ORDINARY DIFFERENTIAL EQUATIONS FOR PHYSICISTS

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Lecture Notes

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Contents

Chapter 1

Introduction

We explain what an ordinary differential equation is, how they appear, how they may be interpreted, and further questions.

1.1 Motivation and basic notions

In many instances, a function $y = y(x)$ under interest cannot be explicitly given, but only characterized by some equation that it satisfies in (part of) its domain. If this equation involves the derivative y' of y or even higher derivatives y'', y''' etc., then we say that y satisfies a differential equation.

Example 1.1.1. The acceleration of a car depends on the velocity $v(t)$, the mass m, the power of propulsion $F_{\mathrm{p}}(v) = C_1 v$, the rolling resistance F_{r} and the air resistance $F_{\mathrm{a}}(v) = C_2 v^2$, where C_1, C_2 and m are positive constants. More precisely, we have

$$mv'(t) = F_{\mathrm{p}}(v(t)) - F_{\mathrm{r}} - F_{\mathrm{a}}(v(t)),$$

for some range of times t. The reason is that the power of propulsion is proportional to the velocity, but the air resistance is proportional to its square, and the acceleration is the time derivative of the velocity. $\diamond$

Example 1.1.2. Under rather crude assumptions, the size of a population at time t, $p(t)$, increases linearly in $p(t)$ as t varies, and hence it satisfies the equation

$$p'(t) = Cp(t),$$

for some range of times t, where $C > 0$ is a constant. In this model, the increase of the population is not hampered by anything, and it will increase unboundedly with great velocity. A more realistic model also incorporates the boundedness of the space and assumes that the population satisfies a equation of the type

$$p'(t) = p(t)(a - bp(t)),$$

where a, b are positive parameters. $\diamond$

Example 1.1.3. A chain is fixed at two points (x_1, y_1) and (x_2, y_2) (with $x_1 < x_2$) in the x-y-plane and hangs freely. The chain is described by a curve $y = y(x)$, $x \in [x_1, x_2]$. The derivative and the curvature of this chain line satisfy the equation

$$ay''(x) = \sqrt{1 + y'(x)^2}, \qquad x \in [x_1, x_2],$$

where the parameter $a > 0$ expresses the stiffness of the material of the chain. ◇

In all these examples, there are reasons that it is easier (or even the only possibility) to write down an equation that the function under interest satisfies than finding directly an explicit formula for it. In general, such an equation often involves derivatives of the function if there is a rule that says something about the behavior of the function under the influence of small changes of its argument. In other words, one writes down a differential equation. We now define what this is.

Definition 1.1.4. *Let $n \in \mathbb{N}$, $D \subset \mathbb{R}^{n+2}$ and $G \subset \mathbb{R}^{n+1}$ two domains and $F\colon D \to \mathbb{R}$ and $f\colon G \to \mathbb{R}$ two functions.*

(i) If $y = y(x)$ satisfies the equation

$$F\big(x, y(x), y'(x), y''(x), \dots, y^{(n)}(x)\big) = 0, \tag{1.1.1}$$

then we say that y satisfies an implicit ordinary differential equation *of n-th order.*

(ii) If $y = y(x)$ satisfies the equation

$$y^{(n)}(x) = f\big(x, y(x), y'(x), y''(x), \dots, y^{(n-1)}(x)\big), \tag{1.1.2}$$

then we say that y satisfies an explicit ordinary differential equation *of n-th order.*

Some few remarks are in order. Certainly we assumed that the vector in the argument of F in (1.1.1) lies in D for all x considered. More precisely, y is a solution to (1.1.1) if there is an interval I such that the vector $(x, y(x), y'(x), y''(x), \dots, y^{(n)}(x))$ lies in D for all $x \in I$ and (1.1.1) is satisfied for every $x \in I$. An analogous remark applies to (1.1.2). Hence, it is left open what the precise domain of the solution y is, and often there is no definite or no simple answer to this question.

It is common to drop the argument x if no confusion can arise and to write

$$F\big(x, y, y', y'', \dots, y^{(n)}\big) = 0 \tag{1.1.3}$$

instead of (1.1.1), analogously

$$y^{(n)} = f\big(x, y, y', y'', \dots, y^{(n-1)}\big) \tag{1.1.4}$$

for (1.1.2).

In the following, we shall write for short *ODE* instead of *ordinary differential equation.*

Remark 1.1.5. *Partial* differential equations are equations for functions $y = y(x_1, \dots, x_k)$, depending on more than one argument, which involve one or more partial derivatives of y with respect to different arguments. The theory of partial differential equations is is an enormously large subfield of mathematics and will not be touched in this lecture. ◇

Example 1.1.6. $y' = xy^2$ is an explicit ODE of first order. A solution for $I = \mathbb{R}$ is $y(x) = -2(1 + x^2)^{-2}$. ◇

Example 1.1.7. $(yy^{(5)})^2 + xy'' + \log y = 0$ is an implicit ODE of fifth order. A solution with $I = \mathbb{R}$ is $y \equiv 1$, e. g. ◇

Example 1.1.8. The implicit ODE $1 + (y')^2 = 0$ does not possess any solution. ◇

Example 1.1.9. Solutions of the circle equation $x^2 + y^2 = c$ with $c > 0$ are solutions of the implicit ODE of first order, $yy' + x = 0$. Here one has a solution with domain $I = (0, \sqrt{c})$ and another with domain $I = (-\sqrt{c}, 0)$. ◇

ODEs may have no solution, precisely one solution or many solutions. In the latter case, it may be that all the solutions can be assembled with one or more parameters. A simple example is the ODE $y'' = 0$, which has the two-parameter solution $y(x) = a + bx$. Typically, these parameters are obtained as *integration constants*. One sometimes conceives a family of solution with r parameters as *one* solution with r free parameters. Such a solution is called *general.* If *all* solutions belong to this family, this solution is called *complete.* For a fixed choice of the values of the parameters, the solution is called a *particular solution*, or a *special solution.* If it does not belong to a family of solutions, it is called a *singular solution.*

Example 1.1.10. $y' - 5y = 0$ has on $I = \mathbb{R}$ the general solution $y(x) = c\mathrm{e}^{5x}$ with $c \in \mathbb{R}$. It is not difficult to show (see later) that this is a complete solution. Some particular solutions are $y(x) = 0$ or $y(x) = 17\mathrm{e}^{5x}$. ◇

Example 1.1.11. $|y'| + |y| = 0$ does not possess a general solution, but only the particular solution $y \equiv 0$. ◇

Example 1.1.12. $(y')^2 - 4xy' + 4y = 0$ is an implicit ODE of first order. A general solution is $y(x) = 2cx - c^2$ with $c \in \mathbb{R}$. These lines are the tangents at the parabola $y(x) = x^2$, which itself is a singular solution. ◇

Remark 1.1.13. In comfortable cases, the solution of an ODE may be found by integration. For instance, the ODE $y'' = f$, where $f = f(x)$ is a given nice function, can be explicitly solved by two integrations as follows

$$\begin{aligned} y'(x) &= \int_{x_0}^{x} f(u)\,\mathrm{d}u + c_1, \\ y(x) &= \int_{x_0}^{x} y'(v)\,\mathrm{d}v + c_2 = \int_{x_0}^{x}\int_{x_0}^{v} f(u)\,\mathrm{d}u\,\mathrm{d}v + (x - x_0)c_1 + c_2, \end{aligned}$$

where $c_1, c_2 \in \mathbb{R}$ are integration constants and x_0 is suitably chosen in the domain of f. ◇

Remark 1.1.14. The integrals appearing in Remark (1.1.13) are called *definite* (in fact, they are Riemann-integrals) since the lower and upper boundaries of the integration are specified. It is common to formulate them also as *indefinite* integrals, *primitives*, by writing

$$y'(x) = \int f(x)\,\mathrm{d}x \qquad \text{and} \qquad y(x) = \int\int f(x)\,\mathrm{d}x\,\mathrm{d}x.$$

Here we have to keep in mind that the primitive $\int f(x)\,\mathrm{d}x$ is unique only up to adding a constant, and the double-primitive $\int\int f(x)\,\mathrm{d}x\,\mathrm{d}x$ is unique only up to adding a polynomial of first order. ◇

1.2 Initial-value problems

In order to enforce uniqueness of an ODE of n-th order, one usually imposes n additional hypotheses. The easiest way to do that is to fix the value of the solution and of its derivatives in one certain point.

Definition 1.2.1. *An* initial-value problem, IVP, *is an ODE of the form* (1.1.4)*, together with conditions of the form*

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \dots, \quad y^{(n-1)}(x_0) = y_{n-1}, \tag{1.2.1}$$

where $(x_0, y_0, \dots, y_{n-1}) \in \mathbb{R}^{n+1}$ *is chosen in the domain of* f*.*

We will see later that in many cases the condition in (1.2.1) makes the solution of (1.1.4) unique. The notion 'initial-value problem' stems from applications where $x = t$ plays the role of time, and $(y(t_0), y'(t_0), \dots, y^{(n-1)}(t_0))$ is the initial value of the system.

Example 1.2.2. A movement with one axis is uniquely determined by the place $x(t)$ and the velocity $v(t) = x'(t)$. If we consider a linear pendulum, then x satisfies the IVP

$$x'' + \omega^2 x = 0, \qquad x(0) = x_0, \quad x'(0) = v_0.$$

The unique solution is

$$x(t) = x_0 \cos(\omega t) + \frac{v_0}{\omega} \sin(\omega t).$$

In the (x, y) plane, the *phase plane*, this movement is described by an ellipse. More precisely, the graph of the function $t \mapsto \binom{x(t)}{v(t)}$ (where $v(t) = x'(t)$) describes an ellipse and actually runs through this ellipse periodically after time intervals of length $\frac{2\pi}{\omega}$. This may also be seen by noting that the movement satisfies the equation $\frac{(x')^2}{\omega^2} + x^2 = x_0^2 + v_0^2/\omega^2$, which also describes that ellipse. ◇

Example 1.2.3. The IVP $x' = 1 + x^2$, $x(0) = 0$, is solved by $x = \tan$, but only in the interval $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$. This admits the pysical interpretation that the the movement reaches infinity in finite time if its velocity increases with the square of its distance to the origin. ◇

An IVP (1.1.4) with (1.2.1) is called *locally solvable* and the corresponding y is called a *local solution* if there exists an interval $I = (x_0 - \varepsilon, x_0 + \varepsilon)$ with some $\varepsilon > 0$ such that (1.2.1) is satisfied and (1.1.4) is satisfied in I. The IVP is called *properly posed* if a local solution exists in such an interval, is unique there, and if it depends continuously on the initial values $y_0, \dots, y_{n-1}$. These requirements are obviously natural with respect to a proper use of numerical solution methods.

For a properly posed IVP, there arise two questions: What is the largest interval on which (1.1.4) is satisfied (i.e., how far can the local solution be extended without losing the property of being a solution), and how can the solution be explicitly computed?

A *continuation* of a function $y\colon I \to \mathbb{R}$ on an interval J containing I is a function $\widetilde{y}\colon J \to \mathbb{R}$ that satisfies $\widetilde{y}(x) = y(x)$ for all $x \in I$, i.e., the functions $\widetilde{y}$ and y coincide on I. Continuations do not have to be unique. If y is the local solution to an ODE, the question is to find continuations $\widetilde{y}$ of y that are still solutions on the larger interval, J. This question is difficult in general. However, there are important and handy abstract criteria, see later.

1.3 Orthogonal trajectories

Suppose we have a one-parameter family of curves $y_C = y_C(x)$ in the plane indexed by $C \in \mathbb{R}$, given as the solution to the equation

$$\Phi(x, y, C) = 0, \qquad \text{i.e., } \Phi(x, y_C(x), C) = 0 \quad \text{for all } x. \tag{1.3.1}$$

We are interested to describe the *orthogonal trajectories* of this family of curves, that is, all the lines that intersect every curve y_C at right angles. For finding them, we differentiate (1.3.1) with respect to x using the chain rule and obtain

$$\Phi_x(x, y, C) + \Phi_y(x, y, C)y' = 0, \tag{1.3.2}$$

where Φ_x and Φ_y denote the partial derivatives with respect to x and y. Now we combine (1.3.1) and (1.3.2) to eliminate the dependence on the parameter C. Let us assume that the ODE $F(x, y, y') = 0$ arises in this way (the F is certainly NOT unique). Then the orthogonal trajectories satisfy the characteristic ODE $F(x, y, -1/y') = 0$. This is seen as follows.

First proof. Let the curve y have slope $y'(x)$ in x, then the slope of the orthogonal curve, $\widetilde{y}$, in x is found as follows. Consider the tangents, t and $\widetilde{t}$, of the curves y and $\widetilde{y}$ in x, which are two lines that intersect in $(x, y(x))$ at a right angle. Then $y'(x) = t'(x) = \frac{\Delta t}{\Delta x}$, where $\Delta x = x_1 - x$ and $\Delta t = t(x_1) - t(x)$ for some $x_1 \neq x$. Analogously, $\widetilde{y}'(x) = \widetilde{t}'(x) = \frac{\Delta \widetilde{t}}{\Delta x}$. By orthogonality, $\Delta t - \Delta\widetilde{t}$ is the length of the hypothenuse of a triangle having a right angle at $(x, y(x))$. The other side lengths of this triangle are, according to Pythagoras' theorem, equal to the square roots of $(\Delta t)^2 + (\Delta x)^2$ respectively $(\Delta\widetilde{t})^2 + (\Delta x)^2$. (A figure illustrates this.) Hence, applying Pythagoras' theorem once more yields that $(\Delta t - \Delta\widetilde{t})^2 = (\Delta t)^2 + (\Delta x)^2 + \Delta\widetilde{t})^2 + (\Delta x)^2$. After summarizing, we obtain that $-\Delta t\Delta\widetilde{t} = (\Delta x)^2$, i.e.,

$$-y'(x)\widetilde{y}'(x) = -\frac{\Delta t}{\Delta x}\frac{\Delta\widetilde{t}}{\Delta x} = 1.$$

This shows that $\widetilde{y}' = -1/y'$.

Second proof. We consider the curves $x \mapsto \binom{x}{y(x)}$ and $x \mapsto \binom{x}{\widetilde{y}(x)}$ and consider an x in which the two curves meet. Then the derivative vectors $\binom{1}{y'(x)}$ and $\binom{1}{\widetilde{y}'(x)}$ have to be orthogonal to each other, i.e.,

$$0 = \begin{pmatrix} 1 \\ y'(x) \end{pmatrix} \cdot \begin{pmatrix} 1 \\ \widetilde{y}'(x) \end{pmatrix} = 1 + y'(x)\widetilde{y}'(x).$$

This leads obviously to the same equation.

Example 1.3.1. The tangents of the unit circle are describes by the family of ODEs $\Phi(x, y, C) = 0$, where

$$\Phi(x, y, C) = x \cos C + y \sin C - 1. \tag{1.3.3}$$

Differentiating with respect to x yields $\cos C + y' \sin C = 0$. With the help of $1 = \cos^2 C + \sin^2 C$ we can eliminate the parameter C: First we have $1 - \sin^2 C = \cos^2 C = (-y' \sin C)^2$, i.e., $1/(1 + (y')^2) = \sin^2 C$ and, on the other hand, one easily sees that $(y - xy')^2 = \sin^{-2} C$. Hence, we arrive at

$$1 + (y')^2 = (y - xy')^2,$$

an ODE which is solved by the circle line $y^2 = 1 - x^2$. $\diamond$

Example 1.3.2. All the circles in the plane that contain both the points $(-1,0)$ and $(1,0)$ are given by the equation $\Phi(x,y,C)=0$ with $\Phi(x,y,C)=x^2+(y-C)^2-1-C^2$. By differentiating we obtain $x+y'(y-C)=0$. By elimination we obtain the equation

$$(x^2-y^2-1)y'=2xy.$$

Replacing y' by $-1/y'$, we obtain the equation that describes the orthogonal trajectories:

$$x^2-y^2-1=-2xyy'.$$

The family of circles $(x-c)^2+y^2=c^2-1$ with $c^2>1$ satisfies this equation, hence all these circles are the orthogonal lines for the circles that contain $(-1,0)$ and $(1,0)$. ◇

Example 1.3.3 (Vector fields). Suppose we are given a planar vector field $(v_1,v_2)\colon \mathbb{R}^2\to\mathbb{R}^2$. The differential equation of the field lines of this field is given by $y'=v_2(x,y)/v_1(x,y)$, where $v_1(x,y)\neq 0$, or

$$\frac{\mathrm{d}x}{\mathrm{d}y}=\frac{v_1(x,y)}{v_2(x,y)},\qquad \text{where } v_2(x,y)\neq 0.$$

The field $(-v_2,v_1)$ is orthogonal to (v_1,v_2). Its differential equation is hence given as $v_1(x,y)+v_2(x,y)y'=0$.

As a simple example, if $v_1(x,y)=y$ and $v_2(x,y)=x$, then the field lines of the field (v_1,v_2) satisfy the equation $x-yy'=0$. A general solution is the solution to $x^2-y^2=C$, the ellipses. The equation of the orthogonal trajectories is $y+xy'=0$ and has the general solution $y=\frac{C}{x}$, the hyperbolas. ◇

Example 1.3.4. The equation

$$\frac{x^2}{C^2}+\frac{y^2}{C^2-1}=1$$

describes for $C^2<1$ a family of hyperbolas with focusses in $(-1,0)$ and in $(-1,0)$, and for $C^2>1$ it describes a family of ellipses with the same focusses. These two families are orthogonal to each other as we want to show now in two different ways.

First, we calculate the differential equations of the two curves. Fix $C_1^2>1$ and $C_2^2<1$ and let y_{C_1} and y_{C_2} denote the corresponding ellipse respectively hyperbola. The two curves intersect each other in the point $x^*=C_1^2C_2^2$ with value $y_{C_1}(x^*)=y_{C_2}(x^*)=\sqrt{C_1^2-1}\sqrt{1-C_2^2}$. The differential equations for the curves are eeasily calculated as

$$y'_{C_1}(x)=\mp\frac{C_1^2-1}{C_1^2}\frac{x}{y_{C_1}(x)}\qquad\text{and}\qquad y'_{C_2}(x)=\pm\frac{1-C_2^2}{C_2^2}\frac{x}{y_{C_2}(x)}. \tag{1.3.4}$$

Hence, we easily see that

$$y'_{C_1}(x^*)y'_{C_2}(x^*)=-\frac{C_1^2-1}{C_1^2}\frac{x^*}{y_{C_1}(x^*)}\frac{1-C_2^2}{C_2^2}\frac{x^*}{y_{C_2}(x^*)}=-\frac{C_1^2C_2^2}{(C_1^2-1)(1-C_2^2)}\frac{C_1^2-1}{C_1^2}\frac{1-C_2^2}{C_2^2}=-1.$$

Hence, the two curves are orthogonal in their intersection point x^*.

The second way to see this fact is to consider the curves $x\mapsto\binom{x}{y_{C_1}(x)}$ and $x\mapsto\binom{x}{y_{C_2}(x)}$ and to show that its derivatives, $\binom{1}{y'_{C_1}(x)}$ and $\binom{1}{y'_{C_2}(x)}$, are orthogonal in their intersection point x^*. This amounts to showing that $0=1+y'_{C_1}(x^*)y'_{C_2}(x^*)$, which we have derived already. ◇

Chapter 2

Elementary ODEs

In this chapter, we present a couple of well-known elementary ODEs whose general solution may be found eplicitly by standard methods.

2.1 Exact differential equations

This type of ODEs arises from an equation of the form

$$U(x,y) = C \tag{2.1.1}$$

by differentiation. They have the form

$$U_x(x,y) + U_y(x,y)\, y' = 0, \tag{2.1.2}$$

where $U_x = \frac{\partial}{\partial x}U$ and $U_y = \frac{\partial}{\partial y}U$, as usual, are the partial derivatives.[1] If the function U were known, then one could go back from (2.1.2) to (2.1.1) by integration and would be left to solve the equation $U(x,y) = C$ (which is *not* a differential equation, by the way!). However, the main work consists in finding the function U from a given ODE and to bring this ODE in the form (2.1.2).

An open set $G \subset \mathbb{R}^2$ is called *connected* if any two points $z_1, z_2 \in G$ may be joined with a continuous function $f\colon [0,1] \to G$ satisfying $f(0) = z_1$ and $f(1) = z_2$.

Definition 2.1.1. *Let $G \subset \mathbb{R}^2$ be an open connected set and $A, B\colon G \to \mathbb{R}$ two continuous functions. The differential equation*

$$A(x,y) + B(x,y)\, y' = 0 \tag{2.1.3}$$

is called exact *if there is a continuously differentiable function $U\colon G \to \mathbb{R}$ such that*

$$U_x = \frac{\partial U}{\partial x} = A, \qquad \textit{and} \qquad U_y = \frac{\partial U}{\partial y} = B \quad \textit{in } G.$$

In this case, we call U a primitive *of the ODE in* (2.1.3).

[1] We derived (2.1.2) from (2.1.1) by taking $y = y(x)$ as a function of x and differentiating with respect to x; we also used the general differentiation rule $\frac{\mathrm{d}}{\mathrm{d}x}U(f(x),g(x)) = f'(x)U_x(f(x),g(x)) + g'(x)U_y(f(x),g(x))$.

We now state the connection between (2.1.2) and (2.1.1) and provide an abstract criterion for local uniqueness and existence of a solution.

Lemma 2.1.2 (Existence and uniqueness for exact ODEs). *Let $G \subset \mathbb{R}^2$ be an open connected set, $A, B\colon G \to \mathbb{R}$ two continuous functions, and U a primitive of* (2.1.3)*.*

(a) A function $y\colon I \to \mathbb{R}$ (where I is an interval such that $(x, y(x)) \in G$ for any $x \in I$) is a solution to (2.1.3) *if and only if the map $x \mapsto U(x, y(x))$ is constant on I.*

(b) For any $(x_0, y_0) \in G$ such that $B(x_0, y_0) \neq 0$, the IVP

$$A(x, y) + B(x, y)\, y' = 0, \qquad y(x_0) = y_0, \tag{2.1.4}$$

is locally uniquely solvable. The curve of the solution is contained in $\{(x, y)\colon U(x, y) = U(x_0, y_0))\}$.

Proof. (a) Let I be an interval and $y\colon I \to \mathbb{R}$ such that $(x, y(x)) \in G$ for any $x \in I$. According to the chain rule for derivatives, y is a solution of (2.1.3) on I if and only if

$$0 = A(x, y(x)) + B(x, y(x))\, y'(x) = \frac{\mathrm{d}}{\mathrm{d}x} U(x, y(x)) \qquad \text{for any } x \in I,$$

i.e., if and only if $U(x, y(x)) = c$ for some $c \in \mathbb{R}$ and all $x \in I$.

(b) Assume that $U_y(x_0, y_0) = B(x_0, y_0) \neq 0$. Then, according to the implicit function theorem, there is a neighborhood of x_0 in which precisely one function y may be defined that solves the equation $U(x, y(x)) = U(x_0, y_0)$. According to assertion (a), the proof is finished. □

Now we give a criterion for exactness, which will be of practical importance for solving an exact ODE. An open set $G \subset \mathbb{R}^2$ is called *simply connected* if it is connected, but does not have 'holes', i.e., the complement of G in $\mathbb{R}^2 \cup \{\infty\}$ is also connected. Here ∞ is the north pole if one would wrap the plane $\mathbb{R}^2$ around a ball. For example, the infinite strip $\{(x, y) \in \mathbb{R}^2 \colon |x| < 1\}$ is simply connected: the two points $(1, 0)$ and $(-1, 0)$, e.g., can be joined together outside the strip by a line that goes through ∞.

Lemma 2.1.3 (Test on exactness). *Let $G \subset \mathbb{R}^2$ be an open simply connected set and $A, B\colon G \to \mathbb{R}$ two continuously differentiable functions. Then the differential equation* (2.1.3) *is exact if and only if the* integrability condition $A_y = B_x$*, i.e.,*

$$\frac{\partial}{\partial y} A(x, y) = \frac{\partial}{\partial x} B(x, y), \qquad \textit{for any } (x, y) \in G, \tag{2.1.5}$$

is satisfied.

Sketch of proof. If (2.1.3) is exact, then U is twice continuously differentiable, and Schwarz's theorem states that $U_{xy} = U_{yx}$, i.e., the order of the two derivatives may be interchanged. This is the same as $A_y = B_x$.

On the contrary, in the case that $A_y = B_x$ is satisfied, one constructs the function U as the line integral

$$U(x, y) = \int_{(x_0, y_0)}^{(x, y)} \big(A(x, y)\, \mathrm{d}x + B(x, y)\, \mathrm{d}y\big), \qquad (x, y) \in G,$$

where $(x_0, y_0) \in G$ is fixed. The domain of integration is some curve in G from (x_0, y_0) to (x, y). Since G is simply connected and A and B continuously differentiable, the function U can be shown to be well-defined and not to depend on the chosen curve. In particular, we can take a polygon line whose segments are parallel to the axes. Then one can differentiate U and derive that indeed $U_x = A$ and $U_y = B$ hold. The details are involved. □

Example 2.1.4. The equation $2x + 3\cos y + (2y - 3x\sin y)y' = 0$ is exact, as is seen by picking $A(x, y) = 2x + 3\cos y$ and $B(x, y) = 2y - 3x\sin y$: we have $A_y = B_x$. ◇

Example 2.1.5. The equation $(y^2-x)+(x^2-y)y' = 0$ is not exact, since $\frac{\partial}{\partial y}(y^2-x) \neq \frac{\partial}{\partial x}(x^2-y)$ for all $x \neq y$. ◇

Now we give a general recipe:

Solving the exact ODE (2.1.3).

(a) Check that $A_y = B_x$ holds.

(b) Solve $U_x = A$ by integrating A with respect to x:

$$U(x, y) = \int A(x, y)\,\mathrm{d}x + c(y). \tag{2.1.6}$$

(c) Differentiate (2.1.6) with respect to y and make the ansatz $U_y = B$:

$$B(x, y) = U_y(x, y) = \frac{\partial}{\partial y}\int A(x, y)\,\mathrm{d}x + c'(y). \tag{2.1.7}$$

(d) Find $c(y)$ by integrating (2.1.7) with respect to y. Via (2.1.6), this gives U.

The point is that the integration constant c that arises in (2.1.6) depends on y (but not on x, though). Alternately to the above recipe, one can also start with solving the equation $U_y = B$ by integrating with respect to y.

Once one has U, one can try to solve the equation $U(x, y) = C$, which is NOT a differential equation. This task may be nasty to solve, since one has to find all constants C for which there is a solution, and then one has to determine the domain of the solution. If there is some initial value (x_0, y_0) given, then one has to put $C = U(x_0, y_0)$. It may be possible that the solution to the equation $U(x, y) = C$ is not the graph of a function $y = y(x)$, but the graph of a function $x = x(y)$. It may also be possible that the solution set $\{(x, y) \colon U(x, y) = C\}$ is the union of several such graphs.

Example 2.1.6. We want to solve the IVP

$$2xy + (2y + x^2)\,y' = 0, \qquad y(0) = 1.$$

Hence $A(x, y) = 2xy$ and $B(x, y) = 2y + x^2$.

(a) This ODE is exact in $G = \mathbb{R}^2$, since $A_y = 2x = B_x$.

(b) $U(x, y) = \int A(x, y)\,\mathrm{d}x + c(y) = \int 2xy\,\mathrm{d}x + c(y) = x^2y + c(y)$.

(c) $2y + x^2 = \frac{\partial}{\partial y}\int A(x, y)\,\mathrm{d}x + c'(y) = x^2 + c'(y)$, i.e., $c'(y) = 2y$.

(d) $c(y) = y^2$ by integration. (We may forget about an integration constant, since we add that later.)

Hence, $U(x,y) = x^2y + y^2$. The equation $U(x,y) = C$ is a quadratic equation in y; an explicit solution is easily found. If, e.g., the initial value $(0,1)$ is given, then the solution is

$$y(x) = \frac{1}{2}\Big(\sqrt{x^4+4} - x^2\Big), \qquad x \in \mathbb{R}.$$

$\diamond$

Example 2.1.7. We want to solve the IVP

$$-\frac{y}{x^2+y^2} + \frac{x}{x^2+y^2}\, y' = 0, \qquad y(1) = 1.$$

Hence, $A(x,y) = -\frac{y}{x^2+y^2}$ and $B(x,y) = \frac{x}{x^2+y^2}$.

(a) We have $A_y = B_x = \frac{y^2-x^2}{(x^2+y^2)^2}$ in the set $G = \mathbb{R}^2 \setminus \{(0,0)\}$, which is not simply connected. Nevertheless, we follow the recipe.

(b) $U(x,y) = -\arctan\frac{x}{y} + c(y)$.

(c) $\frac{x}{x^2+y^2} = B(x,y) = \frac{x}{x^2+y^2} + c'(y)$, i.e., $c'(y) = 0$.

(d) c is constant.

Hence, $U(x,y) = -\arctan\frac{x}{y}$. A general solution of $U = C$ is $y(x) = Kx$ with some appropriate K, if $C \in (-\frac{\pi}{2}, \frac{\pi}{2})$, since the range of arctan is equal to this interval.

The initial value $U(1,1)$ is $\arctan 1 = \frac{\pi}{4}$. Hence, we have to solve the equation $-\arctan\frac{x}{y} = \frac{\pi}{4}$, which has the solution $y(x) = -x$. However, the domain of this solution is only $x \in (0,\infty)$, since the ODE is not defined in $(x,y) = (0,0)$. $\diamond$

Example 2.1.8. We want to solve the ODE

$$2x + 3\cos y + (2y - 3x\sin y)\, y' = 0,$$

i.e., $A(x,y) = 2x + 3\cos y$ and $B(x,y) = 2y - 3x\sin y$.

(a) This equation is exact.

(b) $U(x,y) = x^2 + 3x\cos y + c(y)$.

(c) $2y - 3x\sin y = B(x,y) = -3x\sin y + c'(y)$, i.e., $c'(y) = 2y$.

(d) $c(y) = y^2$.

Hence, $U(x,y) = x^2 + 3x\cos y + y^2$. The equation $U = C$ can be solved for x only; this is a quadratic equation. $\diamond$

Later we will look at ODEs of the form (2.1.3) which are NOT exact, but turn into an exact ODE after multiplication with a suitable function $m(x,y)$, the so-called *integrating factor*.

2.2 Separable ODEs

A *separable ODE* has the form

$$y' = f(x)g(y), \tag{2.2.1}$$

where f and g are two continuous functions that are defined on intervals I respectively J. If $eta \in J$ is a zero of g, then the constant function $y \equiv \eta$ is obviously a particular solution. On the other hand, if $g(y) \neq 0$ for all $y \in J$, then we can turn (2.2.1) into an exact ODE:

$$f(x) - \frac{1}{g(y)}\, y' = 0,$$

and a primitive is easily seen to be given by

$$U(x, y) = \int f(x)\,\mathrm{d}x - \int \frac{1}{g(y)}\,\mathrm{d}y.$$

Hence, in this case, the local existence and uniqueness result for exact ODEs in Lemma 2.1.2 also applies to separable equations. This result may be applied to any restriction of J to some subinterval in which g has no zero, tyically the oopern intervals between the zeros of g.

We give now a general recipe how to solve separable ODEs:

Solving the separable ODE (2.2.1).

(a) Find all zeros $\eta \in J$ of g. Then $y \equiv \eta$ is a particular solution.

(b) Consider an interval $J' \subset J$ with $g(y) \neq 0$ for all $y \in J'$. Bring all x-terms to one side of the equation, and all y-terms to the other:

$$\frac{1}{g(y)}\,\mathrm{d}y = f(x)\,\mathrm{d}x.$$

(c) Integrate each side w.r.t. x resp. y:

$$G(y) := \int \frac{1}{g(y)}\,\mathrm{d}y, \qquad F(x) := \int f(x)\,\mathrm{d}x.$$

The general implicit solution is $G(y) = F(x) + C$ for some integration constant $C \in \mathbb{R}$.

We now would like to solve the equation $G(y) = F(x) + C$ for y. This is possible if G is bijective, i.e., if it is strictly increasing or strictly decreasing. Since $G' = 1/g$ is either positive throughout J' or negative throughout J', it indeed is invertible, i.e., the inverse G^{-1} of G exists. However, the set of x for which $G^{-1}(F(x) + C)$ is meaningful heavily depends on C, which has to be determined on a case-by-case basis.

If one is interested in solving the IVP (2.2.1), together with the condition $y(x_0) = y_0$ for some $x_0 \in I$, $y_0 \in J$, then one has to put $C = G(y_0) - F(x_0)$ in the case $g(y_0) \neq 0$. (In the case $g(y_0) = 0$, $y \equiv y_0$ is solution.)

Example 2.2.1. We want to solve the ODE $y' = y^2$, i.e., $f(x) = 1$ and $g(y) = y^2$ on the real line, i.e., $I = J = \mathbb{R}$.

(a) $y \equiv 0$ is a particular solution.

(b) For $y > 0$ or for $y < 0$, we transform into $y^{-2}\,\mathrm{d}y = \mathrm{d}x$.

(c) Clearly, $G(y) = -y^{-1}$ and $F(x) = x$.

The equation $-y^{-1} = x + C$ can be solved for all $x \in \mathbb{R} \setminus \{-C\}$, which is the union of the two intervals $(-\infty, -C)$ and $(-C, \infty)$. The solution is $y(x) = -\frac{1}{x+C}$.

If the IVP $y(x_0) = y_0 > 0$ is given, then we have to pick $C = -x_0 - \frac{1}{y_0}$, and we obtain the solution

$$y(x) = \frac{y_0}{1 - y_0(x - x_0)}, \qquad x \in \Big(-\infty, x_0 + \frac{1}{y_0} \Big).$$

$\diamond$

Example 2.2.2. Consider $y' = 1 + y^2$. Hence $f(x) = 1$ and $g(y) = 1 + y^2$ on the real line, i.e., $I = J = \mathbb{R}$.

(a) There are no zeros of g.

(b) We transform into $\frac{\mathrm{d}y}{1+y^2} = \mathrm{d}x$.

(c) Integration gives $G(y) = \arctan y$ and $F(x) = x$.

We have obtained the equation $\arctan y = x + C$. Clearly, $\arctan\colon \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$ is bijective. Hence, in order to solve for y, the argument $x + C$ must lie in that interval. Consequently, the solution $y(x) = \tan(x + C)$ is defined only in the interval $x \in (-\frac{\pi}{2} - C, \frac{\pi}{2} - C)$. $\diamond$

Example 2.2.3. We consider $y' = \frac{1-y^2}{x}$ with $f(x) = \frac{1}{x}$ on $I = \mathbb{R} \setminus \{0\}$ and $g(y) = 1 - y^2$ on $J = \mathbb{R}$.

(a) The particular solutions are $y(x) \equiv 1$ and -1.

(b) Outside the zeros -1 and 1, we transform into $\frac{\mathrm{d}y}{1-y^2} = \frac{\mathrm{d}x}{x}$.

(c) Integration gives $G(y) = \log \sqrt{|\frac{1+y}{1-y}|}$ and $F(x) = \log |x|$.

The resulting equation is $|\frac{1+y}{1-y}| = Kx^2$ (after substituting $K = \mathrm{e}^{C^2}$). This is equivalent to $\frac{1+y}{1-y} = \widetilde{K}x^2$, where $\widetilde{K}$ may have both signs.

Now let the IVP $y(x_0) = y_0$ be given. In order to solve this, we have to distinguish cases: If $y_0 = 1$, then the solution is $y(x) \equiv 1$. If $y_0 \neq 1$ and $x_0 \neq 0$, then we see that $\widetilde{K} = \pm \mathrm{e}^{(G(y_0) - F(x_0))^2} = \frac{1+y_0}{(1-y_0)x_0^2}$, and we can solve the equation $\frac{1+y}{1-y} = \widetilde{K}x^2$ to the effect that $y(x) = \frac{\widetilde{K}x^2 - 1}{\widetilde{K}x^2 + 1}$. The domain is equal to the whole line $\mathbb{R}$ if $\widetilde{K} > 0$, i.e., if $y_0 < 1$. Otherwise, the domain consists of the two intervals left and right of the zero of the denominator, $1/\sqrt{-\widetilde{K}}$. $\diamond$

2.3 First-order linear differential equations

These are ODEs of the form

$$y' + a(x)y = f(x), \tag{2.3.1}$$

where $f, a\colon I \to \mathbb{R}$ are two functions on an interval I, and a is called the *coefficient function.* The equation is called *homogeneous* if $f \equiv 0$, and *inhomogeneous* otherwise. The equation

$$y' + a(x)y = 0 \tag{2.3.2}$$

is the homogeneous equation associated with (2.3.1). For continuous a, this is very easily solved:

Lemma 2.3.1 (Existence and uniqueness of (2.3.2)). *If a is continuous, then the linear homogenous equation* (2.3.2) *has the general complete solution*

$$y_{\mathrm{h}}(x) = C\mathrm{e}^{-A(x)}, \qquad A(x) = \int a(x)\,\mathrm{d}x, \tag{2.3.3}$$

and $C \in \mathbb{R}$ is the integration constant.

Proof. (2.3.2) is a separable equation whose solution for $y \neq 0$ is given by $\frac{\mathrm{d}y}{y} = -a(x)\,\mathrm{d}x$, i.e., $\log|y| = -\int a(x)\,\mathrm{d}x + C_1 = -A(x) + C_1$. With $C = \pm\mathrm{e}^{C_1}$, (2.3.3) follows. The case $y \equiv 0$ is included with $C = 0$. □

Remark 2.3.2. Linear differential equations are often used to describe growth of a population as a function of time. If $x(t)$ denotes the size at time t, then the equation $x'(t) = -a(t)x(t)$ describes a situation in which the change of the population size in infinitesimal time intervals is proportional to the size itself, with a proportional factor, $-a(t)$, that depends on the time. For the inhomogeneous equation, $x'(t) = -a(t)x(t) + f(t)$, the increase or decrease of the population size is additionally influenced by a 'controlling function' f. ◇

Example 2.3.3. If the coefficient function $a(x) = a$ is constant, then the solution to (2.3.2) is just the exponential function $y_{\mathrm{h}}(x) = C\mathrm{e}^{-ax}$. As $x \to \infty$, we have an exponential decay to zero if $a > 0$ and an exponential increase if $a < 0$. ◇

The general and complete solution of the inhomogeneous equation (2.3.1) can also be found easily, although it is not an exact equation. We offer two independent proofs, the first of which describes an important solution algorithm, called the *variation of constants.*

Lemma 2.3.4 (General solution of (2.3.1)). *If a and f are continuous functions on an interval I, then the general and complete solution to* (2.3.1) *is given by*

$$y(x) = \mathrm{e}^{-A(x)}\Big(\int_{x_0}^{x} \mathrm{e}^{A(z)} f(z)\,\mathrm{d}z + C\Big), \qquad \textit{where } A(x) = \int a(x)\,\mathrm{d}x, \tag{2.3.4}$$

and $x_0 \in I$ and $C \in \mathbb{R}$ are arbitrary.

First proof: variation of constants. Let y_{h} with $C = 1$ in (2.3.3) be the solution to the homogeneous equation (2.3.2), then we make the ansatz

$$y(x) = C(x)y_{\mathrm{h}}(x). \tag{2.3.5}$$

In other words, we treat the integration constant C in (2.3.3) as a function of x, and we want to derive what this function is. In order to see that, we just substitute in (2.3.1) and obtain, with the help of $y_{\mathrm{h}}' + ay_{\mathrm{h}} = 0$,

$$f(x) = y'(x) + a(x)y(x) = C'(x)y_{\mathrm{h}}(x) + C(x)y_{\mathrm{h}}'(x) + a(x)C(x)y_{\mathrm{h}}(x) = C'(x)y_{\mathrm{h}}(x).$$

This can easily be solved to the effect that

$$C(x) = \int_{x_0}^{x} \frac{1}{y_{\mathrm{h}}(z)} f(z)\,\mathrm{d}z + c. \tag{2.3.6}$$

Since $y_{\mathrm{h}}(x) = \mathrm{e}^{-A(x)}$, the result follows. □

Second proof: making (2.3.1) exact. The equation (2.3.1) is not exact. However, after multiplication with $\mathrm{e}^{A(x)}$, we see that it is equivalent to the exact equation

$$\left[\mathrm{e}^{A(x)} a(x) y - \mathrm{e}^{A(x)} f(x)\right] + \mathrm{e}^{A(x)} y' = 0. \tag{2.3.7}$$

Now our solution algorithm for exact equations derives (2.3.4). Indeed, we first obtain

$$U(x, y) = y \int \mathrm{e}^{A(x)} a(x)\,\mathrm{d}x - \int \mathrm{e}^{A(x)} f(x)\,\mathrm{d}x + c(y) = y\mathrm{e}^{A(x)} - \int \mathrm{e}^{A(x)} f(x)\,\mathrm{d}x + c(y).$$

and then we see that $c'(y) = \mathrm{e}^{A(x)} - \mathrm{e}^{A(x)} = 0$, i.e., we may put $U(x, y) = y\mathrm{e}^{A(x)} - \int \mathrm{e}^{A(x)} f(x)\,\mathrm{d}x$. The equation $U(x, y) = c$ is obviously equivalent to (2.3.4). □

Hence, linear ODEs even admit *global* solutions, i.e., solutions that are defined on the whole real line and satisfy (2.3.1) everywhere:

Corollary 2.3.5. *Let a and f be continuous functions on an interval I, and let $x_0 \in I$ and $y_0 \in \mathbb{R}$ be arbitrary. Then the solution of the IVP* (2.3.1)*, together with $y(x_0) = y_0$, is uniquely given by* (2.3.4) *with $c = y_0$.* ◇

Remark 2.3.6. The general and complete solution of the inhomogeneous equation in (2.3.1) is given as

$$y(x) = y_{\mathrm{p}}(x) + y_{\mathrm{h}}(x) = y_{\mathrm{p}}(x) + c\mathrm{e}^{-A(x)}, \qquad c \in \mathbb{R}. \tag{2.3.8}$$

Here y_{p} is an arbitrary particular solution of (2.3.1), and y_{h} is the homogeneous solution from Lemma 2.3.1. ◇

Remark 2.3.7 (Principle of super position). If y_i is a solution of (2.3.1) with f replaced by f_i for $i = 1, 2$, then $\alpha y_1 + \beta f_2$ is a solution of (2.3.1) with f replaced by $\alpha f_1 + \beta f_2$, for any $\alpha, \beta \in \mathbb{R}$. This observation may be helpful when the right-hand side f is a sum of several functions of different type. In this case, we may split the task into smaller subtasks. ◇

Remark 2.3.8 ('Ansatz of type of the right-hand side'). If the coefficient function $a(x) \equiv a$ is constant and if the right-hand side $f(x)$ is of one of the forms

$$p(x), \qquad q(x)\mathrm{e}^{kx}, \qquad p(x)\sin(kx) + q(x)\cos(kx), \qquad p, q \text{ polynomials}, k \in \mathbb{R}, \tag{2.3.9}$$

then an ansatz of type of this respective function often leads quickly to a particular solution.

More precisely, we make the ansatz that $y_{\mathrm{p}}(x)$ is an arbitrary polynomial of the same degree of p resp. q times the exponential, resp. a combination of such polynomial with the sine and cosine functions with the same value of k. The idea is that, for any $y_{\mathrm{p}}(x)$ of this form, the left-hand side, $y_{\mathrm{p}}' + ay_{\mathrm{p}}$, is again of the same type, and we only have to compare the coeffients of the polynomials in order to systematically find what $y_{\mathrm{p}}(x)$ is. Note that it is important to take an *arbitrary* polynomial of the same degree, even if not all lower-oder coefficients of p or q are non-zero.

For example, if $f(x) = x^2\mathrm{e}^{2x}$, then an appropriate ansatz is $y_{\mathrm{p}}(x) = (ax^2 + bx + c)\mathrm{e}^{2x}$. The unknown values of a, b and c are derived by a comparison of the coefficients after substituting y_{p} in (2.3.1). ◇

The general algorithm for finding the solution of (2.3.1) is now clear from the preceding: Find the primitive $A(x)$ of a, provide the homogeneous solution y_h in (2.3.3), find a particular solution y_p by the method of variation of constants or (if $a(x)$ is constant and f is of the type in (2.3.9)) by an ansatz of this type, and obtain the general complete solution $y_\mathrm{p} + y_\mathrm{h}$. An associated IVP is solved by deriving the integration constant c from the equation $y(x_0) = y_0$.

Example 2.3.9. We want to solve $y' + \frac{1}{x}y = x^3$ for $x > 0$. Hence, $a(x) = \frac{1}{x}$ and therefore $A(x) = \log x$. Hence, the homogeneous solution is $y_\mathrm{h}(x) = c\mathrm{e}^{-\log x} = \frac{c}{x}$. In order to find a particular solution, we use variation of the constants and make the ansatz $y(x) = c(x)\frac{1}{x}$; see (2.3.5). Substituting this in (2.3.1), we arrive at (see (2.3.6))

$$c(x) = \int x^3 \cdot x \,\mathrm{d}x = \frac{1}{5}x^5 + c, \qquad \text{i.e.,} \quad y_\mathrm{p}(x) = \frac{1}{5}x^4.$$

Hence, the general complete solution is $y(x) = c\frac{1}{x} + \frac{1}{5}x^4$. ◇

Example 2.3.10. We want to solve $y' + 2y = 3\mathrm{e}^{5x} + x^3 - 1$. It is clear that $y_\mathrm{h}(x) = c\mathrm{e}^{-2x}$ with some $c \in \mathbb{R}$. We use the transposition principle and solve first the equation $y' + 2y = 3\mathrm{e}^{5x}$ with the ansatz $y_1(x) = c_1\mathrm{e}^{5x}$. Substituting this in $y' + 2y = 3\mathrm{e}^{5x}$ easily yields that $y_1(x) = \frac{3}{7}\mathrm{e}^{5x}$. In a second step, we solve $y' + 2y = x^3 - 1$ with the ansatz $y_2(x) = d_3x^3 + d_2x^2 + d_1x + d_0$. Substituting this in $y' + 2y = x^3 - 1$, we find, after some elementary calculation, that $y_2(x) = \frac{1}{2}x^3 - \frac{3}{4}x^2 + \frac{3}{4}x - \frac{7}{8}$. Then $y_\mathrm{p} = y_1 + y_2$ is a particular solution of $y' + 2y = 3\mathrm{e}^{5x} + x^3 - 1$. Hence, the general complete solution is $y(x) = c\mathrm{e}^{-2x} + \frac{3}{7}\mathrm{e}^{5x} + \frac{1}{2}x^3 - \frac{3}{4}x^2 + \frac{3}{4}x - \frac{7}{8}$. ◇

Example 2.3.11 (Electric circuit). The current $I(t)$ at time t satisfies

$$LI'(t) + RI(t) = U(t), \qquad t \in [0, \infty),$$

where $L, R > 0$ are parameters. The solution, $I_\mathrm{h}(t) = c\mathrm{e}^{-\frac{R}{L}t}$, of the homogeneous equation, $LI'(t) + RI(t) = 0$, describes the exponential decay of the current after switching off the source of the voltage. In the case of direct current, we have a constant right-hand side, $U(t) = U_0$. A particular solution is $I_\mathrm{p}(t) = \frac{U_0}{R}$. This yields the general solution

$$I(t) = \frac{U_0}{R} - \Big(\frac{U_0}{R} - I_0\Big)\mathrm{e}^{-\frac{R}{L}t},$$

where $I_0 = I(0)$ is the initial value of the current. Independently of I_0, the current approaches the value $\frac{U_0}{R}$ at large times with an exponential speed.

In the case of alternating current, we have $U(t) = U_0\cos(\omega t)$ for some $U_0, \omega > 0$. We make the ansatz $I_\mathrm{p}(t) = c_1\cos(\omega t) + c_2\sin(\omega t)$ and substitute in the ODE, to get

$$(Rc_1 + \omega Lc_2)\cos(\omega t) + (Rc_2 - \omega Lc_1)\sin(\omega t) = U_0\cos(\omega t).$$

The comparison of the coefficients, i.e., $Rc_1 + \omega Lc_2 = U_0$ and $Rc_2 - \omega Lc_1 = 0$, yields, also using the well-known addition theorems for the cosine of a sum of the arguments,

$$I_\mathrm{p}(t) = \frac{U_0}{R^2 + \omega^2L^2}\big(R\cos(\omega t) + \omega L\sin(\omega t)\big) = \frac{U_0}{\sqrt{R^2 + \omega^2L^2}}\cos(\omega t - \delta),$$

where $\delta = \arctan\frac{\omega L}{R}$. Hence,

$$I(t) = c\mathrm{e}^{-\frac{R}{L}t} + \frac{U_0}{\sqrt{R^2 + \omega^2L^2}}\cos(\omega t - \delta).$$

For large times, $I(t)$ is rather close to the phase-shifted alternating current. ◇

2.4 Integrating factors

I our second proof of Lemma 2.3.4, we saw an interesting idea: we succeeded in turning a non-exact equation into an exact one. Even better, the two equations are obviously equivalent since they differ by multiplication with a function that is positive everywhere. In this section, we systematically investigate this idea. Let the (non-exact, in general) equation

$$A(x,y) + B(x,y)\,y' = 0 \qquad \text{in } G \subset \mathbb{R}^2, \tag{2.4.1}$$

be given. A function $M\colon G \to \mathbb{R} \setminus \{0\}$ is called an *integrating factor* or an *Euler multiplicator* if the equation

$$M(x,y)A(x,y) + M(x,y)B(x,y)\,y' = 0 \qquad \text{in } G \subset \mathbb{R}^2, \tag{2.4.2}$$

is exact. Since we required $M(x,y) \neq 0$ for all (x,y), the solutions to (2.4.1) and (2.4.2) are obviously identical.

Example 2.4.1. The family of straight lines through the origin is given by the equation $\Phi(x,y) = C$, where C is a constant, and $\Phi(x,y) = \frac{x}{y}$ for $(x,y) \in G = \mathbb{R} \times (0,\infty)$. They satisfy the exact equation $0 = \Phi_x + \Phi_y\,y' = \frac{1}{y} - \frac{x}{y^2}\,y'$. By multiplication with the nasty denominator, we obtain the equation $0 = y - x\,y'$, which looks much nicer, but is not exact. Hence, $M(x,y) = \frac{1}{y^2}$ is an integrating factor for $0 = y - x\,y'$. ◇

Assume that A and B are continuously differentiable on G, and G is a simply connected open set in $\mathbb{R}^2$. Then M is an integrating factor if and only if

$$\frac{\partial}{\partial y}\big[M(x,y)A(x,y)\big] = \frac{\partial}{\partial x}\big[M(x,y)B(x,y)\big] \qquad \text{in } G. \tag{2.4.3}$$

This is equivalent to the partial differential equation

$$B\,M_x - A\,M_y = (A_y - B_x)\,M. \tag{2.4.4}$$

In this lecture, we will not study general methods to solve this partial differential equation systematically. Instead, we will look at some particular cases in which we are lucky enough to find the integrating factor explicitly.

Let us look at cases where an ansatz of the form $M(x,y) = m(u(x,y))$ is successful, i.e., where the integrating factor is a function m of some simple function $u(x,y)$ of x and y. (Examples for u that appear frequently are $u(x,y) = x + y$ or xy or just x.) Using the chain rule, we see that (2.4.4) is equivalent to

$$\frac{A_y(x,y) - B_x(x,y)}{B(x,y)u_x(x,y) - A(x,y)u_y(x,y)} = \frac{m'(u(x,y))}{m(u(x,y))}. \tag{2.4.5}$$

We will also not study all the cases of possible functions u and m that realize this equation, but we want to point out that, given an explicit choice of $u(x,y)$ (given or guessed), we just have to check whether or not the left hand side of (2.4.5) is equal to a function of $u(x,y)$, say it is equal to $h(u(x,y))$. In this (lucky) case, we just find m via the equation $h = \frac{m'}{m}$ and find $m(u) = \mathrm{e}^{\int h(u)\,\mathrm{d}u}$. Therefore, we have found the Euler multiplicator $M(x,y) = m(u(x,y)) = \mathrm{e}^{\int h(u)\,\mathrm{d}u}\,|_{u=u(x,y)}$.

It should be stressed that the above method is not systematic and leads to a success only in rather limited cases. Nevertheless, here is a list of standard ansatzes for $u(x,y)$ that one can try:

$$u(x,y) = \qquad x, \qquad y, \qquad x+y, \qquad x-y, \qquad xy, \qquad x^2+y^2, \qquad x^2-y^2. \tag{2.4.6}$$

We summarize:

Solving (2.4.1).

(i) If $A_y = B_x$ then proceed with the recipe in Section 2.1.

(ii) Otherwise, try step by step the functions u in (2.4.6) and calculate $H(x,y) = \frac{A_y - B_x}{B\,u_x - A\,u_y}$. If $H(x,y) = h(u(x,y))$ for some h, then go to Step (iii), otherwise choose a new $u(x,y)$ or give up.

iii) With the function h from Step (ii), calculate $M(x,y) = m(u(x,y)) = \mathrm{e}^{\int h(u)\,\mathrm{d}u}\,|_{u=u(x,y)}$. Then (2.4.2) is exact. Apply the recipe in Section 2.1.

Example 2.4.2. Consider $y + x(2xy-1)\,y' = 0$, i.e., $A(x,y) = y$ and $(B(x,y) = x(2xy-1)$.

(i) This is not exact since $A_y - B_x = 1 - 4xy + 1 = 2 - 4xy \neq 0$.

(ii) We try $u(x,y) = x$ and have to look at $H(x,y) = \frac{A_y - B_x}{B} = -\frac{2}{x}$, which is fine, since it is $h(u(x,y))$ with $h(u) = -\frac{2}{u}$.

(iii) We find the Euler multiplicator $M(x,y) = m(x) = \mathrm{e}^{-\int 2/x\,\mathrm{d}x} = \frac{1}{x^2}$. The exact version of $y + x(2xy-1)\,y' = 0$ is

$$\frac{y}{x^2} + \frac{2xy-1}{x}\,y' = 0.$$

It has the general implicit solution $y^2 - \frac{y}{x} = C$ for $C in \mathbb{R}$.

$\diamond$

Example 2.4.3. Consider $(x^2y^3 + y) + (x^3y^2 - x)\,y' = 0$, i.e., $A(x,y) = x^2y^3 + y$ and $B(x,y) = x^3y^2 - x$.

(i) This equation is not exact since $A_y - B_x = 3y^2x^2 + 1 - 3x^2y^2 + 1 = 2 \neq 0$.

(ii) As an example, $u(x,y) = x$ does not yield an Euler multiplicator since $\frac{A_y - B_x}{B} = \frac{2}{x^3y^2 - x}$ is not a function of $u(x,y) = x$. However, with $u(x,y) = xy$, we are successful since $\frac{A_y - B_x}{yB - xA} = \frac{2}{-2xy} = h(u(x,y))$ with $h(u) = -\frac{1}{u}$.

(iii) We find $m(u) = \mathrm{e}^{\int h(u)\,\mathrm{d}u} = \frac{1}{u}$ and hence $M(x,y) = \frac{1}{xy}$. Hence, we arrive at the exact equation

$$\Big(xy^2 + \frac{1}{x}\Big) + \Big(x^2y - \frac{1}{y}\Big)\,y' = 0, \qquad \text{in } G = (0,\infty)^2, \text{say.}$$

The general implicit solution is $x^2y^2 + \log\frac{x^2}{y^2} = C$ for some $c \in \mathbb{R}$. $\diamond$

2.5 Substitution

2.5.1 $y' = f(ax + by + c)$, where $b \neq 0$

In this case, we certainly substitute $v(x) = ax + by(x) + c$. Using that $v' = a + by'$, the equation is turned into the separable equation

$$v' = a + bf(v).$$

Example 2.5.1. Consider $y' = (x + y + 1)^3$. We put $v = x + y + 1$ and obtain $v' = 1 + v^3$, which has the general implicit solution

$$\frac{1}{6}\log\frac{(1+v)^2}{1-v+v^2} + \frac{1}{\sqrt{3}}\arctan\frac{2v-1}{\sqrt{3}} = x + c.$$

Re-substitution of $v = x + y + 1$ gives the general implicit solution in x and y. $\diamond$

2.5.2 Similarity equations

If x and y appear only in the form y/x, we have an equation of the form

$$y' = f\left(\tfrac{y}{x}\right), \qquad x \neq 0, \tag{2.5.1}$$

for some function f of one variable. This equation is called a *similarity equation* since it is invariant under the map $(x, y) \mapsto (\alpha x, \alpha y)$ for any $\alpha \neq 0$. This map is called a *similarity transformation*, and this explains the name. This means that, if y is a solution, also the map $x \mapsto \frac{1}{\alpha}y(\alpha x)$ is. In particular, taking $\alpha = -1$, if we have a solution y on the positive axis only, then $x \mapsto -y(-x)$ is a solution on the negative axis.

In order to solve (2.5.1), we substitute $v(x) = \frac{y(x)}{x}$ and obtain the separable equation

$$v' = \frac{1}{x}\left(f(v) - v\right), \qquad x \neq 0. \tag{2.5.2}$$

In particular, for any zero η of f, a particular solution of (2.5.1) is $y(x) = \eta x$. Once one has found the general solution $v(x)$ of (2.5.2), we know that $y(x) = xv(|x|)$ is the general solution of (2.5.1) on $(-\infty, 0)$ and on $(0, \infty)$.

Example 2.5.2. Consider $y' = \frac{y}{x} - \sqrt{1 - \frac{y}{x}}$, where $x \neq 0$ and $\frac{y}{x} \leq 1$. We may assume for a while that $x > 0$. Clearly, $f(\eta) = \eta - \sqrt{1 - \eta}$, whose only zero is $\eta = 1$. Hence, $y(x) = x$ is a particular solution.

The substitution $v(x) = y(x)/x$ turns the equation into

$$-\frac{\mathrm{d}v}{\sqrt{1-v}} = \frac{\mathrm{d}x}{x},$$

which has the solution $2\sqrt{1-v} = C + \log x$, i.e., $v(x) = 1 - \frac{1}{4}(C + \log x)^2$ for $x > 0$.

Hence, $y(x) = x(1 - \frac{1}{4}(C + \log|x|)^2)$ is the general solution for $x > 0$ and for $x < 0$. Note that this solution is not complete since the particular solution $y(x) = x$ does not belong to this family. Furthermore, the (quite special) IVP $y' = \frac{y}{x} - \sqrt{1 - \frac{y}{x}}$ with $y(x_0) = x_0$ is not uniquely solvable, since $y(x) = x$ and $y(x) = x(1 - \frac{1}{4}(C_0 + \log|x|)^2)$ with $C_0 = -\log|x_0|$ are solutions. $\diamond$

2.5.3 Bernoulli equation

The *Bernoulli differential equation* is of the form

$$y' + a(x)y = b(x)y^\alpha, \tag{2.5.3}$$

where a and b are continuous functions and $\alpha \in \mathbb{R} \setminus \{0,1\}$ is a parameter. (The cases $\alpha = 0$ or $\alpha = 1$ are just linear equations of type (2.3.1).) We multiply (2.5.3) with $(1-\alpha)y^{-\alpha}$ and substitute

$$\eta(x) = y(x)^{1-\alpha}.$$

Then we have $\eta'(x) = (1-\alpha)y^{-\alpha}y'$, and hence (2.5.3) turns into the linear equation

$$\eta' + (1-\alpha)a(x)\eta = (1-\alpha)b(x).$$

Now (2.5.3) is easily solved by solving this equation and re-substituting $y(x) = \eta(x)^{\frac{1}{1-\alpha}}$.

Example 2.5.3 (Harmonic oscillator with air resistance). We look at a movement $x = x(t)$ with air resistance, i.e., the acceleration is proportional to the square of the velocity:

$$x'' + \omega^2 x + r x'|x'| = 0. \tag{2.5.4}$$

where ω^2 and r are two positive parameters. For the velocity, we make the ansatz as a function of the place: $x'(t) = v(x(t))$. Then, according to the chain rule, $x''(t) = v'(x(t))x'(t) = v'(x)v(x)$. Hence, (2.5.4) is equivalent to $v'v + \omega^2 x + rv|v| = 0$, i.e., after dividing by v,

$$v' + \sigma r v = -\omega^2 x v^{-1}, \tag{2.5.5}$$

where $\sigma = |v|/v$. (Observe that σ is just the sign of v.) Now, (2.5.5) is a Bernoulli equation with $\alpha = -1$ and $a(x) = \sigma r$ and $b(x) = -\omega^2 x$. The substitution $\eta(x) = v(x)^2$ leads to the linear equation $\eta' + 2\sigma r\eta = -2\omega^2 x$, whose general solution is

$$\eta(x) = -\frac{\omega^2}{\sigma r}x + \frac{\omega^2}{2r^2} + c\mathrm{e}^{-2r\sigma x}.$$

The phase curves, i.e., the curves $t \mapsto \binom{x(t)}{x'(t)}$, are spirals. Indeed, re-substituting $v = \sqrt{\eta}$, we see that

$$x'(t) = v(x(t)) = \sqrt{\eta(x(t))} = \begin{cases} \sqrt{c_1\mathrm{e}^{-2rx} - \frac{\omega^2}{r}x + \frac{\omega^2}{2r^2}} & \text{if } v > 0, \\ \sqrt{c_2\mathrm{e}^{2rx} + \frac{\omega^2}{r}x + \frac{\omega^2}{2r^2}} & \text{if } v < 0. \end{cases}$$

We obtain the spiral by concatenating the respective pieces at the times at which $v(x(t)) = 0$. $\diamond$

2.5.4 The Riccati equation

This is an equation of the type

$$y' = a(x)y + b(x)y^2 + f(x). \tag{2.5.6}$$

This type appears in the description of growth models with an external influence (via f). (In the special case $a > 0$ and $b < 0$, we mentioned this equation in Example 1.1.2 as a model for growth in a bounded region.)

In general, there is no systematic explicit method to solve (2.5.6). However, if a particular solution y_0 is known, then the substitution

$$v(x) = \frac{1}{y(x) - y_0(x)}, \qquad \text{i.e.,} \quad y(x) = y_0(x) + \frac{1}{v(x)},$$

turns (2.5.6) into the linear equation

$$v' + \big(a(x) + 2y_0(x)b(x)\big)v + b(x) = 0. \tag{2.5.7}$$

Hence, for v the general solution to (2.5.7), the general solution to (2.5.6) is given as $y = y_0 + \frac{1}{v}$.

Example 2.5.4. $y' = 4x^2y - xy^2 + 4$ possesses the particular solution $y_0(x) = 4x$. For $v(x) = 1/(y(x) - 4x)$, we obtain $v' - 4x^2v = x$. Using the methods from Section 2.3, we easily obtain that

$$v(x) = \mathrm{e}^{\frac{4}{3}x^3}\Big(\int_{x_0}^{x} \mathrm{e}^{-\frac{4}{3}\zeta^3}\zeta\,\mathrm{d}\zeta + C\Big).$$

The general solution to $y' = 4x^2y - xy^2 + 4$ is hence $y(x) = 4x + \frac{1}{v(x)}$. ◇

Example 2.5.5. $x' = x - x^2 + t^2 - t - 1$ has the particular solution $x(t) = 1 - t$. Substituting $v(t) = 1/(x(t)+t-1)$, we obtain the linear equation $v' + (2t-1)v - 1 = 0$, which has the general solution

$$v(t) = \mathrm{e}^{t-t^2}\Big(\int_{t_0}^{t} \mathrm{e}^{\tau^2-\tau}\,\mathrm{d}\tau + C\Big).$$

Hence, the general solution is $x(t) = 1 - t + \frac{1}{v(t)}$. ◇

2.6 Solution by inversion of y'

Consider a general first-order ODE, $F(x, y, y') = 0$. If it is not possible to find a one-parameter family $y = y_c(x)$ of solutions, and if also a representation as level lines, $U(x, y) = C$, fails, then one could try to represent a solution in the form of a parameter description $x = x(p), y = y(p)$, where p is a new parameter. Here we want to consider only the case where y' is throughout positive in the interval considered, such that we can choose $p = y'$. This may be conceived as inversion of the map $x \mapsto y'(x)$, by writing $x = x(y') = x(p)$. Then y is also a function of p, since $y = y(x) = y(x(p))$, which means that we have a parameter description as desired.

After substituting

$$p = y', \qquad x = x(y') = x(p), \qquad y = y(x(p)),$$

we obtain the equation $F(x(p), y(p), p) = 0$, for a range of parameter values of p. Writing $\dot{x} = \frac{\mathrm{d}}{\mathrm{d}p}x$, we have to find the two functions $x(p)$ and $y(p)$ from the equations

$$0 = F_x\dot{x} + F_y\dot{y} + F_p, \qquad \dot{y} = p\dot{x}, \tag{2.6.1}$$

where the first equation comes from differentiating $F(x(p), y(p), p) = 0$ with respect to p, and the last equation comes from the chain rule: $\dot{y} = \frac{\mathrm{d}}{\mathrm{d}p}y(x(p)) = \dot{y}(p)\dot{x}(p) = p\dot{x}$. We can easily solve (2.6.1) to the effect that

$$\dot{x} = -\frac{F_p}{F_x + pF_y}, \qquad \dot{y} = -\frac{pF_p}{F_x + pF_y}. \tag{2.6.2}$$

Once we have found the solution curves $x(p)$ and $y(p)$ via (2.6.2), we can try to eliminate the auxiliary parameter p by a combination of the two equations for $x(p)$ and $y(p)$ and obtain (hopefully) an equation for y in terms of x.

However, (2.6.2) can be solved only in certain cases. We want to discuss three particular ones, in which x and y appear only linearly.

2.6.1 $x = f(y')$

In this case, $F(x, y, y') = f(y') - x$, and we have $\dot{x} = -\frac{F_p}{F_x + pF_y} = F_p = \dot{f}(p)$ and $\dot{y} = -\frac{pF_p}{F_x + pF_y} = p\dot{f}(p)$ from (2.6.2). Intergration gives $x(p) = f(p)$ and $y(p) = \int p\dot{f}(p)\,\mathrm{d}p$.

Example 2.6.1. Consider $x(1 + (y')^2) = 1$. Since the term in the outer brackets is larger than 1, we have solutions only in $x \in (0, 1]$. With $p = y'$ we obtain the solution curves

$$x(p) = \frac{1}{1+p^2}, \qquad y(p) = \int p\dot{x}(p)\,\mathrm{d}p = px(p) - \int \frac{1}{1+p^2}\,\mathrm{d}p = \frac{p}{1+p^2} - \arctan p + C.$$

We can express p in terms of x as $p = \pm\sqrt{(1-x)/x}$ and have therefore

$$y(x) = C \pm \left(\sqrt{x(1-x)} - \arctan\sqrt{\frac{1-x}{x}}\right), \qquad x \in (0, 1].$$

◇

2.6.2 $y = f(y')$

Here we obtain from (2.6.2) that $x(p) = \int \frac{f'(p)}{p}\,\mathrm{d}p$ and $y(p) = f(p)$.

2.6.3 d'Alembert's equation, $y = xf(y') + g(y')$

Hence, $F(x, y, p) = xf(p) + g(p) - y$. If $f(p)$ is not the identical function, i.e., $f(p) \not\equiv p$, then the first equation in (2.6.2) gives the linear first-order equation

$$(f(p) - p)\dot{x} + f'x = -g'.$$

In the other case, i.e., $f(p) \equiv p$, we have the *Clairaut differential equation*, $y = xy' + g(y')$. A general solution to this equation is the family of straight lines $y(x) = cx + g(c)$ with $c \in \mathbb{R}$. A singular solution, which satisfies $F(x(p), y(p), p) = 0$, is given by

$$x(p) = -g'(p), \qquad y(p) = -pg'(p) + g(p).$$

This solution is in many cases equal to the enveloping curve of the above straight lines (see, e.g., Example 1.1.12).

Example 2.6.2. Consider $y = (x + 1)(y')^2$, which is of d'Alembert type. It has the singular solution $y = 0$. The first equation in (2.6.2) is the linear equation $\dot{x}(p) = \frac{2x(p)+2}{1-p}$, which has the general solution $x(p) = -1 + C(1-p)^{-2}$. This we can easily solve for $y' = p = 1 \pm \sqrt{C/(x+1)}$. (In the case $C > 0$ only for $x > -1$, and in the case $C < 0$ only for $x < -1$.) We obtain explicitly $y(x) = x + 1 \pm 2\sqrt{C(x+1)} + C_2$ for some integration constant C_2. Checking with the equation $y = (x+1)(y')^2$ shows that $C_2 = 0$, since

$$(x+1)(y')^2 = (1+x)\Big(1 \pm \sqrt{\frac{C}{1+x}}\Big)^2 = 1 + x \pm 2\sqrt{C(1+x)}.$$

Hence, the explicit general solution is $y(x) = x + 1 \pm 2\sqrt{C(x+1)}$. This solution may also be expressed implicitly by the equation

$$(x + 1 - y - C)^2 = 4Cy,$$

as is shows with the help of $y' = p = 1 \pm \sqrt{C/(x+1)}$ and $y = (x+1)(y')^2$. This is the family of parabolas which touch the straight lines $y = 0$ and $x = -1$ and are opened in the $\binom{1}{1}$ resp. $\binom{-1}{-1}$ direction. $\diamond$

Chapter 3

Linear second-order ODEs

This type of differential equations appears in the description of mechanical and electrical oscillations:

$$y'' + ay' + by = f(x), \tag{3.0.1}$$

where $a, b \in \mathbb{R}$ are constants, and f is a continuous function. (The case of non-constant coefficient functions $a(x), b(x)$ is much more difficult and will not be considered in this lecture.) As in the case of linear first-order equations in Section 2.3, we call (3.0.1) *homogeneous* if $f \equiv 0$ and *inhomogeneous* otherwise. There are some aspects that are the same as in the first-order case, some are analogous, and some aspects will be new.

3.1 Solution of the homogeneous equation

Remark 3.1.1 (Principle of super position). This principle is the same as in the first-order case, see Remark 2.3.7. Indeed, if y_1, y_2 are solutions of $y_i'' + ay_i' + by_i = f_i$ for two functions f_1 resp. f_2, then $y = \alpha y_1 + \beta f_2$ is a solution of $y'' + ay' + by = \alpha f_1 + \beta f_2$, for any $\alpha, \beta \in \mathbb{R}$. We already know that this observation may be helpful when the right-hand side f is a sum of several functions of different type. $\diamond$

We can also use the principle of super position to find the general structure of the set of solutions to the homogeneous equation. In order to express it properly, we need some few notions from Linear Algebra.

Definition 3.1.2 (Linear independence). *A set of n functions, $\{y_1, \ldots, y_n\}$, is called* linearly independent *over an interval $I \subset \mathbb{R}$ if for any $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ the following implication holds:*

$$\sum_{i=1}^{n} \lambda_i y_i = 0 \qquad \Longrightarrow \qquad \lambda_1 = \cdots = \lambda_n = 0.$$

(Here we mean by $\sum_{i=1}^{n} \lambda_i y_i = 0$ that $\sum_{i=1}^{n} \lambda_i y_i(x) = 0$ for any $x \in I$.) Hence, the only way to express the function that is identical to zero as a linear combination of the $y_1, \ldots, y_n$ is the trivial one. Two functions y_1, y_2 are obviously linearly independent if it is not possible to write y_1 as a constant times y_2 or vice versa.

Lemma 3.1.3 (Solution of a homogeneous 2nd order linear ODE). *For $a, b \in \mathbb{R}$, the general and complete solution of the homogeneous 2nd order linear ODE $y'' + ay' + by = 0$ is always of the form $\lambda_1 y_1 + \lambda_2 y_2$, where y_1, y_2 are any two linearly independent solutions of this equation.*

In other words, it is always possible to find two linearly independent solutions, and every solution is then a linear combination of these two. Even better, for *any* pair of linearly independent solutions, every solution is a linear combination of these two. In the language of Linear Algebra, the set of solutions is a *two-dimensional vector space*,and for any two solutions y_1, y_2, the set $\{y_1, y_2\}$ is a *basis* of this set. In the following we give a proof which also explains how to find linearly independent solutions (see (3.1.2)–(3.1.4)) and what cases may occur.

Sketch of proof of Lemma 3.1.3. As in Section 2.3, we write y_h for the solution of the homogeneous equation $y'' + ay' + by = 0$. We will explain how to find two linearly independent solutions. The proof that this is the complete solution requires some means from the theory of n-th order systems of differential equations which we do not treat here.

We make an exponential ansatz, i.e., $y_\mathrm{h}(x) = \mathrm{e}^{\lambda x}$. Then

$$y_\mathrm{h}''(x) + ay_\mathrm{h}'(x) + by_\mathrm{h}(x) = \big(\lambda^2 + a\lambda + b\big)\mathrm{e}^{\lambda x}.$$

This is obviously identical to zero if and only if the

$$\text{characteristic equation} \qquad \lambda^2 + a\lambda + b = 0 \tag{3.1.1}$$

is satisfied, i.e., $\lambda = -\frac{a}{2} \pm \frac{1}{2}\sqrt{a^2 - 4b}$. We distinguish three cases.

1st case: $a^2 > 4b$. Then we have found two real solutions $\lambda_1 \neq \lambda_2$ and have therefore two solutions

$$y_{\mathrm{h}1}(x) = \mathrm{e}^{\lambda_1 x} \qquad \text{and} \qquad y_{\mathrm{h}2}(x) = \mathrm{e}^{\lambda_2 x}. \tag{3.1.2}$$

It is easy to see that these two are linearly independent.

2nd case: $a^2 = 4b$. Here we have just one solution, $\lambda = -\frac{a}{2}$, which is a double zero. Certainly, $y_{\mathrm{h}1}(x) = \mathrm{e}^{\lambda x}$ is a solution, but we need a second one. This is provided by the method of variation of constants: We make the ansatz $y(x) = c(x)y_{\mathrm{h}1}(x)$ and go into $y'' + ay' + by = 0$ to obtain, using that $y_{\mathrm{h}1}'' + ay_{\mathrm{h}1}' + by_{\mathrm{h}1} = 0$, that

$$\begin{aligned} 0 &= (cy_{\mathrm{h}1})'' + a(cy_{\mathrm{h}1})' + bcy_{\mathrm{h}1} = c''y_{\mathrm{h}1} + 2c'y_{\mathrm{h}1}' + cy_{\mathrm{h}1}'' + a(c'y_{\mathrm{h}1} + cy_{\mathrm{h}1}') + bcy_{\mathrm{h}1} = (c'' + ac')y_{\mathrm{h}1} + 2c'y_{\mathrm{h}1}' \\ &= c''y_{\mathrm{h}1}, \end{aligned}$$

where we also used that $\lambda = -\frac{a}{2}$. Since $y_{\mathrm{h}1}$ is never zero, we have that $c'' \equiv 0$, and hence c is a polynomial of first order. This means that we may put $y_{\mathrm{h}2}(x) = xy_{\mathrm{h}1}(x) = x\mathrm{e}^{\lambda x}$. Hence,

$$y_{\mathrm{h}1}(x) = \mathrm{e}^{-\frac{a}{2}x} \qquad \text{and} \qquad y_{\mathrm{h}2}(x) = x\mathrm{e}^{-\frac{a}{2}x} \tag{3.1.3}$$

are two linearly independent solutions.

3rd case: $a^2 < 4b$. Here we do not have any real solution, but only two complex solutions, $\lambda_1 = -\frac{a}{2} + \mathrm{i}\frac{1}{2}\sqrt{|a^2 - 4b|}$ and $\lambda_1 = -\frac{a}{2} - \mathrm{i}\frac{1}{2}\sqrt{|a^2 - 4b|}$, which are conjugate to each other. Now we use the complex exponential function, more precisely the famous Euler formula, to write

$$y_{\mathrm{h}i}(x) = \mathrm{e}^{\lambda_i x} = \mathrm{e}^{-\frac{a}{2}x}\,\mathrm{e}^{\pm\mathrm{i}\frac{1}{2}\sqrt{|a^2-4b|}x} = \mathrm{e}^{-\frac{a}{2}x}\Big[\cos\Big(\frac{1}{2}\sqrt{|a^2 - 4b|}\,x\Big) \pm \sin\Big(\frac{1}{2}\sqrt{|a^2 - 4b|}\,x\Big)\Big].$$

One can easily check that both, real part and imaginary part, are solutions to $y'' + ay' + by = 0$, and that they are indeed linearly independent. Hence, we have found the two solutions

$$y_{\mathrm{h}1}(x) = \mathrm{e}^{-\frac{a}{2}x} \cos\left(\frac{1}{2}\sqrt{|a^2-4b|}\, x\right) \qquad \text{and} \qquad y_{\mathrm{h}2}(x) = \mathrm{e}^{-\frac{a}{2}x} \sin\left(\frac{1}{2}\sqrt{|a^2-4b|}\, x\right). \tag{3.1.4}$$

□

Example 3.1.4. $y'' - 6y' + 5y = 0$. The characteristic equation is $\lambda^2 - 6\lambda + 5 = 0$, which has the two zeros $\lambda_1 = 1$ and $\lambda_2 = 5$. According to (3.1.2), a solution basis is formed by $y_1(x) = \mathrm{e}^x$ and $y_2(x) = \mathrm{e}^{5x}$. ◇

Example 3.1.5. $y'' - 6y' + 34y = 0$. The characteristic equation is $\lambda^2 - 6\lambda + 34 = 0$, which has the two zeros $\lambda_1 = 3 + 5\mathrm{i}$ and $\lambda_2 = 3 - 5\mathrm{i}$. According to (3.1.4), a solution basis is formed by $y_1(x) = \mathrm{e}^{3x}\cos(5x)$ and $y_2(x) = \mathrm{e}^{3x}\sin(5x)$. ◇

3.2 Solution of the inhomogeneous equation

Now we turn to the inhomogeneous equation in (3.0.1). With the help of Lemma 3.1.3, it is easy to find the structure of its general complete solution:

Lemma 3.2.1 (Complete solution of (3.0.1)). *The general complete solution to the linear second-order equation* (3.0.1) *is* $y = y_{\mathrm{p}} + c_1 y_{\mathrm{h}1} + c_2 y_{\mathrm{h}2}$*, where* y_{p} *is any particular solution, and* $y_{\mathrm{h}1}$ *and* $y_{\mathrm{h}2}$ *are any two linearly independent solutions to the homogeneous equation* $y'' + ay' + by = 0$*, and* $c_1, c_2 \in \mathbb{R}$.

Proof. According to the principle of superposition in Remark 3.1.1, any such function $y = y_{\mathrm{p}} + c_1 y_{\mathrm{h}1} + c_2 y_{\mathrm{h}2}$ is a solution to (3.0.1). On the other hand, if y is a solution, then $y - y_{\mathrm{p}}$ is a solution of the homogeneous equation and therefore, according to Lemma 3.1.3, of the form $c_1 y_{\mathrm{h}1} + c_2 y_{\mathrm{h}2}$. □

Certainly, we want to know how to derive a particular solution, y_{p}, of (3.0.1). In general, this is much more difficult than finding the homogeneous solutions, but there are some systematic methods that one can try. We assume that $\{y_1, y_2\}$, a basis of the solutions to $y'' + ay' + by = 0$, is already known.

First method: variation of the constants. With two auxiliary functions c_1, c_2, we make the ansatz $y(x) = c_1(x)y_1(x) + c_2(x)y_2(x)$. Using that y_1 and y_2 solve the homogeneous equation, and adding the auxiliary equation $c_1'y_1 + c_2'y_2 = 0$, we arrive at the system

$$c_1'(x)y_1(x) + c_2'(x)y_2(x) = 0, \tag{3.2.1}$$
$$c_1'(x)y_1'(x) + c_2'(x)y_2'(x) = f(x). \tag{3.2.2}$$

This is a linear 2×2-system and can easily be solved:

$$c_1'(x) = -\frac{y_2(x)f(x)}{W(x)}, \qquad c_2'(x) = \frac{y_1(x)f(x)}{W(x)}, \qquad \text{where} \quad W = y_1y_2' - y_2y_1'. \tag{3.2.3}$$

Now a direct integration yields c_1 and c_2, and we obtain $y = c_1y_1 + c_2y_2$. ◇

Second method: ansatz of the type of the right-hand side. If $f(x)$ is of the form $f(x) = p_m(x)\mathrm{e}^{wx}$, where $w \in \mathbb{R}$, and p_m is a polynomial of order m, then one can skip some of the calculations that are necessary in the method of variation of the constants by making the following ansatz. We call $\chi(\lambda) = \lambda^2 + a\lambda + b$ the characteristic polynomial. We put

$$y_{\mathrm{p}}(x) = \begin{cases} P_m(x)\mathrm{e}^{wx} & \text{if } \chi(w) \neq 0, \\ xP_m(x)\mathrm{e}^{wx} & \text{if } \chi(w) = 0, \chi'(w) \neq 0, \\ x^2 P_m(x)\mathrm{e}^{wx} & \text{if } \chi(w) = \chi'(w) = 0, \end{cases} \tag{3.2.4}$$

where $P_m(x)$ is a polynomial of order m. In the first case, $x \mapsto \mathrm{e}^{wx}$ is not a homogeneous solution, in the second it is, but χ possesses two different zeros (case 1, see (3.1.2)), and in the third it is, and w is a zero of multiplicity two of χ (case 2, see (3.1.3)).

Analogously, if $f(x) = p_m(x)\cos(wx)$ or $f(x) = p_m(x)\sin(wx)$, where $w \in \mathbb{R}$, and p_m is a polynomial of order m, then the ansatz is $y_{\mathrm{p}}(x) = x^k P_m(x)(A\cos(wx) + B\sin(wx))$, where $k \in \mathbb{N}_0$ is the multiplicity of the zero $\mathrm{i}w$ (note the i!) of χ. (Here we say that $\mathrm{i}w$ has multiplicity zero if $\chi(\mathrm{i}w) \neq 0$.) The same ansatz may be used if f is a sum of two such functions, i.e., $f(x) = p_m(x)\cos(wx) + q_m(x)\sin(wx)$ with polynomials $p_m(x)$ and $q_m(x)$. $\diamond$

Third method: power series. We assume that the right-hand side admits a power series expansion $f(x) = \sum_{n=0}^{\infty} c_n(x - x_0)^n$ which converges locally uniformly in a neighborhood of some x_0. Then we make the ansatz that also the solution, $y(x) = \sum_{n=0}^{\infty} d_n(x - x_0)^n$, is a power series in that neighborhood. Clearly, $y'(x) = \sum_{n=0}^{\infty}(n+1)d_{n+1}(x - x_0)^n$ and $y''(x) = \sum_{n=0}^{\infty}(n+2)(n+1)d_{n+2}(x-x_0)^n$. Substituting all these series in the equation $f(x) = y''+ay'+by$, we obtain

$$\sum_{n=0}^{\infty} c_n(x - x_0)^n = \sum_{n=0}^{\infty} \big[bd_n + a(n+1)d_{n+1} + (n+2)(n+1)d_{n+2}\big](x - x_0)^n.$$

A comparison of the coefficients yields the infinite system of linear equations

$$\text{for all } n \in \mathbb{N}_0: \qquad c_n = bd_n + a(n+1)d_{n+1} + (n+2)(n+1)d_{n+2}. \tag{3.2.5}$$

The first of these equations is $c_0 = bd_0 + ad_1 + 2d_2$, the second is $c_1 = bd_1 + 2ad_2 + 6d_3$ and so on. One can choose two initial values for d_0 and d_1 and calculate iteratively d_2 from the first equation, d_3 from the second and so on. In lucky cases, one is able to systematically determine a solution $d_0, d_1, d_2, \dots$ from this system of equations, and in even more lucky cases, one can identify also the solution, $y(x) = \sum_{n=0}^{\infty} d_n(x - x_0)^n$, for these values of the d_n. $\diamond$

Fourth method: Laplace transform. See Section 3.4. $\diamond$

3.3 Examples

Example 3.3.1. Consider $y'' - 4y' + 4y = 7$. The characteristic equation, $\lambda^2 - 4\lambda + 4\lambda = 0$, has a two-fold solution, $\lambda = 2$. Hence, the general solution of the homogeneous equation is $(c_1 + c_2x)\mathrm{e}^{2x}$ with $c_1, c_2 \in \mathbb{R}$. For finding the solution of the inhomogeneous equation, we make the ansatz of the type of the right-hand side: $y_{\mathrm{p}}(x) = c$. Substituting this, we obtain $c = \frac{7}{4}$. Hence, the general and complete solution is $y(x) = \frac{7}{4} + (c_1 + c_2x)\mathrm{e}^{2x}$.

For the sake of comparison, we solve the inhomogeneous equation once more from scratch with the method of variation of constants. The ansatz is then

$$c_1'(x)e^{2x} + c_2'(x)xe^{2x} = 0, \qquad c_1'(x)2e^{2x} + c_2'(x)(1+2x)e^{2x} = 7.$$

From the first of the two equations, we get $c_1'(x) = -xc_2'(x)$. Using this in the second, we see that $c_2'(x) = 7e^{-2x}$, i.e., $c_2(x) = -\frac{7}{2}e^{-2x}$. This in turn gives that $c_1'(x) = \frac{7}{2}xe^{-2x}$, i.e., $c_1(x) = \frac{7}{2}xe^{-2x} + \frac{7}{4}e^{-2x}$. Hence, the particular solution is $y_p(x) = c_1(x)e^{2x} + c_2(x)xe^{2x} = \frac{7}{4}$. ◇

Example 3.3.2. We search for all *periodic* solutions to the equation $y'' - 7y' + 6y = \sin x$. The characteristic equation, $\lambda^2 - 7\lambda + 6 = 0$, has the solutions 6 and 1. Hence, all the homogeneous solutions, e^{6x} and e^x, are not periodic. We will obtain a particular solution using the ansatz of the type of the right-hand side: $y_p(x) = A\cos x + B\sin x$. Substituting this in the equation, we obtain

$$\sin x = \big(-A - 7B + 6A\big)\cos x + \big(-B + 7A + 6B\big)\sin x.$$

It is clear that the two functions cos and sin are linearly independent, since none can be written as a constant multiple of the other. Hence, a comparison of the two coefficients gives $-A-7B+6A = 0$ and $-B+7A+6B = 1$, which enforces that $A = 7/74$ and $B = 5/74$. Hence, the only periodic solution is $y(x) = \frac{1}{74}(7\cos x + 5\sin x)$. ◇

Example 3.3.3. Consider $y'' + 4y = x^2 + 5\cos(2x)$. The characteristic equation, $\lambda^2 + 4\lambda = 0$, has no real solution, but only the solution $\pm 2i$. Hence, $\cos(2x)$ and $\sin(2x)$ form a basis of the homogeneous solution space. To solve the equation, we use the super position principle and split the tasks into two.

First the equation $y'' + 4y = x^2$. The correct ansatz of the type of the right-hand side is $y_{p1}(x) = a + bx + cx^2$. Substituting this, we obtain $x^2 = 2c + 4(a + bx + cx^2)$, and a comparison of the coefficients yields $2c + 4a = 0$, $4b = 0$ and $4c = 1$, i.e., $y_{p1}(x) = -\frac{1}{8} + \frac{1}{4}x^2$.

Now the equation $y'' + 4y = 5\cos(2x)$. The right-hand side is one of the basis solutions, in other words, we are in the case of the second line of (3.2.4). Hence, the ansatz is $y_{p2}(x) = x(A\sin(2x) + B\cos(2x))$. Substituting this and $y_{p2}''(x) = 4(A - Bx)\cos(2x) - 4(B + 4A)\sin(2x)$ into the equation, we get $5\cos(2x) = 4A\cos(2x) - 4B\sin(2x)$ and hence $B = 0$ and $A = \frac{5}{4}$. This gives $y_{p2}(x) = \frac{5}{4}x\sin(2x)$.

Summarizing, the general and complete solution of $y'' + 4y = x^2 + 5\cos(2x)$ is $y(x) = -\frac{1}{8} + \frac{1}{4}x^2 + \frac{5}{4}x\sin(2x) + c_1\cos(2x) + c_2\sin(2x)$. ◇

Example 3.3.4. Consider the IVP

$$y'' - 4y' + 5y = e^{2x}\tan x, \qquad y(0) = 0, y'(0) = 0.$$

The characteristic equation, $\lambda^2 - 4\lambda + 5 = 0$, has the two solutions $2 \pm i$. Hence, a basis of the homogeneous equation is given by $y_1(x) = e^{2x}\cos x$ and $y_2(x) = e^{2x}\sin x$. We use the method of variation of the constants to solve the inhomogeneous equation. The ansatz is

$$c_1'(x)e^{2x}\cos x + c_2'(x)e^{2x}\sin x = 0,$$
$$c_1'(x)(2\cos x - \sin x)e^{2x} + c_2'(x)(2\sin x + \cos x)e^{2x} = e^{2x}\tan x.$$

From the first equation, we get $c_1'(x) = -c_2'(x)\tan x$. Substituting this in the second and using that $\sin^2 + \cos^2 = 1$, we obtain $c_2'(x) = \sin x$, i.e., $c_2(x) = -\cos x$. This gives that $c_1'(x) = -\frac{\sin^2 x}{\cos x}$

and therefore, making the substitution $u = \sin x$,

$$c_1(x) = \int \Big(\cos x - \frac{1}{\cos x}\Big)\,\mathrm{d}x = \sin x - \int \frac{\mathrm{d}u}{1-u^2} = \sin x - \frac{1}{2}\log\frac{1+\sin x}{1-\sin x}.$$

This means that the general solution is

$$y(x) = \Big(\sin x - \frac{1}{2}\log\frac{1+\sin x}{1-\sin x}\Big)\mathrm{e}^{2x}\cos x - \cos x\mathrm{e}^{2x}\sin x + c_1\mathrm{e}^{2x}\cos x + c_2\mathrm{e}^{2x}\sin x. \tag{3.3.1}$$

We want to determine $c_1, c_2 \in \mathbb{R}$ such that $y(0) = 0$ and $y'(0) = 0$. It is clear that the first equation means that $c_1 = 0$. The second equation leads to

$$0 = \frac{\mathrm{d}}{\mathrm{d}x}\Big|_{x=0}\Big(\sin x - \frac{1}{2}\log\frac{1+\sin x}{1-\sin x}\Big) - 1 + c_2 = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\Big|_{x=0}\log\frac{1+\sin x}{1-\sin x} + c_2 = 1 - c_2.$$

Hence, the solution is the function $y(x)$ in (3.3.1) with $c_1 = 0$ and $c_2 = 1$. ◇

Example 3.3.5. Consider $x''(t) + 2\alpha x'(t) + \omega_0^2 x(t) = \cos(\omega t)$, where $\alpha, \omega_0, \omega \in \mathbb{R}\setminus\{0\}$ are parameters satisfying $\alpha^2 < \omega_0^2$ and $\omega \neq 0$. We are in case (3) and easily see that two basis solutions of the homogeneous equation are given by $t \mapsto \cos(\omega_1 t)$ and $t \mapsto \sin(\omega_1 t)$, where $\omega_1^2 = \omega_0^2 - \alpha^2$. To solve the inhomogeneous equation, we make the ansatz of the type of the right hand side,

$$x_{\mathrm{p}}(t) = A\cos(\omega t) + B\sin(\omega t),$$

and substitute this in the equation, to obtain

$$\cos(\omega t) = \Big(-A\omega^2 + 2\alpha B\omega + A\omega_0^2\Big)\cos(\omega t) + \Big(-B\omega^2 - 2\alpha A\omega + B\omega_0^2\Big)\sin(\omega t).$$

A comparison of the coefficients gives the two equations

$$-A\omega^2 + 2\alpha B\omega + A\omega_0^2 = 1 \qquad \text{and} \qquad -B\omega^2 - 2\alpha A\omega + B\omega_0^2 = 0,$$

which is easily solved to the effect that

$$A = V^2(\omega_0^2 - \omega^2) \qquad \text{and} \qquad B = V^2 2\alpha\omega, \qquad \text{where } V = \big(4\alpha^2\omega^2 + (\omega_0^2 - \omega^2)^2\big)^{-1/2}.$$

Using the addition theorems for cosine, we obtain the solution

$$x_{\mathrm{p}}(t) = V^2\Big((\omega_0^2 - \omega^2)\cos(\omega t) + 2\alpha\omega\sin(\omega t)\Big) = V\cos(\omega t + \varphi), \qquad \text{where } \varphi = \arctan\frac{2\alpha\omega}{\omega_0^2 - \omega^2}.$$

◇

Example 3.3.6 (Damped oscillation). A spring is stimulated by an external force $F(t)$, which creates a movement $x = x(t)$. We assume that the force that brings the spring into its original form is proportional to the distance $x(t)$, and the damping is proportional to the velocity $x'(t)$. Then the movement is described by the equation

$$x''(t) + 2\alpha x'(t) + \omega_0^2 x(t) = F(t), \qquad t \in [0, \infty),$$

where $\alpha \geq 0$ is a damping coefficient describing the elasticity of the spring, and $\omega_0 > 0$ is a parameter. If the system oscillates freely, then we have the homogeneous equation with $F \equiv 0$, i.e., we consider the eigen oscillations of the spring. We have the three cases appearing in the proof of Lemma 3.1.3: If $\alpha^2 > \omega_0^2$, then we have a very strong damping. The general solution, $x_{\rm h}(t) = c_1 {\rm e}^{(-\alpha+\beta)t} + c_2 {\rm e}^{(-\alpha-\beta)t}$ (where $\beta = \sqrt{\alpha^2 - \omega_0^2}$), decays exponentially to zero at late times, and the movement crosses the x-axis only once. In the second case $\beta = 0$, the solution has the form $x_{\rm h}(t) = (c_1 + c_2 t){\rm e}^{-\alpha t}$, and the qualitative behavior is basically the same. However, in the case of weak damping, $\alpha^2 < \omega_0^2$, we have periodic oscillations with the *eigen frequency* of the spring, $\omega_1^2 = \omega_0^2 - \alpha^2$, and the solutions may be written (using the addition theorems for the cosine of a sum of angles)

$$x_{\rm h}(t) = {\rm e}^{-\alpha t}\big(c_1 \cos(\omega_1 t) + c_2 \sin(\omega_1 t)\big) = C{\rm e}^{-\alpha t}\cos(\omega_1 t - \delta),$$

where $C, \delta \in \mathbb{R}$ are suitable coefficients. The amplitude is damped by the leading term, ${\rm e}^{-\alpha t}$, which lets the solution decay to zero exponentially if $\alpha > 0$, but in an periodically oscillating way. For zero damping, $\alpha = 0$, there is even no decay, and we just have harmonic oscillations, the system oscillates back and forth for ever.

Now we consider a particular type of oscillation with external force: the case of $F(t) = \cos(\omega t)$, where $\omega \in \mathbb{R} \setminus \{0\}$. As is explained in Example 3.3.5, we have the particular solution

$$x_{\rm p}(t) = V(\omega)\cos(\omega t - \varphi), \qquad \text{where} \quad V(\omega) = \big((\omega_0^2 - \omega^2)^2 + 4\alpha^2\omega^2\big)^{-1/2}, \varphi = \arctan\frac{2\alpha\omega}{\omega_0^2 - \omega^2}.$$

This oscillation has the same frequency as the external force, but is in a different phase, if $\alpha > 0$. The amplitude, $V(\omega)$, depends on ω and vanishes if the frequency ω gets large. It is maximal for the so-called *resonance frequency*, $\omega = \omega_* = \sqrt{\omega_0^2 - 2\alpha^2} = \sqrt{\omega_1^2 - \alpha^2}$, which can be reached only for sufficiently small damping, more precisely, only for $\alpha < \omega_0/\sqrt{2}$. Then the amplitude is $V(\omega_*) = 1/(2\alpha\omega_1)$. If the damping α is small, then the resonance frequency is close to the eigen frequency of the spring, and the resulting amplitude is large. This effect is sometimes very welcome for reinforcing reasons, but may result in the famous *resonance catastrophy.* ◇

Example 3.3.7 (Laguerre equation). With some parameter $m \in \mathbb{R}$, we consider

$$xy'' + (1 - x)y' + my = 0.$$

Even if the coefficients are not constant, we make the power series ansatz, $y(x) = \sum_{n=0}^\infty d_n(x - x_0)^n$, and substitute $y'(x) = \sum_{n=1}^\infty n d_n (x - x_0)^{n-1}$ and $y''(x) = \sum_{n=2}^\infty n(n-1)d_n(x - x_0)^{n-2}$, to obtain

$$0 = md_0 + d_1 + \sum_{n=1}^{\infty}\big[m - nd_n + (n+1)^2 d_{n+1}\big]x^n.$$

Hence, all the coefficients on the right-hand side must be equal to zero. We may choose $d_0 \in \mathbb{R}$ freely, say $d_0 = 1$. Iteratively, we obtain the solution

$$d_n = (-1)^n\binom{m}{n}\frac{1}{n!}, \qquad \text{where } \binom{m}{n} = \frac{m(m-1)(m-2)\cdots(m-n+1)}{n!}.$$

It is easy to see that the series $y(x) = \sum_{n=0}^\infty (-1)^n\binom{m}{n}\frac{x^n}{n!}$ converges for any $x \in \mathbb{R}$. This gives a one-parameter family of solutions, cy with $c \in \mathbb{R}$. If $m \in \mathbb{N}$, then the series stops at $n = m$, and we obtain a polynomial, the so-called *Laguerre polynomial*, $L_m(x)$. It is known that

$$L_m(x) = \frac{{\rm e}^x}{m!}\frac{{\rm d}^m}{{\rm d}x^m}\big(x^m{\rm e}^{-x}\big) \qquad \text{and} \qquad \int_0^\infty L_n(x)L_m(x){\rm e}^{-x}\,{\rm d}x = \delta_{nm}.$$

One may wonder why we only have a *one*-parameter family, even though it is an equation of second order. The reason is that the power-series ansatz is good only for solutions that are power series around the origin. In fact, there is a second solution which is not regular at zero. $\diamond$

Example 3.3.8 (Failure of power series ansatz). If we want to expand the solution of

$$x^2y'' + (x^2 - x)y' + 2y = 0$$

into a power series around zero, then we only obtain the trivial solution $y \equiv 0$, since one can iteratively derive that all the derivatives of y at zero are zero. This means that $y \equiv 0$ is the only solution that is regular at the origin. $\diamond$

3.4 Laplace transform

In this section we introduce and apply a certain integral transform that maps a function $f\colon \mathbb{R} \to \mathbb{R}$ on a transformed function $\mathcal{L}f\colon \mathbb{R} \to \mathbb{R}$. The advantages of this transform will be the following: (1) the transform $\mathcal{L}f$ is explicitly known for many important special choices of f, (2) the transform of f' is a simple function of $\mathcal{L}f$, and ODEs look easier after applying the transform, (3) the function f can (at least theoretically) be derived from $\mathcal{L}f$. We will use these nice properties for solving ODEs by first transforming the ODE, solving it for the transforms, and transforming back. For technical reasons, we will first restrict to functions on the domain $[0, \infty)$ instead of $\mathbb{R}$.

3.4.1 Definition and basic properties

Definition 3.4.1. *The* Laplace transform *of a function $f\colon [0, \infty) \to \mathbb{R}$ is the function*

$$\mathcal{L}f(s) = \int_0^\infty \mathrm{e}^{-st} f(t)\,\mathrm{d}t, \qquad s \in \mathbb{R},$$

for any s such that the integral converges.

Certainly, the Laplace transform does not exist for general functions, but in many important cases:

Example 3.4.2. (a) The Laplace transform of the exponential function, $f(t) = \mathrm{e}^{\alpha t}$ for $\alpha \in \mathbb{R}$ is

$$\mathcal{L}f(s) = \int_0^\infty \mathrm{e}^{-st}\mathrm{e}^{\alpha t}\,\mathrm{d}t = \int_0^\infty \mathrm{e}^{-(s-\alpha)t}\,\mathrm{d}t = \frac{1}{s-\alpha}, \qquad s > \alpha.$$

This transform exists only on the interval (α, ∞).

(b) The transform of $f(t) = \cos(\omega t)$ is $\mathcal{L}f(s) = \frac{s}{s^2+\omega^2}$ for $s > 0$, and the one of $f(t) = \sin(\omega t)$ is $\mathcal{L}f(s) = \frac{\omega}{s^2+\omega^2}$ for $s > 0$. This is most elegantly seen as follows. We use the above calculation for $\alpha \in \mathbb{C}$ and obtain that $\int_0^\infty \mathrm{e}^{-st}\mathrm{e}^{\alpha t}\,\mathrm{d}t = \frac{1}{s-\alpha}$ for any $s \in \mathbb{C}$ satisfying $\Re(s-\alpha) > 0$. Now we use this for $\alpha = \mathrm{i}\omega$ and note that $\mathrm{e}^{\mathrm{i}\omega t} = \cos(\omega t) + \mathrm{i}\sin(\omega t)$. Certainly, also the Laplace transform splits in real and imaginary part, and a little calculation shows that

$$\frac{1}{s-\mathrm{i}\omega} = \frac{s}{s^2+\omega^2} + \mathrm{i}\frac{\omega}{s^2+\omega^2}.$$

Hence, the real part, $\frac{s}{s^2+\omega^2}$ is the transform of the real part of $\mathrm{e}^{\mathrm{i}\omega t}$, and the same is true for the imaginary part.

$\diamond$

Here is a general criterion for finiteness of the Laplace transform. Its proof is clear.

Lemma 3.4.3. *If $f\colon [0,\infty) \to \mathbb{R}$ is integrable over any compact set and if there exist constants $M, k \in \mathbb{R}$ such that $|f(t)| \leq M\mathrm{e}^{kt}$ for any $t > 0$, then the Laplace transform $\mathcal{L}f(s)$ exists for any $s > k$.*

Hence, any polynomial and any exponential function and their linear combinations possess a Laplace transform, but not the function $t \mapsto \mathrm{e}^{t^2}$, e.g. A property of the Laplace transform that will be very important for our applications is the following.

Theorem 3.4.4 (Laplace inversion). *If $f\colon [0,\infty) \to \mathbb{R}$ possesses a Laplace transform $\mathcal{L}f(s)$ on a non-trivial interval, then the function f is uniquely determined by $\mathcal{L}f$, up to finitely many points.*

Unfortunately, our means are not sufficient to prove this statement, and it would take too much time to provide them here. Theorem 3.4.4 says that, modulo just finitely many exceptions, any Laplace transformable function and its Laplace transform are uniquely determined by each other. This means that we may, at least theoretically, derive f almost uniquely from its transform $\mathcal{L}f$, i.e., we can obtain by re-transforming $\mathcal{L}f$. However, solving this re-transformation explicitly is difficult for most functions $\mathcal{L}f$, and we will content ourselves with a long list of examples of explicit functions whose re-transform we register.

Lemma 3.4.5 (Basic properties of the Laplace transform). *Let $f, g\colon [0,\infty) \to \mathbb{R}$ be functions such that their Laplace transforms F resp. G exist on some common domain. We also assume that f possesses all properties necessary for stating the following rules (i.e., sufficient differentiability resp. integrability). Then the following Laplace transforms exist on this domain, and the following formulas hold.*

(i) *For any $\alpha, \beta \in \mathbb{R}$, we have $\mathcal{L}(\alpha f + \beta g) = \alpha F + \beta G$ (linearity of the Laplace transform).*

(ii) *For any $c > 0$, the transform of the map $t \mapsto f(ct)$ is the map $s \mapsto \frac{1}{c}F(\frac{s}{c})$.*

(iii) *$\mathcal{L}(f')(s) = sF(s) - f(0+)$, where $f(0+) = \lim_{t\downarrow 0} f(t)$.*

(iv) *For any $n \in \mathbb{N}$, $\mathcal{L}(f^{(n)})(s) = s^n F(s) - s^{n-1} f(0+) - s^{n-2} f'(0+) - \cdots - f^{(n-1)}(0+)$.*

(v) *The Laplace transform of the primitive of f, i.e., of $t \mapsto \int_0^t f(x)\,\mathrm{d}x$, is the map $s \mapsto \frac{1}{s}F(s)$.*

(vi) *For any $n \in \mathbb{N}$, the Laplace transform of $t \mapsto t^n f(t)$ is $(-1)^n F^{(n)}$.*

(vii) *The Laplace transform of $t \mapsto \frac{1}{t} f(t)$ is the map $s \mapsto \int_s^\infty F(u)\,\mathrm{d}u$.*

Proof. (i): clear. (ii): easy

(iii): A partial integration yields that

$$\int_0^\infty \mathrm{e}^{-st} f'(t)\,\mathrm{d}t = \mathrm{e}^{-st} f(t)\Big|_0^\infty + s\int_0^\infty \mathrm{e}^{-st} f'(t)\,\mathrm{d}t = 0 - \lim_{t\downarrow 0} f(t) + sF(s) = sF(s) - f(0+).$$

(iv) is derived with the help of (iii) using an induction.

(v) We have

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-st}\int_0^t f(u)\,\mathrm{d}u\,\mathrm{d}t &= \int_0^\infty \mathrm{d}u\, f(u)\int_u^\infty \mathrm{d}t\,\mathrm{e}^{-st} = \int_0^\infty \mathrm{d}u\, f(u)\Big(-\frac{1}{s}\mathrm{e}^{-st}\Big)\Big|_{t=u}^{t=\infty}\\ &= \int_0^\infty \mathrm{d}u\, f(u)\frac{1}{s}\mathrm{e}^{-us} = \frac{1}{s}F(s).\end{aligned}$$

(vi): It suffices to do this for $n = 1$ only, the general case is done via a simple induction. It suffices to show that, if $t \mapsto tf(t)\mathrm{e}^{-st}$ is integrable, then F is differentiable with $F'(s) = -\int_0^\infty tf(t)\mathrm{e}^{-st}\,\mathrm{d}t$. It is clear that $\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{e}^{-st} = \lim_{n\to\infty} g_n(s,t)$, where $g_n(s,t) = n(\mathrm{e}^{-(s+\frac{1}{n})t} - \mathrm{e}^{-st})$. Furthermore, for any $n \in \mathbb{N}$, we have the bound $|g_n(s,t)| \le |\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{e}^{-st}| = t\mathrm{e}^{-st}$, by the convexity of the map $s \mapsto \mathrm{e}^{-st}$. By assumption, the function $t \mapsto tf(t)\mathrm{e}^{-st}$ is integrable. According to Lebesgue's theorem[1], we have

$$\begin{aligned}F'(s) &= \lim_{n\to\infty} n\big(F(s+\tfrac{1}{n}) - F(s)\big) = \lim_{n\to\infty}\int_0^\infty f(t)g_n(s,t)\,\mathrm{d}t\\ &= \int_0^\infty f(t)\lim_{n\to\infty} g_n(s,t)\,\mathrm{d}t = \int_0^\infty f(t)\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{e}^{-st}\,\mathrm{d}t = -\int_0^\infty tf(t)\mathrm{e}^{-st}\,\mathrm{d}t.\end{aligned}$$

(vii): We invert the assertion (vi): Let H be the Laplace transform of $t \mapsto \frac{1}{t}f(t)$, then (vi) says that the one of f is equal to $-H'$. Hence, $F = -H'$, i.e., $H(s) = \int_s^{s_0} F(u)\,\mathrm{d}u + c$ for some s_0, c. Putting $s = s_0$, we obtain that $c = H(s_0) = \int_0^\infty \frac{1}{t}f(t)\mathrm{e}^{-st}\,\mathrm{d}t$, which converges to zero as $s_0 \to \infty$. Hence, we may put $c = 0$ and $s_0 = \infty$. □

With the help of Lemma 3.4.5, we have already a long list of examples:

Example 3.4.6. (a) the Laplace transform of the polynomial $t \mapsto t^n$ is $s \mapsto \frac{n!}{s^{n+1}}$.

(b) The Laplace transform of $t \mapsto \cosh(\omega t) = \frac{1}{2}(\mathrm{e}^{\omega t} + \mathrm{e}^{-\omega t})$ is $s \mapsto \frac{1}{2}\frac{1}{s-\omega} + \frac{1}{2}\frac{1}{s+\omega} = \frac{s}{s^2-\omega^2}$.

(c) The Laplace transform of $t \mapsto \mathrm{e}^t + 2\mathrm{e}^{3t}$ is $s \mapsto \frac{1}{s-1} + \frac{2}{s-3}\frac{3s-5}{s^2-4s+3}$.

(d) The Laplace transform of $t \mapsto t\sin(\omega t)$ is $s \mapsto -\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}(\sin(\omega t))(s) = \frac{2\omega s}{(s^2+\omega^2)^2}$.

(e) The Laplace transform of $t \mapsto t\mathrm{e}^{2t}$ is $s \mapsto -\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{s-2} = -\frac{1}{(s-2)^2}$.

(f) The Laplace transform of $t \mapsto \frac{1}{t}\sin(\omega t)$ is $s \mapsto \int_s^\infty \frac{\omega}{u^2+\omega^2}\,\mathrm{d}u = \frac{\pi}{2} - \arctan\frac{s}{\omega} = \arctan\frac{\omega}{s}$.

(g) The Laplace transform of $t \mapsto 1 - \cos(at)$ is $s \mapsto \frac{1}{s} - \frac{s}{s^2+a^2} = \frac{a^2}{s(s^2+a^2)}$.

(h) $\mathcal{L}(\sinh)(s) = \frac{1}{s^2-1}$.

(i) The Laplace transform of $t \mapsto \mathrm{e}^{-at}\cos(bt))$ is $s \mapsto \frac{s+a}{(s^2+a^2)+b^2}$.

◇

[1] Lebesgue's theorem says that, if $\lim_{n\to\infty} g_n(t) = g(t)$ for any t and if there is an integrable function h such that $|g_n(t)| \le h(t)$ for any t and any $n \in \mathbb{N}$, then $\lim_{n\to\infty}\int g_n(t)\,\mathrm{d}t = \int g(t)\,\mathrm{d}t$.

3.4.2 Application to 2nd order ODES

Important conclusions from Lemma 3.4.5 are the rules (iii) and (iv), which relate the Laplace transform to the derivative. Taking derivatives is equivalent to an elementary multiplication with a certain polynomial and subtraction of certain limiting values of f at zero. This will be used now for solving the 2nd order ODE of (3.0.1), $y'' + ay' + by = f(x)$. More precisely, we look at the IVP

$$y'' + ay' + by = f(x), \qquad y(0) = y_0, \qquad y'(0) = y_1. \tag{3.4.1}$$

Let $Y = \mathcal{L}y$ denote the Laplace transform of y and F the one of f. With the help of Lemma 3.4.5(iii) and (iv), we can transform this ODE into

$$(s^2 Y(s) - sy_0 - y_1) + a(sY(s) - y_0) + bY(s) = F(s).$$

Noet that the initial values have been already incorporated in this equation. This equation is indeed very simple to solve, and we obtain the explicit solution

$$Y(s) = \frac{F(s)}{s^2 + as + b} + y_0 \frac{s + a}{s^2 + as + b} + y_1 \frac{1}{s^2 + as + b}. \tag{3.4.2}$$

Observe that the latter two functions correspond to the fundamental solutions of the homogeneous equation, actually, they are their Laplace transforms. Furthermore, note that the denominator is equal to the characteristic polynomial of the ODE $y'' + ay' + by = f(x)$, and it is clear that the re-transform of the latter two terms crucially depends on the structure of the zeros of this polynomial.

Now we have to derive the target function y from Y. This is theoretically possible by the virtue of Theorem 3.4.4, but in general a difficult task if one wants to have an explicit expression for y. Usually, one uses a table of explicit functions whose Laplace transforms are explicitly known.

If one has to solve, instead of the IVP in (3.4.1), just some 2nd order ODE $y'' + ay' + by = f(x)$, then one makes the ansatz of an IVP with initial values $y(0) = c_1$ and $y'(0) = c_2$ for constants $c_1, c_2 \in \mathbb{R}$ and obtains a general solution with two parameters.

Example 3.4.7. We search for the general solution to $y'' + 4y = \sin(\omega x)$. Recall from Example 3.4.2 that the Laplace transform of $x \mapsto \sin(\omega x)$ is $F(s) = \frac{\omega}{s^2+\omega^2}$. We can immediately go into (3.4.2) with general initial values $c_1 = y_0$ and $c_2 = y_1$ and obtain, for the Laplace transform Y of y,

$$Y(s) = \frac{1}{s^2 + 4}\Big(\frac{\omega}{s^2 + \omega^2} + sc_1 + c_2\Big) = \frac{\omega}{(s^2 + \omega^2)(s^2 + 4)} + c_1 \frac{s}{s^2 + \omega^2} + c_2 \frac{1}{s^2 + \omega^2}.$$

We know from Example 3.4.2 that the re-transforms of the latter two terms are $x \mapsto c_1 \cos(2x)$ and $x \mapsto \frac{c_2}{2} \sin(2x)$, respectively. Hence, we recover the fundamental solution of the homogeneous equation in the present case, where the characteristic polynomial has two purely imaginary roots. In order to find the re-transform of the first term, we have to distinguish the cases $\omega^2 \neq 4$ and $\omega^2 = 4$. In the first case, we write $\frac{\omega}{(s^2+\omega^2)(s^2+4)}$ as $\frac{A}{s^2+\omega^2} - \frac{A}{s^2+4}$ with an appropriate constant A, and in the second case, we write it in terms of $\frac{1}{s}$ times the derivative of $s \mapsto (s^2 + \omega^2)^{-1}$ (which is $-\frac{2s}{(s^2+\omega^2)^2}$), and use Lemma 3.4.5.

Summarizing, with the help of an appropriate table, or with repeated applications of the rules in Lemma 3.4.5, or just by checking, one obtains that the first term is the Laplace transform of the function

$$x \mapsto \begin{cases} \frac{1}{2(\omega^2-4)}(\omega \sin(2x) - 2\sin(\omega x)), & \text{if } \omega^2 \neq 4, \\ \frac{1}{8}(\sin(2x) - 2x\cos(2x)) & \text{otherwise.} \end{cases}$$

Hence, the general solution to $y'' + 4y = \sin(\omega x)$ is the sum of this function with $c_1 \cos(2x)$ and $\frac{c_2}{2}\sin(2x)$ for $c_1, c_2 \in \mathbb{R}$. $\diamond$

Example 3.4.8. The Laplace transform even enables us to solve some 2nd order ODEs with *non-constant* coefficients, if these are just monomials, say. E.g., if we look for the general solution to the equation $xy''(x) - y'(x) = 0$, then the Laplace transform yields, with the help of Lemma 3.4.5 (where we again write $Y = \mathcal{L}(y)$):

$$\begin{aligned} 0 &= -\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}(y'')(s) - \mathcal{L}(y')(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\big(s^2Y(s) - sy(0) - y'(0)\big) - \big(sY(s) - y(0)\big) \\ &= -2sY(s) - s^2Y'(s) + y(0) - sY(s) + y(0) = -3sY(s) - s^2Y'(s) + 2y(0). \end{aligned}$$

Hence, we obtain a linear first-order equation, which is easily solved: $Y(s) = \frac{y(0)}{s} + \frac{c}{s^3}$, for some $c \in \mathbb{R}$. Transforming back yields that $y(x) = y(0) + \frac{c}{2}x^2$. $\diamond$

Chapter 4

Existence and uniqueness theorems

In this part, we present general existence and uniqueness results for first-order ODEs. Their proofs will provide us in particular with an explicit approximative construction scheme, which is of practical interest. We first have to provide some basics about metric spaces.

4.1 Metric spaces

We summarize the most important facts and assertions about metric spaces, some of which are already known from basic calculus.

Definition 4.1.1 (Metric space). *Let X be a non-empty set, and* $\mathrm{d}\colon X \times X \to [0, \infty)$ *a* metric *on X, i.e., a symmetric function that satisfies the triangle inequality* $\mathrm{d}(x, y) + \mathrm{d}(y, z) \leq \mathrm{d}(x, z)$ *for any $x, y, z \in X$ and satisfies* $\mathrm{d}(x, y) = 0$ only *for $x = y$. Then we call the pair (X, d) a* metric space.

A standard example of a metric space is $\mathbb{R}^n$ with the metric $\mathrm{d}(x, y) = \|x - y\|$ coming from any norm $\|\cdot\|$ on $\mathbb{R}^n$. In the connection with ODEs, we will be mostly concerned with $\mathcal{C}(U)$, the set of continuous functions $U \to \mathbb{R}$, where $U \subset \mathbb{R}^n$ is a compact domain, i.e., bounded and closed. The natural metric on $\mathcal{C}(U)$ is the supremum metric, $\mathrm{d}_\infty(f, g) = \sup_{x \in U} |f(x) - g(x)|$. Certainly, the supremum is a maximum by compactness of U.

An important question is whether or not every Cauchy sequences possesses a limit.

Definition 4.1.2 (Convergence, Cauchy sequences, completeness). *Let (X, d) be a metric space.*

(i) We say, a sequence $(x_n)_{n \in \mathbb{N}}$ in X converges *towards $x \in X$, if, for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $\mathrm{d}(x_n, x) < \varepsilon$ for any $n \in \mathbb{N}$ satisfying $n \geq N$.*

(ii) A Cauchy sequence *is a sequence $(x_n)_{n \in \mathbb{N}}$ in X such that, for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $\mathrm{d}(x_n, x_m) < \varepsilon$ for any $n, m \in \mathbb{N}$ satisfying $n, m \geq N$.*

(iii) If any Cauchy sequence in X possesses a limit in X, then (X, d) (or just X) is called complete.

It is well-known that $\mathbb{R}^n$ is complete with the standard norm and with the p-norm for any $p \in [1, \infty)$. It is also widely known that $\mathcal{C}(U)$ is complete with the supremum metric. This will be important for us when constructing sequences of functions whose limit (if it exists) satisfies some limiting equation, but only its Cauchy sequence property can be proven *a priori*. The base for arguments like that is the following famous fixed point theorem. A *fixed point* a of a map $A\colon X \to X$ is a point $x^* \in X$ satisfying $A(x^*) = x^*$.

Theorem 4.1.3 (Banach's fixed point theorem). *Let (X, d) be a complete metric space, and let $A\colon X \to X$ be a* contraction*, i.e., there is $q \in (0,1)$ such that $\mathrm{d}(A(x), A(y)) \le q\mathrm{d}(x, y)$ for any $x, y \in X$. Then A possesses a unique fixed point x^* in X. Furthermore, for any initial value $x_0 \in X$, the iterating sequence $(x_n)_{n\in\mathbb{N}_0}$, defined by $x_{n+1} = A(x_n)$, converges towards x^*. We even have the error estimate $\mathrm{d}(x^*, x_n) \le \frac{q^n}{1-q}\mathrm{d}(x_0, x_1)$ for any $n \in \mathbb{N}$.*

Proof. The sequence $(x_n)_{n\in\mathbb{N}_0}$ is a Cauchy sequence, since we have, for any $n < m$,

$$\begin{aligned}\mathrm{d}(x_n, x_m) &\le \sum_{i=1}^{m-n} \mathrm{d}(x_{n+i-1}, x_{n+i}) = \sum_{i=1}^{m-n} \mathrm{d}(A(x_{n+i-2}), A(x_{n+i-1})) \le q \sum_{i=1}^{m-n} \mathrm{d}(x_{n+i-2}, x_{n+i-1}) \\ &\le \cdots \le \sum_{i=1}^{m-n} q^{n+i-1}\mathrm{d}(x_0, x_1) \le \frac{q^n}{1-q}\mathrm{d}(x_0, x_1).\end{aligned} \tag{4.1.1}$$

By completeness of X, there is a limiting point x^* of $(x_n)_{n\in\mathbb{N}_0}$. Since d is a continuous map and A as well, we then have

$$\mathrm{d}(x^*, A(x^*)) = \lim_{n\to\infty} \mathrm{d}(x_n, A(x_n)) \le \lim_{n\to\infty} \mathrm{d}(x_n, x_{n+1}) \le \lim_{n\to\infty} \frac{q^n}{1-q}\mathrm{d}(x_0, x_1) = 0. \tag{4.1.2}$$

Hence, x^* is a fixed point of A. The error estimate now follows from (4.1.1) after letting $m \to \infty$. The uniqueness of the fixed point is shown in a similar way as in (4.1.2) (exercise). □

Remark 4.1.4 (Possible improvement). The proof of Banach's fixed point theorem may be improved to obtain stronger a stronger assertion. E.g., A does not have to be a contraction, but it is sufficient to have an estimate of the form

$$\mathrm{d}(A^n(y_1), A^n(y_2)) \le \alpha_n \mathrm{d}(y_1, y_2), \qquad y_1, y_2 \in X, n \in \mathbb{N},$$

where A^n is the n-th iterate of A (i.e., $A^1 = A$ and $A^{n+1} = A \circ A^n$), and $\sum_{n\in\mathbb{N}} \alpha_n < \infty$. ◇

4.2 Picard iteration

We want to apply Banach's fixed point theorem to the IVP

$$y' = f(x, y), \qquad y(x_0) = y_0, \tag{4.2.1}$$

where $f\colon R \to \mathbb{R}$ is a continuous function on the rectangle $R = \{(x, y) \in \mathbb{R}^2\colon |x - x_0| \le a, |y - y_0| \le b\}$, and $x_0, y_0 \in \mathbb{R}$ and $a, b > 0$. However, (4.2.1) is not a fixed point equation, but after some elementary transformation, it is turned into such an equation. The proof of the following lemma is clear.

Lemma 4.2.1. *The IVP in* (4.2.1) *is equivalent to the integral equation*

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\,\mathrm{d}t. \tag{4.2.2}$$

Hence, if we define a map

$$A\colon \mathcal{C}([x_0 - a, x_0 + a]) \to \mathcal{C}([x_0 - a, x_0 + a]), \qquad A(y)(t) = y_0 + \int_{x_0}^{x} f(t, y(t))\,\mathrm{d}t, \tag{4.2.3}$$

then (4.2.2) is the fixed point equation $y = A(y)$ in the space $\mathcal{C}([x_0 - a, x_0 + a])$, more precisely in the space $\mathcal{C}_{0,b}$, the space of those functions y in $\mathcal{C}([x_0 - a, x_0 + a])$ that satisfy $y(x_0) = y_0$ and $y(t) \in [y_0 - b, y_0 + b]$ for any $t \in [x_0 - a, x_0 + a]$. (The latter condition in the definition of $\mathcal{C}_{0,b}$ is necessary in order that $f(t, y(t))$ is well-defined.) It turns out later that we will have to restrict the domain $[x_0 - a, x_0 + a]$ to some smaller interval $[x_0 - a_1, x_0 + a_1]$ such that $A\colon \mathcal{C}_{0,b} \to \mathcal{C}_{0,b}$ is well-defined and such that A is even a contraction.

In order to apply Banach's fixed point theorem to A, we have to find conditions on f such that A becomes a contraction. A useful concept for this is a certain continuity property:

Definition 4.2.2 (Lipschitz continuity). *Let $G \subset \mathbb{R}^2$ be a domain (i.e., an open and connected set) and $f\colon G \to \mathbb{R}$ a function.*

(i) We say that f satisfies a Lipschitz condition *or that f is* Lipschitz continuous *with respect to y in G if there exists a constant $L \geq 0$ (a* Lipschitz constant*) such that*

$$\big|f(x, y_1) - f(x, y_2)\big| \leq L|y_1 - y_2| \qquad (x, y_1), (x, y_2) \in G.$$

(ii) We say that f satisfies a local Lipschitz condition *or that f is* locally Lipschitz continuous *with respect to y in G if for any point in G there is a neighborhood U of the point in which f is Lipschitz continuous (with Lipschitz constant depending on U).*

Lipschitz continuity is a stronger property than continuity, but weaker than differentiability. A Lipschitz continuous function has local slopes of bounded size everywhere.

Example 4.2.3. (i) $f(x, y) = |y|$ satisfies a Lipschitz condition everywhere with Lipschitz constant $L = 1$.

(ii) $f(x, y) = y^2$ satisfies a local Lipschitz condition in $\mathbb{R}^2$, but not a global one, since

$$\Big|\frac{f(x, y_1) - f(x, y_2)}{y_1 - y_2}\Big| = |y_1 + y_2|.$$

(iii) $f(x, y) = \sqrt{|y|}$ is not Lipschitz continuous in any interval containing 0, since

$$\Big|\frac{f(0, y) - f(0, 0)}{y - 0}\Big| = \frac{1}{\sqrt{|y|}}.$$

◇

In particular, all continuously differentiable functions are Lipschitz continuous:

Lemma 4.2.4. *Let $G \subset \mathbb{R}^2$ and $f\colon G \to \mathbb{R}$ continuously differentiable with respect to y (i.e., the map $(x, y) \mapsto f_y(x, y)$ is continuous in G). Then f satisfies a local Lipschitz condition in G.*

Proof. Let $(x_0, y_0) \in G$ and let $a, b > 0$ be so small that the rectangle $R = [x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b]$ lies in G. According to the mean value theorem, for $(x, y_1), (x, y_2) \in R$, there is an η between y_1 and y_2 such that $|f(x, y_1) - f(x, y_2)| \le |f_y(x, \eta)|\,|y_1 - y_2|$. Since $(x, y) \mapsto f_y(x, y)$ is continuous in the bounded and closed set R, this map is also bounded, say by $L = \max_{(x,y)\in R} |f_y(x, \eta)|$. Hence, f satisfies a Lipschitz condition in R. □

Now we can give sufficient criteria for the map A in (4.2.3) being a contraction. We always consider the supremum metric on the set of continuous functions. By $\mathcal{C}_{0,b} = \mathcal{C}_{0,b}([x_0 - a_1, x_0 + a_1])$ we denote the set of continuous functions $[x_0 - a_1, x_0 + a_1] \to [y_0 - b, y_0 + b]$ having value y_0 at x_0.

Lemma 4.2.5. *Let $x_0, y_0 \in \mathbb{R}$, $a, b > 0$ and $R = \{(x, y) \in \mathbb{R}^2 \colon |x - x_0| \le a, |y - y_0| \le b\}$ a rectangle, and let $f\colon R \to \mathbb{R}$ be continuous. Put $M = \max_R |f|$ and assume that f is Lipschitz continuous in y with Lipschitz constant $L > 0$. Then, for any $0 < a_1 < \min\{a, \frac{b}{M}, \frac{1}{L}\}$, the map A defined in (4.2.3) is a contraction on the set $\mathcal{C}_{0,b}([x_0 - a_1, x_0 + a_1])$ (after properly restricting the definition in (4.2.3) to the smaller domain $[x_0 - a_1, x_0 + a_1]$ instead of $[x_0 - a, x_0 + a]$).*

Proof. First we show that A is indeed a map from $\mathcal{C}_{0,b}$ into itself. It is clear that, for $y \in \mathcal{C}_{0,b}$, the map $A(y)\colon [x_0 - a_1, x_0 + a_1] \to \mathbb{R}$ is continuous with value y_0 in x_0. Hence, we have to show that $|A(y)(x) - y_0| \le b$ for any $x \in [x_0 - a_1, x_0 + a_1]$. This is clear from

$$|A(y)(x) - y_0| \le \int_{x_0}^{x} \big|f(t, y(t))\big| \,\mathrm{d}t \le a_1 \max_R |f| = a_1 M \le b,$$

by our choice of a_1.

Now we show that A is a contraction. For any $y_1, y_2 \in \mathcal{C}_{0,b}$, we have

$$\begin{aligned}
\mathrm{d}_\infty(A(y_1), A(y_2)) &= \sup_{x\in[x_0-a_1,x_0+a_1]} \big|A(y_1)(x) - A(y_2)(x)\big| \\
&\le \sup_{x\in[x_0-a_1,x_0+a_1]} \int_{x_0}^{x} \big|f(t, y_1(t)) - f(t, y_2(t))\big| \,\mathrm{d}t \\
&\le L \int_{x_0}^{x} \sup_{t\in[x_0-a_1,x_0+a_1]} |y_1(t) - y_2(t)| \,\mathrm{d}t \le La_1 \mathrm{d}_\infty(y_1, y_2).
\end{aligned}$$

By our assumption on a_1, the contraction coefficient $q = La_1$ is smaller than one, and this shows that A is a contraction. □

Now we apply Banach's fixed point theorem to the map A defined in (4.2.3) and obtain that there is a unique solution y to the integral equation in (4.2.2) (which is equivalent to the IVP in (4.2.1) by Lemma 4.2.1). Even better, we also obtain an explicit iteration method that approaches the solution uniformly (i.e., in supremum metric). This method is often also called the *method of successive approximation*.

Theorem 4.2.6 (Picard Iteration). *Let $x_0, y_0 \in \mathbb{R}$, $a, b > 0$ and $R = \{(x,y) \in \mathbb{R}^2 \colon |x-x_0| \leq a, |y-y_0| \leq b\}$ a rectangle, and let $f \colon R \to \mathbb{R}$ be continuous. Put $M = \max_R |f|$ and assume that f is Lipschitz continuous in y with Lipschitz constant $L > 0$. Fix any $a_1 \in (0, \min\{a, \frac{b}{M}, \frac{1}{L}\})$. Then, for an arbitrary initial function $y_1 \in \mathcal{C}_{0,b}([x_0 - a_1, x_0 + a_1])$, the sequence of functions $y_1, y_2, y_3, \ldots$ defined by*

$$y_{n+1}(x) = A(y_n)(x) = y_0 + \int_{x_0}^{x} f(t, y_n(t))\,\mathrm{d}t, \qquad n \in \mathbb{N}, x \in [x_0 - a_1, x_0 + a_1], \tag{4.2.4}$$

converges uniformly on $[x_0 - a_1, x_0 + a_1]$ towards the unique solution to the IVP in (4.2.1).

Remark 4.2.7 (Possible improvement). Using the fixed point theorem of Remark 4.1.4, it is possible to slightly improve Theorem 4.2.6. Indeed, it turns out that one does not have to assume that $a_1 < \frac{1}{L}$. This simplification makes the interval on which we can prove the existence and uniqueness of the solution independent of the Lipschitz constant of f. ◇

In the following examples of applications of Theorem 4.2.6, we do not care about the precise domain $[x_0 - a_1, x_0 + a_1]$ on which we construct a solution. The restriction of a_1 due to the Lipschitz constant of f is purely technical (see Remark 4.2.7), and the others (due to the domain and maximal size of f) max be chosen arbitrarily in our examples, since f is continuously differentiable and therefore Lipschitz continuous on any bounded set. However, once we have found the successive approximations $(y_n)_{n\in\mathbb{N}}$, we do register on what intervals they converge uniformly to the solution.

Example 4.2.8. We apply the Picard iteration to the equation $y' = xy$ with initial value $y(0) = 1$. Hence, $f(x,y) = xy$, $x_0 = 0$ and $y_0 = 1$. We begin with $y_0(x) = 1$. Then we have

$$y_1(x) = 1 + \int_0^x t y_0(t)\,\mathrm{d}t = 1 + \frac{1}{2}x^2, \qquad y_2(x) = 1 + \frac{1}{2}x^2 + \frac{1}{8}x^4,$$

and we wonder what the general building principle of the sequence $(y_n)_n$ is. The answer is given by the formula

$$y_n(x) = \sum_{k=0}^{n} \frac{x^{2k}}{2^k k!},$$

which is easily verified using an induction. Obviously, this sequence converges uniformly on any bounded interval to the function $y(x) = \mathrm{e}^{\frac{1}{2}x^2}$, which is obviously the unique solution. ◇

Example 4.2.9. We apply the Picard iteration to the equation $y' = 2 + 3y$ with initial value $y(0) = \frac{1}{2}$. We begin with the constant function $y_0(x) = 2$ and obtain

$$y_1(x) = \frac{1}{2} + \int_0^x (2 + 3y_0(t))\,\mathrm{d}t = \frac{1}{2} + \frac{7}{2}x, \qquad y_2(x) = \frac{1}{2} + \int_0^x (3y_1(t) + 2)\,\mathrm{d}t = \frac{1}{2} + \frac{7}{2}x + \frac{21}{4}x^2.$$

In order to find the general structure of this iteration scheme, we note that the iterating rule is

$$y_{n+1}(x) = \frac{1}{2} + 2x + 3\int_0^x y_n(t)\,\mathrm{d}t, \qquad n \in \mathbb{N}_0.$$

Hence, it is clear that y_n is a polynomial of degree precisely n. Therefore we make the ansatz $y_n(x) = \sum_{k=0}^n \alpha_k^{(n)} x^k$ and substitute in the iterating rule. After a small calculation and a comparison of the coefficients, we obtain, for any $n \in \mathbb{N}_0$,

$$\alpha_0^{(n+1)} = \frac{1}{2}, \quad \alpha_1^{(n+1)} = 2 + 3\alpha_0^{(n)} = \frac{7}{2}, \qquad \alpha_k^{(n+1)} = \frac{3}{k}\alpha_{k-1}^{(n)}, \qquad k \in \{2, \dots, n+1\}.$$

Hence, $\alpha_k^{(n)} = \frac{7}{2}\frac{3^{k-1}}{k!}$ for $k \in \{2, \dots, n\}$. Hence,

$$y_n(x) = \frac{1}{2} + \frac{7}{2}x + \frac{7}{2}\sum_{k=2}^n \frac{3^{k-1}}{k!}x^k = -\frac{2}{3} + \frac{7}{6}\sum_{k=0}^n \frac{(3x)^k}{k!}.$$

Obviously, this converges towards $y(x) = -\frac{2}{3} + \frac{7}{6}\mathrm{e}^{3x}$, which is certainly the unique solution of the IVP $y' = 2 + 3y$ with initial value $y(0) = \frac{1}{2}$. ◇

Example 4.2.10. We apply the Picard iteration to the equation $y' = y + x$ with initial value $y(0) = y_0$. Then the general rule of the successive approximations is

$$y_{n+1}(x) = y_0 + \frac{x^2}{2} + \int_0^x y_n(t)\,\mathrm{d}t, \qquad n \in \mathbb{N}.$$

This gives $y_0(x) = y_0$, $y_1(x) = y_0 + \frac{x^2}{2} + xy_0$, $y_2(x) = y_0(1 + x + \frac{x^2}{2}) + \frac{x^2}{2} + \frac{x^3}{3!}$ and so on. This leads us to the conjecture that

$$y_n(x) = y_0\sum_{k=0}^n \frac{x^k}{k!} + \sum_{k=2}^n \frac{x^k}{k!}, \qquad n \in \mathbb{N},$$

which we easily verify using the above recursive rule. Hence, it is clear that $\lim_{n\to\infty} y_n(x) = y_0\mathrm{e}^x + \mathrm{e}^x - x - 1$, which is also obviously the unique solution to the IVP $y' = y + x$, $y(0) = y_0$. We note that the convergence is uniform on any bounded subset of $\mathbb{R}$. Theorem 4.2.6 only tells us that the convergence is uniform on any closed subinterval of $(-1, 1)$ since $L = 1$ is the best Lipschitz constant of $f(x, y) = x + y$. However, this restriction is dropped in the improvement remarked in Remark 4.2.7, and actually, it is much less than the truth. ◇

4.3 Peano's existence theorem

In this section, we present another existence result for the IVP in (4.2.1), which is famous and fundamental. It does not require f to be Lipschitz continuous, but the construction of the solution (which is again approximative) is less suitable for explicit use.

Theorem 4.3.1 (Peano). *Let $x_0, y_0 \in \mathbb{R}$, $a, b > 0$ and $R = \{(x, y) \in \mathbb{R}^2 \colon |x - x_0| \le a, |y - y_0| \le b\}$ a rectangle, and let $f \colon R \to \mathbb{R}$ be continuous. Then the IVP*

$$y' = f(x, y), \qquad y(x_0) = y_0, \tag{4.3.1}$$

admits at least one solution that extends to the boundary of R.

To be sure, we do not require that the solution is a map $[x_0-a, x_0+a] \to [y_0-b, y_0+b]$, but a map $[x_0-a_1, x_0+a_2] \to [y_0-b, y_0+b]$, where $a_1, a_2 \in (0, a]$ are maximal, i.e., the solution is at the boundary $y_0 \pm b$ in $x_0 - a_1$ and in $x_0 + a_2$. The proof shows that we may take $a_1 = a_2 = a$ if $\max_{(x,y)\in R} |f(x,y)| \leq \frac{b}{a}$.

Sketch of proof. Of course, we may and shall assume that $x_0 = y_0 = 0$. Abbreviate $M = \max |f| = \max_{(x,y)\in R} |f(x,y)|$. Furthermore, we also may assume that $f \geq 0$, since we otherwise substitute $u(x) = y(x) + Mx$, which solves the IVP $u' = f(x, u - Mx) + M \equiv g(x,u)$ with $u(0) = 0$ and $g \geq 0$.

For technical reasons, we extend f continuously to the infinite strip $[-a,a] \times \mathbb{R}$ by putting $f(x,y) = f(x,b)$ for $y > b$ and $f(x,y) = f(x,-b)$ for $y < -b$, i.e., outside the rectangle R we put f equal to the value at the boundary of R.

Now we decompose the right half of the strip $[-a,a] \times \mathbb{R}$, i.e., the set $[0,a] \times \mathbb{R}$, into disjoint small rectangles

$$R_{k,l}^{(n)} = \Big[a\frac{k}{n}, a\frac{k+1}{n}\Big) \times \Big[\frac{l}{2^n}, \frac{l+1}{2^n}\Big), \qquad k \in \{0, \dots, n-1\}, l \in \mathbb{Z},$$

where $n \in \mathbb{N}$ is a parameter which will eventually be sent to ∞. For any $n \in \mathbb{N}$, we denote by $y_n \colon [0,a] \to \mathbb{R}$ the continuous polygon line with $y_n(0) = 0$ that is a straight line within every sub-rectangle $R_{k,l}^{(n)}$ that it hits and has slope

$$m_{k,l}^{(n)} = \min_{(x,y)\in R_{k,l}^{(n)}} f(x,y)$$

within that rectangle. This construction is done step by step, starting with the first rectangle, $R_{0,0}^{(n)} = [0, a\frac{1}{n}] \times [0, 2^{-n}]$, where y_n is the straight line from $y_n(0) = 0$ with slope $m_{0,0}^{(n)}$. If this slope is small enough, then the graph of y_n in $R_{0,0}^{(n)}$ does not leave this rectangle, but ends at $y_n(a\frac{1}{n}) = a\frac{1}{n} m_{0,0}^{(n)}$. Then we proceed in the right-neighboring rectangle, $R_{1,0}^{(n)}$, by adding a piece with slope $m_{1,0}^{(n)}$ until it leaves also $R_{1,0}^{(n)}$, and so on. Otherwise, the graph of y_n leaves $R_{0,0}^{(n)}$ somewhere in $(0, a\frac{1}{n})$, and it is extended to the upper-neighboring rectangle, $R_{0,1}^{(n)}$, with slope $m_{0,1}^{(n)}$ until it leaves that rectangle. In this way, we proceed.

The function y_n is a discrete approximation to a function that solves the ODE $y' = f(x,y)$, since it solves, by construction, a discrete version of that ODE. It is not too difficult to see that $y_n(x) \leq y_{n+1}(x) \leq Mx \leq Ma$ for any $n \in \mathbb{N}$ and any $x \in [0,a]$. Hence, the sequence $(y_n(x))_{n\in\mathbb{N}}$ posseses a limit $y(x)$ for any $x \in [0,a]$. In this way, we have constructed a function $y \colon [0,a] \to \mathbb{R}$. In the same way, we construct a function $y \colon [-a, 0] \to \mathbb{R}$. Putting these two functions together, we obtain a function $y \colon [-a,a] \to \mathbb{R}$. Now the main work of the proof (which we will omit) consists of showing that y is differentiable in $[-a,a]$, solves the ODE $y' = f(x,y)$ and satisfies $y(x) \in [0,b]$ for any $x \in [-a,a]$. □

Note that Peano's Theorem gives only the existence of a solution of the IVP in (4.2.1), not its uniqueness. As a trivial example, the IVP $y' = 2\sqrt{|y|}$, $y(0) = 0$, has more than one solution, the function identical to zero and the function $x \mapsto x^2 \operatorname{sign}(x)$.

4.4 Further uniqueness theorems

We recall from Lemma 4.2.1 that the IVP in (4.2.1), which is the object of our interest, is equivalent to the integral equation in (4.2.2). Based on this equivalence, we present another uniqueness result for (4.2.1), which is based on Lipschitz continuity and the following important tool.

Lemma 4.4.1 (Gronwall's Lemma). *Let $\varphi\colon [a,b] \to \mathbb{R}$ be continuous, and fix $L \geq 0$. If, for some $C > 0$,*

$$0 \leq \varphi(x) \leq C + L\int_a^x \varphi(t)\,\mathrm{d}t, \qquad x \in [a,b],$$

then φ satisfies the bound $\varphi(x) \leq C\mathrm{e}^{L(x-a)}$ for any $x \in [a,b]$.

Proof. Since $\varphi \geq 0$, $L \geq 0$ and $C > 0$, also $f(y) \equiv C + L\int_a^y \varphi(t)\,\mathrm{d}t$ is positive. Observe that from the assumption we have that

$$\frac{f'(y)}{f(y)} = \frac{L\varphi(y)}{C + L\int_a^y \varphi(t)\,\mathrm{d}t} \leq L, \qquad y \in [a,b].$$

Now integrate the left hand side, $f'(y)/f(y)$, from $y = a$ to $y = x$, to obtain

$$\begin{aligned}\log\Big(C + L\int_a^x \varphi(t)\,\mathrm{d}t\Big) = \log f(x) = \log f\Big|_a^x + \log f(a) = \int_a^x \frac{f'(y)}{f(y)}\,\mathrm{d}y + \log C \\ \leq L(x-a) + \log C.\end{aligned}$$

Applying the exponential function and using once more that $\varphi \leq f$, we arrive at the assertion. □

Applying Gronwall's lemma to arbitrarily small $C > 0$, we obtain the following strong assertion:

Corollary 4.4.2. *In particular, if φ satisfies the bound $0 \leq \varphi(x) \leq L\int_a^x \varphi(t)\,\mathrm{d}t$ for any $x \in [a,b]$, then φ is the function identically equal to zero.*

Now it is easy to obtain a uniqueness result for the integral equation in (4.2.2):

Lemma 4.4.3. *Let f be continuous and satisfy a Lipschitz condition on the domain G. Assume that there is a non-trivial interval I such that there are two solutions $y_1, y_2\colon I \to \mathbb{R}$ of the integral equation in (4.2.2). Then $y_1(x) = y_2(x)$ for any $x \in I$.*

Proof. We consider $\varphi = y_1 - y_2$. Then we have from (4.2.2) that $\varphi(x) = \int_{x_0}^x (f(t, y_1(t)) - f(t, y_2(t)))\,\mathrm{d}t$ for any $x \in I$. We may assume that I is bounded and closed, hence f satisfies a Lipschitz condition in any compact neighborhood K of the curves of y_1 and y_2. This is seen via a standard compactness argument as follows. By assumption, for any $z \in K$, there is some neighborhood U_z of z in which f satisfies a Lipschitz condition with constant L_z. By compactness of K, there are $n \in \mathbb{N}$ and $z_1, \ldots, z_n \in K$ such that K is covered by $U_{z_1}, \ldots, U_{z_n}$. Then f is Lipschitz continuous in K with Lipschitz constant $L = \max_{i=1}^n L_{z_i}$, as one can show with some bit of work.

Using the Lipschitz continuity in K, we obtain that

$$|\varphi(x)| \leq \int_{x_0}^x \big|f(t, y_1(t)) - f(t, y_2(t))\big|\,\mathrm{d}t \leq \int_{x_0}^x L|y_1(t) - y_2(t)|\,\mathrm{d}t \leq L\int_{x_0}^x |\varphi(t)|\,\mathrm{d}t.$$

Now Corollary 4.4.2 yields that $\varphi = 0$, i.e., $y_1 = y_2$. □

Summarizing Theorem 4.3.1, and Lemmas 4.2.1 and 4.4.3, we arrive at the announced uniqueness result:

Theorem 4.4.4 (Existence and uniqueness for (4.2.1)**).** *Let $G \subset \mathbb{R}^2$ be a domain, and let $f : G \to \mathbb{R}$ satisfy a local Lipschitz condition. Then, for any $(x_0, y_0) \in G$, there is a unique solution y for the IVP*

$$y' = f(x, y), \qquad y(x_0) = y_0, \tag{4.4.1}$$

whose graph extends from the left boundary of G to the right boundary of G.

Corollary 4.4.5. *Let $a < b$ and $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ be continuous satisfying a Lipschitz condition in y. Then, for any $(x_0, y_0) \in [a, b] \times \mathbb{R}$, there is precisely one solution of the IVP $y' = f(x, y)$, $y(x_0) = y_0$.*

Example 4.4.6. The IVP $y' = \sin y$, $y(x_0) = y_0$, possesses a unique solution on $\mathbb{R}$. The function $f(x, y) = \sin y$ satisfies a Lipschitz condition with respect to y on $\mathbb{R}$ with Lipschitz constant $L = 1$. The solutions are the constant functions $x \mapsto k\pi$ with $k \in \mathbb{Z}$ and the implicit solution $\tan \frac{y}{2} = \mathrm{e}^{x-c}$ with $c \in \mathbb{R}$. ◇

Example 4.4.7. The IVP $y' = y^2$, $y(0) = y_0 > 0$, possesses the solution $y(x) = \frac{y_0}{1-y_0 x}$, which is defined only in the interval $(-\frac{1}{y_0}, \frac{1}{y_0})$. The function $f(x, y) = y^2$ satisfies a Lipschitz condition on any bounded subinterval w.r.t. y, but in no infinite stripe. ◇

4.5 Continuous dependence of the solution

We can generalize the question of uniqueness of the solution to (4.2.1) to the question if two solutions will be 'close' together if their initial values are. Under the assumption of Lipschitz continuity, this question is easily answered in the affirmative:

Lemma 4.5.1. *Let $G \subset \mathbb{R}^2$ be a domain and $f : G \to \mathbb{R}$ continuous satisfying a Lipschitz condition in G with Lipschitz constant L. Then, for any two solutions y_1, y_2 of the equation $y' = f(x, y)$ and for any $x_0 \in G^{(1)} \equiv \{x : \exists y \in \mathbb{R} : (x, y) \in G\}$,*

$$|y_1(x) - y_2(x)| \le |y_1(x_0) - y_2(x_0)|\mathrm{e}^{L|x-x_0|}, \qquad x \in G^{(1)}.$$

Proof. It is sufficient to consider the case $x > x_0$. Put $\varphi(x) = |y_1(x) - y_2(x)|$, then

$$\begin{aligned} 0 \le \varphi(x) &= \Big| \int_{x_0}^{x} \big[f(t, y_1(t)) - f(t, y_2(t))\big] \,\mathrm{d}t + y_1(x_0) - y_2(x_0) \Big| \\ &\le |y_1(x_0) - y_2(x_0)| + L \int_{x_0}^{x} \varphi(t) \,\mathrm{d}t \end{aligned}$$

Now the assertion follows from Gronwalls Lemma. □

A similar question is whether or not two solutions for different functions f and f^* will be close to each other if f and f^* are close to each other. Also this may be answered positively:

Lemma 4.5.2. *Let $G \subset \mathbb{R}^2$ be a domain and $f\colon G \to \mathbb{R}$ continuous satisfying a Lipschitz condition in G with Lipschitz constant L. Furthermore, let $f^*\colon G \to \mathbb{R}$ be a function that differs from f only by $\varepsilon > 0$, i.e., $|f(x,y) - f^*(x,y)| \le \varepsilon$ for any $(x,y) \in G$. Let $(x_0, y_0) \in G$, and let y and y^* be the solutions of $y' = f(x,y)$ resp. $(y^*)' = f^*(x,y^*)$ in G with the same initial condition $y(x_0) = y^*(x_0) = y_0$. Then, for any sufficiently small $\delta > 0$,*

$$|y(x) - y^*(x)| \le \varepsilon\delta \mathrm{e}^{L(x-x_0)}, \qquad x \in [x_0, x_0+\delta].$$

Proof. We use that y and y^* satisfy the corresponding integral equations (recall Lemma 4.2.1) and see that

$$\begin{aligned} |y(x) - y^*(x)| &\le \int_{x_0}^{x} \big|f(t,y(t)) - f(t,y^*(t)) + f(t,y^*(t)) - f^*(t,y^*(t))\big| \,\mathrm{d}t \\ &\le \int_{x_0}^{x} \big|f(t,y(t)) - f(t,y^*(t))\big| \,\mathrm{d}t + \int_{x_0}^{x} \big|f(t,y^*(t)) - f^*(t,y^*(t))\big| \,\mathrm{d}t \\ &\le L \int_{x_0}^{x} |y(t) - y^*(t)| \,\mathrm{d}t + \varepsilon\delta. \end{aligned}$$

Now Gronwall's Lemma implies the assertion. □

Example 4.5.3. The function $f(x,y) = x + \sin y$ differs in $G = (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}$ from the function $f^*(x,y) = x + \frac{2}{\pi}y$ by less than $\varepsilon = 0.211$. The function $y^*(x) = -\frac{\pi}{2}x + \frac{\pi^2}{4}(\mathrm{e}^{2x/\pi} - 1)$ solves the IVP $(y^*)' = f^*(x,y^*)$, $y^*(0) = 0$. The function $f(x,y) = x + \sin y$ has Lipschitz constant $L = 1$. According to Lemma 4.5.2, for any sufficiently small $\delta > 0$, the solution $y(x)$ of the IVP $y' = f(x,y)$, $y(0) = 0$, differs from $y^*(x)$ in the interval $x \in [0, \delta]$ by no more than $0.211\,\delta\mathrm{e}^{x}$. ◇

Chapter 5

ODE systems

In this chapter, we extend our study of ODEs for single functions to *systems* of ODEs for several functions, which are also of importance in applications. In particular, we will see that higher-order ODEs may be seen as special cases of ODE systems. This will enable us to give existence and uniqueness results for higher-order ODEs, which we postponed in Chapter 2.

5.1 Basics

Consider a function $F\colon D \to \mathbb{R}^n$ on a $(n+1)$-dimensional domain $D \subset \mathbb{R}^{n+1}$ giving a vector of n values, $F(x, y_1, y_2, \dots, y_n) = (F_1(x, y_1, \dots, y_n), \dots, F_n(x, y_1, \dots, y_n))$. Sometimes $F = (F_1, \dots, F_n)$ is called a *vector field*. Then the system of n equations,

$$y_i' = F_i\bigl(x, y_1(x), \dots, y_n(x)\bigr), \qquad i = 1, \dots, n, \tag{5.1.1}$$

will be conceived as one n-dimensional ODE system of first order. It may be abbreviated

$$y' = F(x, y), \qquad y = (y_1, \dots, y_n). \tag{5.1.2}$$

In analogy with the notions introduced at the beginning of Chapter 1, we call y a solution to (5.1.1) if, for some interval I, there is a function $y\colon I \to \mathbb{R}^n$ such that $(x, y(x)) \in D$ for any $x \in I$ an such that (5.1.1) is satisfied for $x \in I$. If this curve runs through a given point $(x_0, y_0) \in D$, then we say that the initial-value problem (IVP) $y' = F(x, y)$, $y(x_0) = y_0$, is solved by y. We speak of a *general* solution if there are n free parameters, and we call the solution *complete* if all the solutions are comprised. We can conceive y as a curve in the n-dimensional space, indexed by 'time' $x \in I$.

Example 5.1.1 (Predator-prey model). Let $y_1(t)$ and $y_2(t)$ denote the number of predators respectively preys at time $t \geq 0$. The death-rate of the predators is $-(\alpha - \beta y_2(t))y_1(t)$, where $\alpha, \beta > 0$, decays without prey and increases linearly with the amount of available prey. The growth rate of the prey, $(\gamma - \delta y_1(t))y_2(t)$, where $\gamma, \delta > 0$, is positive without predators and decreases linearly with the presence of predators. Hence, the vector $y = (y_1, y_2)$ of the two functions satisfies the system

$$\begin{aligned} y_1'(t) &= -(\alpha - \beta y_2(t))y_1(t), \\ y_2'(t) &= (\gamma - \delta y_1(t))y_2(t). \end{aligned}$$

(Remark: With an appropriate choice of the parameters, one could argue why after The Great War much more sharks were caught in the Mediterranian Sea and much less eatable fish than in the years before 1914, even though during The Great War fishing was severely restricted.) $\diamond$

It is important to note that certain higher-order ODEs are special cases of systems of ODEs:

Remark 5.1.2. Consider an explicit n-th order ODE

$$y^{(n)} = f(x, y, y', y'', \dots, y^{(n-1)}). \tag{5.1.3}$$

Putting $y_1 = y$, $y_2 = y_1' = y'$, $y_3 = y'', \dots, y_n = y^{(n-1)} = y_{n-1}'$, then (5.1.3) is equivalent to the following ODE system:

$$y_i' = y_{i+1}, \quad \text{for } i = 1, \dots, n-1, \qquad y_n' = f(x, y_1, y_2, \dots, y_n). \tag{5.1.4}$$

Hence, y is a solution of (5.1.3) if and only if $(y, y', y'', \dots, y^{(n-1)})$ is a solution of (5.1.4). $\diamond$

It is often a good idea to turn a first-order ODE system into a higher-order ODE by differentiation and elimination:

Example 5.1.3. With real coefficients a_1, a_2, a_3 and b_1, b_2, b_3 and ω, we look at the transformator equations,

$$\begin{aligned} I_1'' &= a_1 I_1 + a_2 I_2 + a_3 \cos(\omega t), \\ I_2'' &= b_1 I_1 + b_2 I_2 + b_3 \cos(\omega t). \end{aligned}$$

Differentiating the first equation twice gives

$$I_1^{(4)} = a_1 I_1'' + a_2 I_2'' - \omega^2 a_3 \cos(\omega t).$$

Now substitute the second equation for I_2'' and use the first equation to substitute I_2 (here we assume that $a_2 \neq 0$), then we obtain, for some coefficients c_1, c_2, c_3,

$$I_1^{(4)} = c_1 I_1'' + +c_2 I_1 = c_3 \cos(\omega t),$$

which is a fourth-order ODE for I_1. An analogous equation may be derived for I_2. $\diamond$

5.2 Existence and uniqueness

The results of Chapter 4 may be extended to systems of ODEs with almost no changes. We summarize the most important facts.

The n-dimensional version of *Peano's Theorem* says that the IVP

$$y' = F(x, y), \qquad y(x_0) = y_0, \tag{5.2.1}$$

for any $x_0 \in \mathbb{R}$ and $y_0 \in \mathbb{R}^n$ admits at least one solution y in a given neighborhood of x_0 if the function F is a continuous vector field such that the graph of y is contained in the domain of F.

We denote by $|x| = (x_1^2 + \cdots + x_n^2)^{\frac{1}{2}}$ the Euclidean norm of the vector $x = (x_1, \dots, x_n)$. We say that $f\colon D \subset \mathbb{R}^{n+1} \to \mathbb{R}^n$ satisfies a Lipschitz condition w.r.t. y in the domain D with Lipschitz constant $L > 0$ if $|f(x, y_1) - f(x, y_2)| \le L\,|y_1 - y_2|$ for any $(x, y_1), (x, y_2) \in D$. We say that f satisfies a local Lipschitz condition if for any point in D there is a neighborhood in which f satisfies a Lipschitz condition, with Lipschitz constant possibly depending on the neighborhood. It is clear from Lemma 4.2.4 that a vector field $F = (F_1, \dots, F_n)$ satisfies a local Lipschitz condition if it is continuously partially differentiable w.r.t. y. It is also clear from the triangle inequality that $f = (f_1, \dots, f_n)$ is Lipschitz continuous if and only if every single component $f_1, \dots, f_n$ is.

An n-dimensional version of the existence and uniqueness result of Lemma 4.4.3 says that, if F is continuously partially differentiable w.r.t. y in some domain $D \subset \mathbb{R}^{n+1}$, the IVP in (5.2.1) possesses precisely one maximal solution on any interval such that the solution curve $x \mapsto y(x)$ lies entirely in D. Here we call the solution *maximal* if it cannot be extended to any larger interval.

We also note that continuity results like in Lemmas 4.5.1 and 4.5.2 are valid in the n-dimensional setting.

5.3 Linear systems

The simplest nontrivial ODE systems are linear, i.e., of the form

$$y'(t) = A(t)\,y(t) + b(t), \qquad t \in I \subset \mathbb{R}, y\colon I \to \mathbb{R}^n, \tag{5.3.1}$$

with a *coefficient matrix* $A(t) = (a_{i,j}(t))_{i,j=1,\dots,n} \in \mathbb{R}^{n\times n}$ and a vector $b(t) = (b_1(t), \dots, b_n(t))^{\mathrm{T}} \in \mathbb{R}^n$, for $t \in I$. We call (5.3.1) *homogeneous* if $b(t) = 0$ for any $t \in I$, and *inhomogeneous* otherwise.

Example 5.3.1. The system

$$\begin{aligned} y_1'(t) &= 3y_1(t) + 2y_2(t) + t^2 y_3(t) + \cos(\omega t), \\ y_2'(t) &= (\sin t) y_1(t) + 5y_3(t), \\ y_3'(t) &= t y_1(t) + t^2 y_2(t) + t^3 y_3(t) + \sqrt{t}, \end{aligned}$$

is equivalent to (5.3.1), where

$$A(t) = \begin{pmatrix} 3 & 2 & t^2 \\ \sin t & 0 & 5 \\ t & t^2 & t^3 \end{pmatrix}, \qquad b(t) = \begin{pmatrix} \cos(\omega t) \\ 0 \\ \sqrt{t} \end{pmatrix}.$$

◇

Remark 5.3.2 (Superposition principle). Clearly, also a linear system satisfies the *superposition principle*: if y and z are solutions to the systems $y' = A(t)y + b(t)$ respectively $z' = A(t)z + c(t)$, then $w = \alpha y + \beta z$ is a solution to the equation $w' = A(t) + (\alpha b(t) + \beta c(t))$. ◇

The theory of the structure of the set of the solutions to (5.3.1) is reminiscent of the results of Chapter 2 on ODEs of second order, but there are some features that were hidden there because of the low-dimensionality. We are going to present now the general principle. In particular, we will finally give the proof of the fact that the solutions of the homogeneous equation $y'' + ay' + by = 0$ which we constructed in the proof of Lemma 3.1.3 are indeed complete. We will have to rely on some notions and facts from Linear Algebra.

We recall the important notion of linear independence of Definition 3.1.2. Recall that a subset V of a vector space X is itself a vector space, a *linear subspace*, if and only if it is closed against taking linear combinations. The *dimension* of a vector space is the smallest number of vectors that span this vector space, i.e., such that every element of this space is an appropriate linear combination of these vectors. Equivalently, the dimension is the largest number of linearly independent elements of the space. The vector space X we will be working with is the space of all functions $I \to \mathbb{R}^n$, where I is the interval of (5.3.1).

Definition 5.3.3 (Wronski determinant). *Let $I \subset \mathbb{R}$ be an interval, $y_1, \dots, y_n \colon I \to \mathbb{R}^n$ functions and $Y = (y_1, \dots, y_n)$ the corresponding vector field. We consider $t \mapsto Y(t) \in \mathbb{R}^{n\times n}$ as a continuous, matrix-valued map. Then we call $W(t) = \det(Y(t))$ the* Wronski determinant *of $y_1, \dots, y_n$.*

It is easy to see that

$$W(t_0) \neq 0 \text{ for some } t_0 \in I \qquad \Longrightarrow \qquad y_1, \dots, y_n \text{ are linearly independent.}$$

Indeed, if $\sum_{i=1}^n a_i y_i = 0$ for some $a_1, \dots, a_n \in \mathbb{R}$, then, in particular, $0 = \sum_{i=1}^n a_i y_i(t_0) = Y(t_0)a$, where $a = (a_1, \dots, a_n)^{\mathrm{T}}$. By assumption, $Y(t_0)$ is a regular matrix, and therefore we have $a = 0$. This implies that $y_1, \dots, y_n$ are linearly independent.

Our next goal is to strengthen this criterion under the assumption that $y_1, \dots, y_n$ are solutions of the homogeneous equation. We recall the *trace* $\mathrm{Tr}(A) = \sum_{i=1}^n a_{i,i}$ of a matrix $A = (a_{i,j})_{i,j=1,\dots,n}$, the sum of the diagonal elements of A.

Lemma 5.3.4 (Liouville formula). *Let $I \subset \mathbb{R}$ be an interval and $A\colon I \to \mathbb{R}^{n\times n}$ a continuous matrix-valued mapping. Let $y_1, \dots, y_n$ be n solutions (not necessarily linearly independent) to the homogeneous equation $y'(t) = A(t)\, y(t)$. By $W(t) = \det(Y(t))$ we denote the Wronski determinant of $y_1, \dots, y_n$. Then, for any $t, t_0 \in I$,*

$$W(t) = W(t_0) \exp\Big(\int_{t_0}^t \mathrm{Tr}(A(s))\, \mathrm{d}s \Big). \tag{5.3.2}$$

Proof. With the help of multidimensional calculus, we are going to derive that W satisfies the equation $W'(t) = W(t)\mathrm{Tr}(A(t))$, which implies the result. To this aim, we use the multilinearity of the determinant in the columns to obtain

$$\begin{aligned} W'(t) &= \det\big(y_1'(t), y_2(t), \dots, y_n(t)\big) + \det\big(y_1(t), y_2'(t), y_3(t) \dots, y_n(t)\big) + \dots \\ &\qquad\qquad\qquad + \det\big(y_1(t), \dots, y_{n-1}(t), y_n'(t)\big) \\ &= \det\big(A(t)y_1(t), y_2(t), \dots, y_n(t)\big) + \dots + \det\big(y_1(t), \dots, y_{n-1}(t), A(t)y_n(t)\big) \\ &= \sum_{i,j=1}^n a_{i,j}(t) \det_{i,j}\big(Y(t)\big), \end{aligned}$$

where $\det_{i,j}(Y(t))$ is the determinant of $Y(t)$, after replacing in the i-th column y_i by the j-th, y_j. Hence, for $i \neq j$, $\det_{i,j}(Y(t))$ is the determinant of a matrix having two equal columns and is therefore equal to zero. Therefore, only the sum over the diagonal survives, i.e., $W'(t) = \sum_{i=1}^n a_{i,i}(t) \det(Y(t)) = W(t)\mathrm{Tr}(A(t))$, where we also used that $\det_{i,i} = \det$. □

This nice formula, together with (5.3.2), gives us a nice simple criterion for linear independence of solutions $y_1, \dots, y_n$ of the homogeneous equation $y'(t) = A(t)\, y(t)$:

Corollary 5.3.5. *In the situation of Lemma 5.3.4, if $W(t_0) \neq 0$ for some $t_0 \in I$, then $W(t) \neq 0$ for all $t \in I$. In particular, the functions $y_1, \dots, y_n$ are linearly independent over I.*

Example 5.3.6. Consider $A(t) = \frac{1}{t}\begin{pmatrix} 1 & 2t^2 \\ 0 & 1 \end{pmatrix}$ for $I = (0, \infty)$. Then we have two solutions, $y_1(t) = \binom{t}{0}$ and $y_2(t) = \binom{t^3}{t}$. Furthermore, $W(t) = t^2 \neq 0$ in I. On the other hand, one could have derived this fact using that $\mathrm{Tr}(A(t)) = \frac{2}{t}$ and $W(1) = 1$. ◇

Theorem 5.3.7 (Solutions of a linear ODE system). *Let $I \subset \mathbb{R}$ be an interval and $A\colon I \to \mathbb{R}^{n\times n}$ a continuous matrix-valued mapping. Furthermore, let $b\colon I \to \mathbb{R}^n$ be a continuous vector-valued mapping.*

(i) The set of solutions y of the homogeneous equation,

$$\mathcal{L} = \{y\colon I \to \mathbb{R}^n \mid y'(t) = A(t)\, y(t)\, \forall t \in I\}$$

is a vector space of dimension equal to n.

(ii) The general solution of the linear equation $y' = A(t)\, y + b(t)$ is of the form $y_{\mathrm{p}}(t) + y_{\mathrm{h}}(t)$, where y_{p} is a particular solution, and $y_{\mathrm{h}} \in \mathcal{L}$.

Proof. It is clear that the set $\mathcal{L}$ is a vector space. It is also clear from the superposition principle that any solution of the inhomogeneous equation is a sum of a particular solution and a solution to the homogeneous equation. It is left to show that $\mathcal{L}$ is not empty, and we have to identify its dimension.

Since the map $t \mapsto a_{i,j}(t)$ is continuous for any $i, j \in \{1, \ldots, n\}$, it is also bounded on any bounded subinterval of I. Hence, the map $(t, y) \mapsto A(t)y$ is Lipschitz continuous with respect to y on any bounded subset of $I \times \mathbb{R}^n$. According to the general existence and uniqueness result for first-order ODE systems mentioned in Section 5.1, there is precisely one solution to the IVP $y' = A(t)\, y$, $y(t_0) = y_0$, for any $t_0 \in I$ and any $y_0 \in \mathbb{R}^n$. Hence, $\mathcal{L}$ is not empty.

We apply this to $y_0 = \mathrm{e}_i$, the i-th unit vector: for $i = 1, \ldots, n$, there is precisely one solution y_i to the IVP $y_i' = A(t)\, y_i$, $y_i(t_0) = \mathrm{e}_i$. Since the matrix $(y_1(t_0), \ldots, y_n(t_0))$ is the identity matrix, it is regular, and $W(t_0) = 1 \neq 0$. By Corollary 5.3.5, the functions $y_1, \ldots, y_n$ are linearly independent. In particular, we have found n linearly independent elements of $\mathcal{L}$, and hence its dimensions is at least n.

By the superposition principle, for any $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$, the IVP $y' = A(t)\, y + b(t)$, $y(t_0) = a$, possesses the unique solution $\sum_{i=1}^n a_i y_i$. Hence, every element in $\mathcal{L}$ may be written as a linear combination of $y_1, \ldots, y_n$, which shows that the dimension of $\mathcal{L}$ is at most n. This finishes the proof. □

Combining this result with Remark 5.1.2, we have now a proof for the fact mentioned in Lemma 3.1.3 that the two solutions which we constructed in that proof form indeed a basis of the solution set.

Corollary 5.3.8. *In the situation of Theorem 5.3.7, if $y_1, \dots, y_n$ are any solutions of the homogeneous equation $y' = A(t)\,y$ satisfying $W(t_0) \neq 0$ for at least one $t_0 \in I$, then the general solution of the equation $y' = A(t)\,y + b(t)$ is of the form $y_\mathrm{p} + \sum_{i=1}^n \alpha_i y_i$, where y_p is a particular solution, and $\alpha_1, \dots, \alpha_n \in \mathbb{R}$. Any set of such functions $y_1, \dots, y_n$ is called a* fundamental system *or a* solution basis *of $y' = A(t)\,y$.*

Remark 5.3.9 (Fundamental matrix). If $y_1, \dots, y_n$ are n linearly independent solutions to $y' = A(t)\,y$, then the matrix-valued map $t \mapsto Y(t) = (y_1(t), \dots, y_n(t))$ is called a *fundamental matrix*. Since the Wronski determinant does not vanish in I, $Y(t)$ is regular for any $t \in I$. Hence, the general solution of $y' = A(t)\,y$ may be compactly written as $Y(t)a = \sum_{i=1}^n a_i y_i(t)$, for a vector $a \in \mathbb{R}^n$. In other words, the solution of the IVP $y' = A(t)\,y$, $y(t_0) = y_0$, may be written as

$$Y(t)Y(t_0)^{-1}y_0. \tag{5.3.3}$$

◇

Let us remark here that the explicit systematic computation of the fundamental matrix is messy and often even not possible.

Example 5.3.10. The ODE system

$$y' = \begin{pmatrix} 2t & -3t^2 \\ 3t^2 & 2t \end{pmatrix} y \tag{5.3.4}$$

possesses the fundamental matrix

$$Y(t) = \mathrm{e}^{t^2} \begin{pmatrix} \cos t^3 & \sin t^3 \\ -\sin t^3 & \cos t^3 \end{pmatrix}.$$

As we saw in the proof of Lemma 5.3.4, the Wronski determinant, $W(t) = \mathrm{e}^{2t^2}$, is a solution to $W'(t) = 4tW(t)$. Using (5.3.3), we find the solution of the IVP in (5.3.4), together with $y(t_0) = y_0$, as

$$y(t) = \mathrm{e}^{t^2 - t_0^2} \begin{pmatrix} \cos(t^3 - t_0^3) & \sin(t^3 - t_0^3) \\ -\sin(t^3 - t_0^3) & \cos(t^3 - t_0^3) \end{pmatrix} y_0.$$

◇

Remark 5.3.11 (Variation of constants). The n-dimensional version of this method is easy to explain. As in Section 3.2, we have already a fundamental matrix $Y(t)$ for the homogeneous equation $y'(t) = A(t)\,y(t)$ and want to derive a particular solution y_p for the inhomogeneous equation $y'(t) = A(t)\,y(t) + b(t)$. The ansatz is $y_\mathrm{p}(t) = Y(t)c(t)$ for some vector-valued function $t \mapsto c(t) \in \mathbb{R}^n$, which replaces the constant vector c of the homogeneous solution. Since $y' = A\,y$ and $y_\mathrm{p}' = Y'c + Yc'$, this ansatz is equivalent to $b(t) = Y(t)c'(t)$, and we obtain

$$c(t) = \int Y(t)^{-1} b(t)\,\mathrm{d}t,$$

where the integral is meant componentwise. Now we can summarize:

General solution of a linear first order ODE system. The general solution y of $y'(t) = A(t)\,y(t) + b(t)$ is given by

$$Y(t)\Big[\int_{t_0}^{t} Y(s)^{-1} b(s)\,\mathrm{d}s + c\Big], \qquad c \in \mathbb{R}^n, \tag{5.3.5}$$

where $Y(t)$ is a fundamental matrix of $y'(t) = A(t)\,y(t)$, and $t_0 \in I$. The unique solution of the IVP $y'(t) = A(t)\,y(t) + b(t)$ with $y(t_0) = y_0$ is obtained for choosing $c = Y(t_0)^{-1} y_0$.

◇

Example 5.3.12. The IVP

$$y'(t) = \frac{1}{t}\begin{pmatrix} 1 & -2t^2 \\ 0 & 1 \end{pmatrix} y + t^3\begin{pmatrix} 3 \\ 0 \end{pmatrix}, \qquad (t > 0), \qquad y(1) = \begin{pmatrix} 2 \\ 1 \end{pmatrix},$$

possesses the fundamental matrix $Y(t) = \begin{pmatrix} t & t^3 \\ 0 & t \end{pmatrix}$. The inverse is $Y(t)^{-1} = \frac{1}{t}\begin{pmatrix} 1 & -t^2 \\ 0 & 1 \end{pmatrix}$. Hence, (5.3.5) implies that the unique solution of the IVP is given by

$$\begin{aligned} y_*(t) &= \begin{pmatrix} t & t^3 \\ 0 & t \end{pmatrix} \Big[\int_1^t \frac{1}{s}\begin{pmatrix} 1 & -s^2 \\ 0 & 1 \end{pmatrix} s^3 \begin{pmatrix} 3 \\ 0 \end{pmatrix} \mathrm{d}s + \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}\Big] \\ &= \begin{pmatrix} t & t^3 \\ 0 & t \end{pmatrix}\begin{pmatrix} t^3 \\ 1 \end{pmatrix} = \begin{pmatrix} t^4 + t^3 \\ t \end{pmatrix}. \end{aligned}$$

◇

We already mentioned that, in general, a fundamental matrix cannot be found systematically. The situation is much better when the coefficient matrix is constant, i.e., does not depend on t. The following remark is a very short summary of the technique one applies in that case.

Remark 5.3.13 (ODE systems with constant coefficients). If the coefficient matrix A does not depend on t, then (at least, theoretically) one can use basic knowledge in Linear Algebra to systematically find a fundamental matrix, which we want to indicate here.

The main idea is that, in the special case where A is a diagonal matrix, the system $y' = Ay$ splits into n independent equations for the functions $y_1, \dots, y_n$, without any coupling between them. Hence, it is easy to find a fundamental matrix here. The next step is that, in the special case where A can be transformed into a diagonal matrix, i.e., where SAS^{-1} is a diagonal matrix D for some suitable invertibel matrix S, then the system $y' = Ay$ is transformed into the decoupled system $z' = Dz$ for $z = Sy$. Hence, also this case is easy. We note that this case applies to any normal matrix A, i.e., for any matrix A satisfying $AA^* = A^*A$. In the general case, A can be transformed only into a Jordan normal form, which differs from a diagonal matrix by a number of ones on the next-to-main diagonal. In this case, the transformed system is not really decoupled, but only weakly coupled since in each of the equations, only at most two of the functions y_i appear. ◇

In Remark 5.1.2 we already explained that linear ODEs of higher order may be seen as a special case of ODE systems. This we want to make more explicit now.

Remark 5.3.14 (Linear ODEs of order n). We consider a general n-th order linear ODE of the form $(L\varphi)(t) = f(t)$, where the operator L is given by

$$(L\varphi)(t) = \varphi^{(n)}(t) + a_{n-1}(t)\varphi^{(n-1)}(t) + \cdots + a_1(t)\varphi'(t) + a_0(t)\varphi(t), \tag{5.3.6}$$

with continuous functions $a_i, b\colon I \to \mathbb{R}$ on an interval I. According to Remark 5.1.2, the equation $(L\varphi)(t) = b(t)$ is equivalent to the first-order linear ODE system $y'(t) = A(t)\,y(t) + f(t)$, where $y = (\varphi, \varphi', \varphi'', \dots, \varphi^{(n-1)})^{\mathrm{T}}$ and

$$A(t) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 \\ \vdots & \vdots & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1 & 0 \\ 0 & 0 & 0 & & 0 & 1 \\ -a_0(t) & -a_1(t) & -a_2(t) & \dots & -a_{n-2}(t) & -a_{n-1}(t) \end{pmatrix} \quad \text{and} \quad f(t) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ (t) \end{pmatrix}.$$

Let us consider the homogeneous equation $(L\varphi)(t) = 0$, i.e., $y'(t) = A(t)\,y(t)$. A set of n solutions $y_1, \dots, y_n\colon I \to \mathbb{R}^n$ to $y'(t) = A(t)\,y(t)$ corresponds to a set of n solutions $\varphi_1, \dots, \varphi_n\colon I \to \mathbb{R}$ to the equation $L\varphi = 0$. They are linearly independent if and only if the vectors $(\varphi_i, \varphi_i', \varphi_i'', \dots, \varphi_i^{(n-1)})$ for $i = 1, \dots, n$ are, i.e., if and only if the *Wronski determinant*

$$W(t) = \det \Phi(t), \qquad \text{where} \qquad \Phi(t) = \begin{pmatrix} \varphi_1(t) & \varphi_2(t) & \dots & \varphi_n(t) \\ \varphi_1'(t) & \varphi_2'(t) & \dots & \varphi_n'(t) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_1^{(n-1)}(t) & \varphi_2^{(n-1)}(t) & \dots & \varphi_n^{(n-1)}(t) \end{pmatrix} \tag{5.3.7}$$

does not vanish in I. Hence, as a corollary of Theorem 5.3.7, we obtain the following statements.

Theorem 5.3.15 (Solution of a linear ODE of order n). *Let L be as in* (5.3.6)*.*

1. *The solutions of the homogeneous equation $L\varphi = 0$ form a vector space of dimension equal to n.*
2. *For any $(y_{0,1}, y_{0,2}, \dots, y_{0,n}) \in \mathbb{R}^n$ and any $t_0 \in I$, the IVP $L\varphi = 0$, $\varphi^{(k)}(t_0) = y_{0,k+1}$, $k = 0, \dots, n-1$, possesses precisely one solution.*
3. *A set of n solutions $\varphi_1, \dots, \varphi_n\colon I \to \mathbb{R}$ of $L\varphi = 0$ is a basis of the solution space if and only if the Wronski determinant in* (5.3.7) *does not vanish for some (and then for all) $t \in I$.*

◇

The Liouville formula of Lemma 5.3.4 reads for the Wronskian in (5.3.7) as follows:

$$W(t) = W(t_0) \exp\Big(- \int_{t_0}^{t} a_{n-1}(s)\,\mathrm{d}s \Big). \tag{5.3.8}$$

Example 5.3.16 (Legendre equation). One solution of the Legendre ODE

$$(1 - t^2)\varphi'' - 2t\varphi' + 2\varphi = 0$$

is $\varphi_1(t) = t$. Dividing by $(1-t^2)$, we have that $a_1(t) = -\frac{2t}{1-t^2}$ and $a_0(t) = \frac{2}{1-t^2}$. If φ_2 is another solution, linearly independent of φ_1, then $W = \varphi_1\varphi_2' - \varphi_2\varphi_1'$. With the help of the Liouville formula in (5.3.8), we obtain the condition

$$t\varphi_2(t) - \varphi(t) = W(t) = W(0)\exp\Big(\int_0^t \frac{2s}{1-s^2}\,\mathrm{d}s\Big) = W(0)\frac{1}{1-t^2}.$$

Hence,

$$\begin{aligned}\varphi_2(t) &= \mathrm{e}^{\int \frac{1}{t}\,\mathrm{d}t}\Big(C + \int_1^t \mathrm{e}^{-\int \frac{1}{s}\,\mathrm{d}s}\frac{1}{s(1-s^2)}\,\mathrm{d}s\Big) = t\Big(C + \int_1^t \frac{1}{(1-s^2)s^2}\,\mathrm{d}s\Big)\\ &= t\Big(C + \frac{1}{2}\log\frac{1+t}{1-t} - \frac{1}{t}\Big),\end{aligned}$$

as is seen with the help of $\frac{1}{(1-s^2)s^2} = \frac{1}{s^2} + \frac{1}{1-s^2}$. This implies that the general solution of the Legendre equation is

$$\varphi(t) = c_1 t + c_2\Big(\frac{t}{2}\log\frac{1+t}{1-t} - 1\Big),$$

for constants $c_1, c_2 \in \mathbb{R}$. $\diamond$

Remark 5.3.17 (Reduction of the order). If one solution $\varphi\colon I \to \mathbb{R}$ of the n-th order linear ODE $L\varphi = 0$ is known, then the ansatz $y(x) = c(x)\varphi(x)$ transforms $L\varphi = 0$ into a linear homogeneous ODE of order $n-1$ for the function c' (i.e., we have reduced the order by one). $\diamond$

Example 5.3.18. The homogeneous linear ODE $\varphi'' - x\varphi' + \varphi = 0$ possesses the solution $\varphi_1(x) = x$. The ansatz $\varphi(x) = xc(x)$ (hence $\varphi'(x) = c'(x)x + c(x)$ and $\varphi''(x) = c''(x)x + 2c'(x)$) turns $\varphi'' - x\varphi' + \varphi = 0$ into the equation $xc''(x) + (2-x^2)c'(x) = 0$. This is a separable first-order ODE for c' and has the solution

$$c'(x) = \frac{d}{x^2}\mathrm{e}^{\frac{1}{2}x^2}, \qquad \text{for some } d \in \mathbb{R}.$$

Hence, we obtain a second solution (which is linearly independent of $\varphi_1(x) = x$) as

$$\varphi_2(x) = x\int_{x_0}^x \frac{1}{s^2}\mathrm{e}^{\frac{1}{2}s^2}\,\mathrm{d}s.$$

$\diamond$

Remark 5.3.19 (Variation of constants for n-th order equations). Consider the inhomogeneous equation $L\varphi(t) = b(t)$ with some continuous map $t \mapsto b(t) \in \mathbb{R}$ and L as in (5.3.6). We want to assume that we have a fundamental system $\varphi_1, \ldots, \varphi_n$ of the homogeneous equation $L\varphi = 0$, and we want to derive some particular solution φ_p using the method of variation of constants. The ansatz for that is $\varphi_\mathrm{p}(t) = \sum_{i=1}^n c_i(t)\varphi_i(t)$, where we additionally require that $0 = \sum_{i=1}^n c_i'(t)\varphi_i^{(j)}(t)$ for $j = 0, \ldots, n-2$. The we have $\varphi_\mathrm{p}' = \sum_{i=1}^n (c_i'\varphi_i + c_i\varphi_i') = \sum_{i=1}^n c_i\varphi_i'$ and hence $\varphi_\mathrm{p}'' = \sum_{i=1}^n (c_i'\varphi_i' + c_i\varphi_i'') = \sum_{i=1}^n c_i\varphi_i''$ and so on. Hence, our ansatz may be summarized as

$$y_\mathrm{p}(t) = \begin{pmatrix} \varphi_\mathrm{p}(t) \\ \varphi_\mathrm{p}'(t) \\ \vdots \\ \varphi_\mathrm{p}^{(n-1)}(t) \end{pmatrix} = \Phi(t)\begin{pmatrix} c_1(t) \\ \vdots \\ c_n(t) \end{pmatrix},$$

where we used the notation of Remark 5.3.14. Observe that $y' = \Phi' c + \Phi c' = A(t)y_{\mathrm{p}} + \Phi c'$. Hence, we have to solve the equation

$$\Phi(t)\begin{pmatrix} c_1'(t) \\ \vdots \\ c_n'(t) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ b(t) \end{pmatrix}, \qquad \text{i.e.,} \qquad y_{\mathrm{p}}(t) = \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)\begin{pmatrix} 0 \\ \vdots \\ 0 \\ b(s) \end{pmatrix} \mathrm{d}s.$$

We proceed now with $n = 2$, where the formulas are not so cumbersome. Here one can use the formula $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ and obtains the following

Formula for a particular solution of $\boldsymbol{\varphi'' + a_1(t)\varphi' + a_0(t)\varphi = b(t)}$.

$$\varphi_{\mathrm{p}}(t) = -\varphi_1(t)\int_{t_0}^{t} \frac{\varphi_2(s)}{W(s)}\, b(s)\,\mathrm{d}s + \varphi_2(t)\int_{t_0}^{t} \frac{\varphi_1(s)}{W(s)}\, b(s)\,\mathrm{d}s,$$

where $W = \varphi_1\varphi_2' - \varphi_2\varphi_1'$ is the Wronski determinant.

Tis is obviously an extension of the results of Section 3.2. ◇

Example 5.3.20. Consider $t^2\varphi'' - 2t\varphi' + 2\varphi = t^3$. We are so lucky to have seen that $\varphi_1(t) = t$ and $\varphi_2(t) = t^2$ are two solutions to the homogeneous equation, and they are obviously linearly independent.

The variation ansatz is $\varphi_{\mathrm{p}}(t) = tc_1(t) + t^2 c_2(t)$, i.e.,

$$\begin{pmatrix} t & t^2 \\ 1 & 2t \end{pmatrix}\begin{pmatrix} c_1'(t) \\ c_2'(t) \end{pmatrix} = \begin{pmatrix} 0 \\ t \end{pmatrix}.$$

Hence $c_1'(t) = -t$ and $c_2'(t) = 1$, i.e., a particular solution is $\varphi_{\mathrm{p}}(t) = -\frac{1}{2}t^3 + t^3 = \frac{1}{2}t^3$. The complete solution of $t^2\varphi'' - 2t\varphi' + 2\varphi = t^3$ therefore is $\frac{1}{2}t^3 + c_1 t + c_2 t^2$. ◇