{R}^k : x_1 < x_2 < \cdots < x_k\}$$ Weyl chamber

$$\tau = \inf\{n \in \mathbb{N} : X(n) \notin W\}$$ exit time from W,

The Goal

Consider k i.i.d. random walks $X_i = (X_i(n))_{n\in\mathbb{N}_0}$ ($i = 1, \dots, k$) on $\mathbb{R}$.

Questions:

- What is the conditional version given that the walkers stay in strict order for ever?
- What is the asymptotic probability that they stay in strict order until a late time?
- What is the large-time behaviour of the k walkers given that they stay in strict order until a late time or for ever?

Denote $X = (X_1, \dots, X_k)$, starting from $x \in \mathbb{R}^k$ under $\mathbb{P}_x$, and

$$W = \{x \in \mathbb{R}^k : x_1 < x_2 < \dots < x_k\} \qquad \text{Weyl chamber}$$

$$\tau = \inf\{n \in \mathbb{N} : X(n) \notin W\} \qquad \text{exit time from } W,$$

then our questions may be reformulated as follows.

- What is the conditional distribution of X given $\{\tau = \infty\}$?
- What are the asymptotics of $\mathbb{P}_x(\tau > n)$ as $n \to \infty$?
- Does the distribution of $X(n)/\sqrt{n}$ converge under $\mathbb{P}_x(\cdot \mid \tau > n)$ or $\mathbb{P}_x(\cdot \mid \tau = \infty)$?

- Appeared in JOHANSSON'S beautiful analysis of the corner-growth model (2002).
- Have remarkable connections to tandem queues (survey article [O'CONNELL '03]).
- Discrete version of Dyson's Brownian motion (see below).

- Appeared in JOHANSSON'S beautiful analysis of the corner-growth model (2002).
- Have remarkable connections to tandem queues (survey article [O'CONNELL '03]).
- Discrete version of Dyson's Brownian motion (see below).

Only rather special cases handled yet: nearest-neighbor random walks on $\mathbb{Z}^k$ that satisfy the

$$\text{continuity property:} \qquad \mathbb{P}_x(X(\tau) \in \partial W) = 1.$$

Here, 'ordered' is equivalent to 'non-colliding'.

Examples: simple random walk [KATORI/TANEMURA '04], binomial walk, multinomial walk, Poisson walk [K./O'CONNELL/ROCH '02], Yule process [DOUMERC '05].

General random walks not considered before 2008.

Motivation B: Dyson's Brownian motion

Also called non-colliding Brownian motions: the continuous version of our question [Dyson 1962].

$H(t) = (H_{i,j}(t))_{i,j=1,\dots,k}$ Hermitian Brownian motion (GUE at time $t = 1$)

$\lambda_1(t) \le \lambda_2(t) \le \dots \le \lambda_k(t)$ eigenvalues of $H(t)$

$\lambda = (\lambda_1(t), \dots, \lambda_k(t))_{t\in[0,\infty)}$ eigenvalue process in $\overline{W}$

Motivation B: Dyson's Brownian motion

Also called non-colliding Brownian motions: the continuous version of our question [Dyson 1962].

$H(t) = (H_{i,j}(t))_{i,j=1,\dots,k}$ Hermitian Brownian motion (GUE at time $t = 1$)

$\lambda_1(t) \le \lambda_2(t) \le \cdots \le \lambda_k(t)$ eigenvalues of $H(t)$

$\lambda = (\lambda_1(t), \dots, \lambda_k(t))_{t\in[0,\infty)}$ eigenvalue process in $\overline{W}$

Theorem. [DYSON 1962]

λ satisfies, for $\beta = 2$, the SDE

$$\mathrm{d}\lambda_i(t) = \mathrm{d}B_i(t) + \frac{\beta}{2}\sum_{j\neq i}\frac{1}{\lambda_i(t)-\lambda_j(t)}\,\mathrm{d}t, \qquad i = 1,\dots,k.$$

Furthermore, λ is a Brownian motion in $\mathbb{R}^k$, conditioned on being non-colliding for ever.

Motivation B: Dyson's Brownian motion

Also called non-colliding Brownian motions: the continuous version of our question [Dyson 1962].

$H(t) = (H_{i,j}(t))_{i,j=1,\dots,k}$ Hermitian Brownian motion (GUE at time $t = 1$)

$\lambda_1(t) \le \lambda_2(t) \le \cdots \le \lambda_k(t)$ eigenvalues of $H(t)$

$\lambda = (\lambda_1(t), \dots, \lambda_k(t))_{t\in[0,\infty)}$ eigenvalue process in $\overline{W}$

Theorem. [DYSON 1962]

λ satisfies, for $\beta = 2$, the SDE

$$\mathrm{d}\lambda_i(t) = \mathrm{d}B_i(t) + \frac{\beta}{2}\sum_{j\neq i}\frac{1}{\lambda_i(t)-\lambda_j(t)}\,\mathrm{d}t, \qquad i = 1,\dots,k.$$

Furthermore, λ is a Brownian motion in $\mathbb{R}^k$, conditioned on being non-colliding for ever.

Hence, if $T = \inf\{t > 0\colon B(t) \notin W\}$ is the exit time of a BM B in $\mathbb{R}^k$ from the Weyl chamber W, then, formally,

$$\mathcal{L}(\lambda) = \mathcal{L}(B \mid T = \infty).$$

(More about that later)

Motivation C: Fluctuation Theory

The special case $k = 2$ is equivalent to conditioning a random walk S on $\mathbb{R}$ to stay positive at all times. Fluctuation theory studies conditioning on being nonnegative. The answer is given in terms of a Doob h-transform. If the walker's steps have finite mean, then

$$V(x) = \frac{x - \mathbb{E}_x[S_\sigma]}{-\mathbb{E}_0[S_\sigma]}, \qquad \text{where } \sigma = \inf\{n \in \mathbb{N} \colon S_n < 0\},$$

turns out to be a positive regular function for the restriction to $[0, \infty)$, i.e., $V > 0$ and

$$\mathbb{E}_x[V(S_1)\mathbb{1}_{\{\sigma > 1\}}] = V(x), \qquad x \in [0, \infty).$$

Motivation C: Fluctuation Theory

The special case $k = 2$ is equivalent to conditioning a random walk S on $\mathbb{R}$ to stay positive at all times. Fluctuation theory studies conditioning on being nonnegative. The answer is given in terms of a Doob h-transform. If the walker's steps have finite mean, then

$$V(x) = \frac{x - \mathbb{E}_x[S_\sigma]}{-\mathbb{E}_0[S_\sigma]}, \qquad \text{where } \sigma = \inf\{n \in \mathbb{N} \colon S_n < 0\},$$

turns out to be a positive regular function for the restriction to $[0, \infty)$, i.e., $V > 0$ and

$$\mathbb{E}_x[V(S_1)\mathbb{1}_{\{\sigma>1\}}] = V(x), \qquad x \in [0, \infty).$$

Hence, the Doob transform

$$\widehat{\mathbb{P}}_x\big((S_0, \dots, S_n) \in A\big) = \mathbb{P}_x\big((S_0, \dots, S_n) \in A, \sigma > n\big)\frac{V(S_n)}{V(S_0)}, \qquad A \subset [0, \infty)^{n+1},$$

defines a consistent family of path measures; it is even a Markov chain.

Motivation C: Fluctuation Theory

The special case $k = 2$ is equivalent to conditioning a random walk S on $\mathbb{R}$ to stay positive at all times. Fluctuation theory studies conditioning on being nonnegative. The answer is given in terms of a Doob h-transform. If the walker's steps have finite mean, then

$$V(x) = \frac{x - \mathbb{E}_x[S_\sigma]}{-\mathbb{E}_0[S_\sigma]}, \qquad \text{where } \sigma = \inf\{n \in \mathbb{N} \colon S_n < 0\},$$

turns out to be a positive regular function for the restriction to $[0, \infty)$, i.e., $V > 0$ and

$$\mathbb{E}_x[V(S_1)\mathbb{1}_{\{\sigma>1\}}] = V(x), \qquad x \in [0, \infty).$$

Hence, the Doob transform

$$\widehat{\mathbb{P}}_x\big((S_0, \dots, S_n) \in A\big) = \mathbb{P}_x\big((S_0, \dots, S_n) \in A, \sigma > n\big)\frac{V(S_n)}{V(S_0)}, \qquad A \subset [0, \infty)^{n+1},$$

defines a consistent family of path measures; it is even a Markov chain.
Moreover, it is equal to the limiting process S, given that $\{\sigma > n\}$ as $n \to \infty$. Furthermore,

$$\lim_{n\to\infty} \sqrt{n}\,\mathbb{P}_x(\sigma > n) = V(x), \qquad x \in [0, \infty).$$

Main tools: duality and Sparre-Andersen identity (see [FELLER '71], e.g.).

More on non-colliding BMs

Proper definition in terms of Doob h-transform with $h = \Delta$, where

$$\Delta(x) = \prod_{1 \le i < j \le k} (x_j - x_i) = \det\Big[\big(x_i^{j-1}\big)_{i,j=1,\dots,k}\Big], \qquad \text{Vandermonde determinant}$$

Main properties: Δ is harmonic for $\frac{1}{2}\sum_{i=1}^k \partial_i^2$, and $\Delta > 0$ in W.

More on non-colliding BMs

Proper definition in terms of Doob h-transform with $h = \Delta$, where

$$\Delta(x) = \prod_{1\le i<j\le k} (x_j - x_i) = \det\Big[\big(x_i^{j-1}\big)_{i,j=1,\dots,k}\Big], \qquad \text{Vandermonde determinant}$$

Main properties: Δ is harmonic for $\frac{1}{2}\sum_{i=1}^k \partial_i^2$, and $\Delta > 0$ in W.

Transition probability density of the h-transform:

$$\widehat{p}_t(x,y)\,\mathrm{d}y = \mathbb{P}_x(B(t)\in \mathrm{d}y; T>t)\frac{\Delta(y)}{\Delta(x)}, \qquad x,y\in W.$$

Is this formula helpful?

More on non-colliding BMs

Proper definition in terms of Doob h-transform with $h = \Delta$, where

$$\Delta(x) = \prod_{1\leq i<j\leq k} (x_j - x_i) = \det\Big[\big(x_i^{j-1}\big)_{i,j=1,\dots,k}\Big], \qquad \text{Vandermonde determinant}$$

Main properties: Δ is harmonic for $\frac{1}{2}\sum_{i=1}^k \partial_i^2$, and $\Delta > 0$ in W.

Transition probability density of the h-transform:

$$\widehat{p}_t(x,y)\,\mathrm{d}y = \mathbb{P}_x(B(t)\in \mathrm{d}y; T>t)\frac{\Delta(y)}{\Delta(x)}, \qquad x,y\in W.$$

Is this formula helpful? Yes!

Lemma. [KARLIN/MCGREGOR 1958]

$$\mathbb{P}_x(B(t)\in \mathrm{d}y; T>t) = \det\Big[\big(p_t(x_i,y_j)\big)_{i,j=1,\dots,k}\Big]\,\mathrm{d}y.$$

Main tools of the proof: reflection principle and a clever enumeration.

Some properties

Corollary.

(i) $\widehat{p}_t(0,y) = Ct^{-\frac{k}{4}(k-1)}(2\pi t)^{-k/2}\mathrm{e}^{-|y|^2/(2t)}\Delta(y)^2$ Hermite ensemble

(ii) $\mathbb{P}_x(T > t) \sim Ct^{-\frac{k}{4}(k-1)}\Delta(x)$ as $t \to \infty$ non-colliding probability

(iii) $\lim_{t\to\infty} \mathbb{P}_x\big(B(t)/\sqrt{t} \in \mathrm{d}y \mid T > t\big) = C\mathrm{e}^{-|y|^2/2}\Delta(y)$

(iv) $\mathbb{P}_0(B(t) \in \mathrm{d}y \mid T > t) = C_t\mathrm{e}^{-|y|^2/(2t)}\Delta(y)$

Corollary.

(i) $\widehat{p}_t(0,y) = Ct^{-\frac{k}{4}(k-1)}(2\pi t)^{-k/2}\mathrm{e}^{-|y|^2/(2t)}\Delta(y)^2$ Hermite ensemble

(ii) $\mathbb{P}_x(T > t) \sim Ct^{-\frac{k}{4}(k-1)}\Delta(x)$ as $t \to \infty$ non-colliding probability

(iii) $\lim_{t\to\infty}\mathbb{P}_x\big(B(t)/\sqrt{t} \in \mathrm{d}y \mid T > t\big) = C\mathrm{e}^{-|y|^2/2}\Delta(y)$

(iv) $\mathbb{P}_0(B(t) \in \mathrm{d}y \mid T > t) = C_t\mathrm{e}^{-|y|^2/(2t)}\Delta(y)$

Sketch of proof of (i) and (ii): The KMcG-formula gives

$$\widehat{p}_t(x,y) = C(2\pi t)^{-k/2}\mathrm{e}^{-|x|^2/(2t)}\mathrm{e}^{-|y|^2/(2t)}\det\Big[\big(\mathrm{e}^{x_i y_j/t}\big)_{i,j}\Big]\frac{\Delta(y)}{\Delta(x)}.$$

Corollary.

(i) $\widehat{p}_t(0,y) = Ct^{-\frac{k}{4}(k-1)}(2\pi t)^{-k/2}\mathrm{e}^{-|y|^2/(2t)}\Delta(y)^2$ Hermite ensemble

(ii) $\mathbb{P}_x(T>t) \sim Ct^{-\frac{k}{4}(k-1)}\Delta(x)$ as $t\to\infty$ non-colliding probability

(iii) $\lim_{t\to\infty}\mathbb{P}_x\big(B(t)/\sqrt{t}\in \mathrm{d}y \mid T>t\big) = C\mathrm{e}^{-|y|^2/2}\Delta(y)$

(iv) $\mathbb{P}_0(B(t)\in \mathrm{d}y \mid T>t) = C_t\mathrm{e}^{-|y|^2/(2t)}\Delta(y)$

Sketch of proof of (i) and (ii): The KMcG-formula gives

$$\widehat{p}_t(x,y) = C(2\pi t)^{-k/2}\mathrm{e}^{-|x|^2/(2t)}\mathrm{e}^{-|y|^2/(2t)}\det\Big[\big(\mathrm{e}^{x_iy_j/t}\big)_{i,j}\Big]\frac{\Delta(y)}{\Delta(x)}.$$

As $x\to 0$ or $t\to\infty$,

$$\det\Big[\big(\mathrm{e}^{x_iy_j/t}\big)_{i,j}\Big] \sim \det\Big[\Big(\sum_{l=1}^k \frac{x_i^{l-1}}{(l-1)!t^{l-1}}y_j^{l-1}\Big)_{i,j}\Big]$$

$$= \det\Big[\Big(\frac{x_i^{l-1}}{(l-1)!t^{l-1}}\Big)_{i,l}\Big]\det\Big[\big(y_j^{l-1}\big)_{l,j}\big)\Big] = Ct^{-\frac{k}{4}(k-1)}\Delta(x)\Delta(y).$$

Brownian motion in a truncated Weyl chamber

The non-exit probability from the Weyl chamber W is polynomial, but the one from the truncated chamber $W \cap I^k$ with $I = (-\frac{\pi}{2}, \frac{\pi}{2})$ is exponential:

$$\mathbb{P}_x\big(B_{[0,t]} \subset W \cap I^k\big) \sim \mathrm{e}^{-t\lambda^{(W\cap I^k)}} f^{(W\cap I^k)}(x)\langle f^{(W\cap I^k)}, \mathbb{1}\rangle, \qquad t \to \infty, \text{ for } x \in W,$$

where $\lambda^{(U)}$ denotes the principal eigenvalue and $f^{(U)}$ the corresponding positive L^2-normalised eigenfunction of $-\frac{1}{2}\sum_{i=1}^k \partial_i^2$ in $U \subset \mathbb{R}^k$ with zero boundary condition.

Brownian motion in a truncated Weyl chamber

The non-exit probability from the Weyl chamber W is polynomial, but the one from the truncated chamber $W \cap I^k$ with $I = (-\frac{\pi}{2}, \frac{\pi}{2})$ is exponential:

$$\mathbb{P}_x\big(B_{[0,t]} \subset W \cap I^k\big) \sim \mathrm{e}^{-t\lambda^{(W\cap I^k)}} f^{(W\cap I^k)}(x)\langle f^{(W\cap I^k)}, \mathbb{1}\rangle, \qquad t \to \infty, \text{ for } x \in W,$$

where $\lambda^{(U)}$ denotes the principal eigenvalue and $f^{(U)}$ the corresponding positive L^2-normalised eigenfunction of $-\frac{1}{2}\sum_{i=1}^k \partial_i^2$ in $U \subset \mathbb{R}^k$ with zero boundary condition.

How can we interpolate between these regimes?

Theorem [K./Schmid 2011]

For any $x \in W$ and any $r \in (0,\infty)$, as $t \to \infty$,

$$\mathbb{P}_x\big(B_{[0,t]} \subset W \cap r(t) I^k\big) \sim \Delta(x) \begin{cases} K_0 r(t)^{-\frac{k}{2}(k-1)} \mathrm{e}^{-\frac{t}{r(t)^2}\lambda^{(W\cap I^k)}}, & \text{if } 1 \ll r(t) \ll \sqrt{t}, \\ K_r t^{-\frac{k}{4}(k-1)}, & \text{if } r(t) \sim r\sqrt{t}, \\ K_\infty t^{-\frac{k}{4}(k-1)}, & \text{if } \sqrt{t} \ll r(t). \end{cases}$$

Here $K_r \in (0,\infty)$ are constants for $r \in [0,\infty]$ such that

$$\lim_{r\to\infty} K_r = K_\infty \qquad \text{and} \qquad K_r \sim K_0 r^{-\frac{k}{2}(k-1)} \mathrm{e}^{-r^{-2}\lambda^{(W\cap I^k)}} \quad \text{as } r \downarrow 0.$$

Back to the question

Hence, we are looking for a positive regular function $V\colon W \to (0,\infty)$ for the restriction of the kernel of the walk X on $\mathbb{R}^k$. (Recall: W is the Weyl chamber, and τ its exit time.)

Under the above-mentioned continuity property, the Vandermonde determinant Δ is a positive regular function for the restriction to W, and the solution is similar to the Brownian case.

What is a suitable positive regular function in the general case?

Hence, we are looking for a positive regular function $V\colon W \to (0,\infty)$ for the restriction of the kernel of the walk X on $\mathbb{R}^k$. (Recall: W is the Weyl chamber, and τ its exit time.)

Under the above-mentioned continuity property, the Vandermonde determinant Δ is a positive regular function for the restriction to W, and the solution is similar to the Brownian case.

What is a suitable positive regular function in the general case?

Here it is:

$$V(x) = \Delta(x) - \mathbb{E}_x\big[\Delta(X(\tau))\big], \qquad x \in W.$$

Back to the question

Hence, we are looking for a positive regular function $V\colon W \to (0,\infty)$ for the restriction of the kernel of the walk X on $\mathbb{R}^k$. (Recall: W is the Weyl chamber, and τ its exit time.)

Under the above-mentioned continuity property, the Vandermonde determinant Δ is a positive regular function for the restriction to W, and the solution is similar to the Brownian case.

What is a suitable positive regular function in the general case?

Here it is:

$$V(x) = \Delta(x) - \mathbb{E}_x\big[\Delta(X(\tau))\big], \qquad x \in W.$$

Quite easily seen to be regular, i.e.,

$$\mathbb{E}_x\big[\mathbb{1}_{\{\tau>1\}} V(X(1))\big] = V(x), \qquad x \in W.$$

Not easy to see: $V \geq 0$ on W.
more difficult to see: $V > 0$ on W.
very difficult to see: V is well-defined, i.e., $\Delta(X(\tau))$ is integrable!

Why so delicate:

Consider

$$\mathbb{E}_x\big[|\Delta(X(\tau))|\big] = \sum_{n\in\mathbb{N}} \mathbb{E}_x\Big[\prod_{i<j} |X_j(n) - X_i(n)| \mathbb{1}_{\{\tau=n\}}\Big].$$

All the factors $|X_j(n) - X_i(n)|$ are $O(\sqrt{n})$ with the exception of one of them.

Why so delicate:

Consider

$$\mathbb{E}_x\big[|\Delta(X(\tau))|\big] = \sum_{n\in\mathbb{N}} \mathbb{E}_x\Big[\prod_{i<j} |X_j(n) - X_i(n)| 1\!\!1_{\{\tau=n\}}\Big].$$

All the factors $|X_j(n) - X_i(n)|$ are $O(\sqrt{n})$ with the exception of one of them.

We expect (and later prove) that $\mathbb{P}_x(\tau > n) \approx n^{-\frac{1}{4}k(k-1)}$.

Hence, we should have (and do not prove) that $\mathbb{P}_x(\tau = n) \approx n^{-\frac{1}{4}k(k-1)-1}$.

Hence, we should have

$$\mathbb{E}_x\Big[\prod_{i<j} |X_j(n) - X_i(n)| 1\!\!1_{\{\tau=n\}}\Big] \approx n^{-\frac{3}{2}},$$

which is enough.

The main result

Assume that the steps have mean zero and variance one, and that the local central limit theorem holds.

Theorem. [EICHELSBACHER/K. 08]

If sufficiently high step moments are finite, the following hold.

(i) $\Delta(X(\tau))$ is integrable under $\mathbb{P}_x$ for any $x \in W$.

(ii) V is a positive regular function for the restriction of the transition kernel to W.

(iii) The Doob h-transform with $h = V$ is equal to X, given $\{\tau > n\}$ as $n \to \infty$.

(iv)

$$\lim_{n\to\infty} \mathbb{P}_x\big(n^{-\frac{1}{2}} X(n) \in A \mid \tau > n\big) = \frac{1}{Z_1} \int_A \mathrm{e}^{-\frac{1}{2}|y|^2} \Delta(y)\,\mathrm{d}y \qquad \text{weakly.}$$

and, for some $K \in (0, \infty)$,

$$\lim_{n\to\infty} n^{\frac{k}{4}(k-1)} \mathbb{P}_x(\tau > n) = K V(x), \qquad x \in W.$$

(v) Uniformly on compacts, $\lim_{n\to\infty} n^{-\frac{k}{4}(k-1)} V(\sqrt{n}\,x) = \Delta(x)$.

(vi) For any $x \in W$, the distribution of the process $(n^{-\frac{1}{2}} X(\lfloor nt \rfloor))_{t\in[0,\infty)}$ under $\widehat{\mathbb{P}}_{\sqrt{n}x}$ converges towards Dyson's Brownian motions started at x.

Comments

- Main tools of our proof: generalisation of the Karlin-McGregor formula, local central limit theorem and Hölder's inequality.
- Therefore loss of optimal moment condition.

Comments

- Main tools of our proof: generalisation of the Karlin-McGregor formula, local central limit theorem and Hölder's inequality.
- Therefore loss of optimal moment condition.
- DENISOV/WACHTEL (2009) prove the theorem under almost optimal moment condition: finite $(k-1)$-st moments of the steps. The main tools are martingale arguments, a clever stopping time $T = \inf\{n \in \mathbb{N} \colon \Delta(X(n)) < 0\}$ and an embedding of the walks on a Brownian space.

Comments

- Main tools of our proof: generalisation of the Karlin-McGregor formula, local central limit theorem and Hölder's inequality.
- Therefore loss of optimal moment condition.
- DENISOV/WACHTEL (2009) prove the theorem under almost optimal moment condition: finite $(k-1)$-st moments of the steps. The main tools are martingale arguments, a clever stopping time $T = \inf\{n \in \mathbb{N} \colon \Delta(X(n)) < 0\}$ and an embedding of the walks on a Brownian space.
- K. AND SCHMID (2009) extend Denisov/Wachtel's proof to the Weyl chambers of Type C and D,

$$\begin{aligned} W_C &= \{(x_1,\dots,x_k) \in \mathbb{R}^k \colon 0 < x_1 < \dots < x_k\}, \\ W_D &= \{(x_1,\dots,x_k) \in \mathbb{R}^k \colon |x_1| < x_2 < \dots < x_k\}. \end{aligned}$$

  The relevant positive regular functions are

$$V_C(x) = V_D(x) \prod_{i=1}^{k} x_i \quad \text{and} \quad V_D(x) = \prod_{1 \le i < j \le k} (x_j^2 - x_i^2).$$

- DENISOV/WACHTEL (2011) extend the theorem to less integrable steps.

Open questions

- Relation to the eigenvalue processes of some matrix-valued random walks?
- Relation to general corner-growth process?
- How to construct ordered random walks under infinite variance of the steps?
- Is there a useful duality principle?
- Behaviour of the k ordered random walks if $k \longrightarrow \infty$? Convergence of the empirical measure of some marginal distribution (version of WIGNER's semicircle law)? (See [BAIK/SUIDAN 06] for some partial answer.)