THE LONGEST EXCURSION OF A RANDOM INTERACTING POLYMER

Janine Köcher[1] and Wolfgang König[1,2]

(1 March, 2011)

Abstract: We consider a random N-step polymer under the influence of an attractive interaction with the origin and derive a limit law – after suitable shifting and norming – for the length of the longest excursion towards the Gumbel distribution. The embodied law of large numbers in particular implies that the longest excursion is of order $\log N$ long. The main tools are taken from extreme value theory and renewal theory.

MSC 2010. 60F05, 82D60.

Keywords and phrases. Free energy, interacting polymer, longest excursion, extreme value theory, renewal theory.

1. Introduction and main results

Let $(S_n)_{n\in\mathbb{N}_0}$ be a random walk on the lattice $\mathbb{Z}^d$ starting at the origin and having steps of mean zero. By $\mathbb{P}$ and $\mathbb{E}$ we denote the corresponding probability and expectation, respectively. We conceive the walk $(n, S_n)_{n=0,\dots,N}$ as an N-step polymer in the $(d+1)$-dimensional space. We introduce an attractive interaction with the origin by introducing the Gibbs measure $\mathbb{P}_{\beta,N}$ via the density

$$\frac{\mathrm{d}\mathbb{P}_{\beta,N}}{\mathrm{d}\mathbb{P}} = \frac{\mathrm{e}^{\beta L_N}}{Z_{\beta,N}} \qquad \text{with } Z_{\beta,N} = \mathbb{E}\big[\mathrm{e}^{\beta L_N}\big], \tag{1.1}$$

where $\beta \in (0,\infty)$ is a parameter and

$$L_N = |\{k \in \{1,\dots,N\} : S_k = 0\}| \tag{1.2}$$

denotes the walker's local time at the origin, i.e., the number of returns to the origin. The properties of the polymer under $\mathbb{P}_{\beta,N}$ have been studied a lot [dH09, G07]. In particular, the free energy

$$F(\beta) = \lim_{N\to\infty} \frac{1}{N} \log Z_{\beta,N} \in (0,\beta) \tag{1.3}$$

has been shown to exist and to be positive and strictly increasing in β. Furthermore, it has been shown that the polymer is localised in the sense that L_N is of order N under $\mathbb{P}_{\beta,N}$, and

[1]Institute for Mathematics, TU Berlin, Str. des 17. Juni 136, 10623 Berlin, Germany, `JanineKoecher@aol.com` and `koenig@math.tu-berlin.de`

[2]Weierstrass Institute Berlin, Mohrenstr. 39, 10117 Berlin, `koenig@wias-berlin.de`

the density of the set of hits of the origin has been characterised. In particular, the constrained version, i.e., the polymer under

$$\mathbb{P}^{(c)}_{\beta,N}(\cdot) = \frac{1}{Z^{(c)}_{\beta,N}}\mathbb{E}\big[e^{\beta L_N}\mathbb{1}\{\,\cdot\,\}\mathbb{1}_{\{}S_N=0\}\big], \qquad \text{where } Z^{(c)}_{\beta,N} = \mathbb{E}\big[e^{\beta L_N}\mathbb{1}_{\{}S_N=0\}\big], \tag{1.4}$$

has been studied.

In this paper, we consider the length of the longest excursion of the polymer under $\mathbb{P}^{(c)}_{\beta,N}$. To introduce this object, we denote by $\tau=\{\tau_i\colon i\in\mathbb{N}_0\}$ the set of return times to the origin, where

$$\tau_0=0 \qquad \text{and, inductively, } \tau_{i+1}=\inf\{n>\tau_i\colon S_n=0\}, \quad i\in\mathbb{N}_0. \tag{1.5}$$

Then $\mathbb{P}^{(c)}_{\beta,N}$ is the conditional distribution of the polymer given $\{N\in\tau\}$. The length of the longest excursion is now given as

$$\mathrm{maxexc}_N = \max\{\tau_i-\tau_{i-1}\colon i\in\mathbb{N}, \tau_i\le N\}. \tag{1.6}$$

According to [dH09, Theorem 7.3], maxexc_N is of order $\log N$ under $\mathbb{P}^{(c)}_{\beta,N}$, in the sense that the distribution of $\mathrm{maxexc}_N/\log N$ under $\mathbb{P}^{(c)}_{\beta,N}$ is tight in N. The proof gives the upper bound $2/F(\beta)$, which is not sharp, as we will see below. It is the main goal of this note to derive not only the law of large numbers for maxexc_N, but also a non-trivial limit law for maxexc_N after suitable shifting, in the spirit of extreme value theory.

To formulate our main result, we need to fix our assumptions first.

Assumption ($\boldsymbol{\tau}$). There are $D\in(0,\infty)$ and $\alpha\in(1,\infty)$ such that

$$K(n):=\mathbb{P}(\tau_1=n)\sim Dn^{-\alpha}, \qquad n\to\infty.$$

This assumption is fulfilled for most of the aperiodic random walks $(S_n)_{n\in\mathbb{N}_0}$ under consideration in the literature. For random walks with period $p\in\mathbb{N}$, one has to work with $K(pn)$ instead of $K(n)$ and with pN-step polymers and obtains analogous results. Assumption (τ) can be relaxed with the help of slowly varying functions, on cost of a more cumbersome formulation and proof of the main result.

The main result of this paper is the following.

Theorem 1.1. *Suppose that Assumption (τ) is satisfied, and fix $\beta\in(0,\infty)$. Then, as $N\to\infty$, the distribution of*

$$F(\beta)\mathrm{maxexc}_N - \log\frac{N}{\mu_\beta} + \alpha\log\log\frac{N}{\mu_\beta} - C \tag{1.7}$$

under $\mathbb{P}^{(c)}_{\beta,N}$ weakly converges towards the standard Gumbel distribution, where

$$\mu_\beta = e^\beta\sum_{n\in\mathbb{N}} nK(n)e^{-nF(\beta)} \qquad \textit{and} \qquad C=\log\Big(F(\beta)^\alpha D\frac{e^{\beta-F(\beta)}}{1-e^{-F(\beta)}}\Big). \tag{1.8}$$

Explicitly, it is stated that, for any $x\in\mathbb{R}$,

$$\lim_{N\to\infty}\mathbb{P}^{(c)}_{\beta,N}\big(\mathrm{maxexc}_N\le\gamma_x(N/\mu_\beta)\big)=e^{-e^{-x}}, \qquad \text{where } \gamma_x(N)=\frac{x+C+\log N-\alpha\log\log N}{F(\beta)}. \tag{1.9}$$

In particular, we have the law of large numbers: $\mathrm{maxexc}_N/\log N\to 1/F(\beta)$ in $\mathbb{P}^{(c)}_{\beta,N}$-probability as $N\to\infty$.

2. The proof

It is well-known that the free energy $F(\beta)$ is characterised by the equation

$$\mathrm{e}^{\beta} = \sum_{n\in\mathbb{N}} K(n)\mathrm{e}^{-nF(\beta)}, \tag{2.1}$$

and that it actually holds that $Z^{(\mathrm{c})}_{\beta,N} \sim \mathrm{e}^{NF(\beta)}\frac{1}{\mu_\beta}$ as $N\to\infty$. In particular, $F(\beta)$ is also the exponential rate of $Z^{(\mathrm{c})}_{\beta,N}$. The first step, which is basic to all investigations of the polymer, is a change of measure to the measure Q_β, under which the excursion lengths $T_k = \tau_{k+1}-\tau_k$, are i.i.d. in $k\in\mathbb{N}_0$ with distribution

$$Q_\beta(T_1 = n) = \mathrm{e}^{-\beta}K(n)\mathrm{e}^{-nF(\beta)}, \qquad n\in\mathbb{N}.$$

Since maxexc_N is measurable with respect to the family of the T_k's, it is easy to see from the technique explained in [G07, p. 9] that

$$\mathbb{P}^{(\mathrm{c})}_{\beta,N}(\mathrm{maxexc}_N \le \gamma_N) \sim \mu_\beta Q_\beta(\mathrm{maxexc}_N \le \gamma_N, N\in\tau), \qquad N\to\infty, \tag{2.2}$$

for any choice of the sequence $(\gamma_N)_{N\in\mathbb{N}}$, where $\mu_\beta = \sum_{n\in\mathbb{N}} nQ_\beta(T_1=n)\in[1,\infty)$ is the expectation of the length of the first excursion under Q_β. Introducing

$$M_n = \max_{k=1}^{n} T_k \qquad\text{and}\qquad \sigma_N = \inf\{k\in\mathbb{N}\colon \tau_k\ge N\}, \tag{2.3}$$

we see that $\mathrm{maxexc}_N = M_{\sigma_N}$ on $\{N\in\tau\}$ for any $N\in\mathbb{N}$. (Note that $\sigma_N = L_N$ on the event $\{N\in\tau\}$.) Hence, Theorem 1.1 is equivalent to

$$\lim_{N\to\infty} Q_\beta(M_{\sigma_N}\le \gamma_x(N/\mu_\beta), N\in\tau) = \frac{1}{\mu_\beta}\mathrm{e}^{-\mathrm{e}^{-x}}, \qquad x\in\mathbb{R}. \tag{2.4}$$

The proof of this consist of a combination of three fundamental ingredients:

(1) an *extreme value theorem* for M_n under Q_β,
(2) a *law of large numbers* for σ_N under Q_β,
(3) a *renewal theorem* for τ under Q_β.

Items (2) and (3) are immediate: We have from renewal theory that $\sigma_N/N\to 1/\mu_\beta$ in Q_β-probability and $\lim_{N\to\infty} Q_\beta(N\in\tau) = 1/\mu_\beta$. The first item needs a bit more care:

Lemma 2.1.

$$\lim_{N\to\infty} Q_\beta(M_N\le\gamma_x(N)) = \mathrm{e}^{-\mathrm{e}^{-x}}, \qquad x\in\mathbb{R}.$$

Proof. Note that M_N is the maximum of N independent random variables with the same distribution as $T_1=\tau_1$ under Q_β. Observe that the tails of this distribution are given by

$$\begin{aligned} Q_\beta(\tau_1>k) &= \mathrm{e}^{\beta}\sum_{n>k} K(n)\mathrm{e}^{-nF(\beta)} \sim \mathrm{e}^{\beta}D\sum_{n>k} n^{-\alpha}\mathrm{e}^{-nF(\beta)}\\ &= \mathrm{e}^{\beta}D\mathrm{e}^{-kF(\beta)}k^{-\alpha}\sum_{n\in\mathbb{N}}(1+\tfrac{n}{k})^{-\alpha}\mathrm{e}^{-nF(\beta)}\\ &\sim \mathrm{e}^{-kF(\beta)}k^{-\alpha}D\frac{\mathrm{e}^{\beta-F(\beta)}}{1-\mathrm{e}^{-F(\beta)}}, \qquad k\to\infty, \end{aligned}$$

where in the last step we used the monotonous convergence theorem and the geometric series. Hence, replacing k by $\gamma_x(N)$, we see that, as $N\to\infty$,

$$\begin{aligned}
Q_\beta(\tau_1>\gamma_x(N)) &\sim \mathrm{e}^{-\gamma_x(N)F(\beta)}\gamma_x(N)^{-\alpha}D\frac{\mathrm{e}^{\beta-F(\beta)}}{1-\mathrm{e}^{-F(\beta)}}\\
&=\frac{1}{N}\mathrm{e}^{-C-x}(\log N)^\alpha\Big(\frac{x+C+\log N-\alpha\log\log N}{F(\beta)}\Big)^{-\alpha}\mathrm{e}^{C}F(\beta)^{-\alpha}\\
&\sim\frac{\mathrm{e}^{-x}}{N}.
\end{aligned}$$

From this the assertion easily follows. □

Hence, Theorem 1.1 is easily seen to follow from the above three ingredients, as soon as one shows that σ_N may asymptotically be replaced by N/μ_β and that the two events in (2.4) are asymptotically independent. This is what we show now. First we show that M_{σ_N} and M_{N/μ_β} have the same limiting distribution.

Lemma 2.2.

$$\lim_{N\to\infty}Q_\beta(M_{\sigma_N}\le\gamma_x(N/\mu_\beta))=\mathrm{e}^{-\mathrm{e}^{-x}},\qquad x\in\mathbb{R}.$$

Proof. The upper bound is proved as follows. Fix a small $\varepsilon>0$, then we have, as $N\to\infty$,

$$\begin{aligned}
Q_\beta(M_{\sigma_N}\le\gamma_x(N/\mu_\beta))&\le Q_\beta\Big(M_{\sigma_N}\le\gamma_x(N/\mu_\beta),\sigma_N\ge\frac{N}{\mu_\beta+\varepsilon}\Big)+Q_\beta\Big(\sigma_N<\frac{N}{\mu_\beta+\varepsilon}\Big)\\
&\le Q_\beta\big(M_{N/(\mu_\beta+\varepsilon)}\le\gamma_x(N/\mu_\beta)\big)+o(1).
\end{aligned}\tag{2.5}$$

Observe that, as $N\to\infty$,

$$\begin{aligned}
\gamma_x(N/\mu_\beta)-\gamma_x(N/(\mu_\beta+\varepsilon))&=\frac{1}{F(\beta)}\log(1+\tfrac{\varepsilon}{\mu_\beta})+\frac{\alpha}{F(\beta)}\log\frac{\log N-\log(\mu_\beta+\varepsilon)}{\log N-\log\mu_\beta}\\
&=\frac{1}{F(\beta)}\log(1+\tfrac{\varepsilon}{\mu_\beta})+o(1).
\end{aligned}$$

Hence, we may replace, as an upper bound, $\gamma_x(N/\mu_\beta)$ on the right of (2.5) by $\gamma_{x+B\varepsilon}(N/(\mu_\beta+\varepsilon))$ for some suitable $B\in\mathbb{R}$, use Lemma 2.1 for N replaced by $N/(\mu_\beta+\varepsilon)$ and x replaced by $x+B\varepsilon$ and make $\varepsilon\downarrow 0$ in the end. This shows that the upper bound of the assertion holds. The lower bound is proved in the same way. □

Proof of Theorem 1.1. It is convenient to introduce a Markov chain $(Y_n)_{n\in\mathbb{N}_0}$ with

$$Y_n=\big(Y_n^{(1)},Y_n^{(2)}\big)=\big(T_{\sigma_n},\tau_{\sigma_n}-n\big)$$

on the state space $I=\{(i,j)\in\mathbb{N}\times\mathbb{N}_0\colon j\le i\}$, where we recall (2.3). In words, the first component is the size of the step over n, and the last is the size of the overshoot. This Markov chain is ergodic and positiv recurrent with invariant distribution $\pi(i,j)=Q_\beta(\tau_1=i)/\mu_\beta$ for $(i,j)\in I$. We denote by $\widetilde{Q}_{i,j}$ the distribution of this chain given that it starts in $Y_0=(i,j)$; note that $Q_\beta=\widetilde{Q}_{i,0}$ with an unspecified value of i, which we put equal to 1 by default. The event $\{N\in\tau\}$ is identical to $\{Y_N^{(2)}=0\}=\{Y_N\in\mathbb{N}\times\{0\}\}$; by ergodicity, its probability under $\widetilde{Q}_{i,j}$ converges, as $N\to\infty$, to $\pi(\mathbb{N}\times\{0\})=\frac{1}{\mu_\beta}$, for any $(i,j)\in I$, which is one way to prove the renewal theorem.

Now let $\varepsilon>0$ be given. Pick $K_\varepsilon\in\mathbb{N}$ so large that $\pi(I_{K_\varepsilon}^{\mathrm{c}})<\varepsilon/2$, where $I_k=\{(i,j)\in I\colon i\le k\}$ for any $k\in\mathbb{N}$. Furthermore, pick $R_\varepsilon\in\mathbb{N}$ with $R_\varepsilon>K_\varepsilon$ so large that $\widetilde{Q}_{i,j}(R_\varepsilon\in\tau)\le\frac{1}{\mu_\beta}+\varepsilon$ for any $(i,j) \in I_{K_\varepsilon}$. Now pick $N_\varepsilon \in \mathbb{N}$ so large that $N_\varepsilon > R_\varepsilon$ and $\widetilde{Q}_{1,0}(Y_{N-R_\varepsilon} \in I^{\mathrm{c}}_{K_\varepsilon}) < \pi(I^{\mathrm{c}}_{K_\varepsilon}) + \varepsilon/2$ for any $N \geq N_\varepsilon$. The latter is possible, since $\widetilde{Q}_{1,0}(Y_{N-R_\varepsilon} \in I^{\mathrm{c}}_{K_\varepsilon}) = 1 - \widetilde{Q}_{1,0}(Y_{N-R_\varepsilon} \in I_{K_\varepsilon})$ converges towards $1 - \pi(I_{K_\varepsilon}) = \pi(I^{\mathrm{c}}_{K_\varepsilon})$ as $N \to \infty$ by ergodicity.

Recall that we only have to prove (2.4). We calculate, with the help of the Markov property at time $N - R_\varepsilon$, for $N > N_\varepsilon$,

$$
\begin{aligned}
Q_\beta(M_{\sigma_N} \leq \gamma_x(N/\mu_\beta), N \in \tau) &= \widetilde{Q}_{1,0}\Big(\max_{k=1}^{N} Y^{(1)}_k \leq \gamma_x(N/\mu_\beta), Y^{(2)}_N = 0\Big) \\
&\leq \widetilde{Q}_{1,0}\Big(\max_{k=1}^{N-R_\varepsilon} Y^{(1)}_k \leq \gamma_x(N/\mu_\beta), Y_{N-R_\varepsilon} \in I_{K_\varepsilon}, Y^{(2)}_N = 0\Big) + \widetilde{Q}_{1,0}(Y_{N-R_\varepsilon} \in I^{\mathrm{c}}_{K_\varepsilon}) \\
&\leq \sum_{(i,j)\in I_{K_\varepsilon}} \widetilde{Q}_{1,0}\Big(\max_{k=1}^{N-R_\varepsilon} Y^{(1)}_k \leq \gamma_x(N/\mu_\beta), Y_{N-R_\varepsilon} = (i,j)\Big)\, \widetilde{Q}_{i,j}(Y^{(2)}_{R_\varepsilon} = 0) + \pi(I^{\mathrm{c}}_{K_\varepsilon}) + \varepsilon/2 \\
&\leq \widetilde{Q}_{1,0}\Big(\max_{k=1}^{N-R_\varepsilon} Y^{(1)}_k \leq \gamma_x(N/\mu_\beta)\Big)\big(\tfrac{1}{\mu_\beta} + \varepsilon\big) + \varepsilon \\
&\leq Q_\beta\big(M_{\sigma_{N-R_\varepsilon}} \leq \gamma_x(N/\mu_\beta)\big)\big(\tfrac{1}{\mu_\beta} + \varepsilon\big) + \varepsilon.
\end{aligned}
$$

Now apply Lemma 2.2 for N replaced by $N - R_\varepsilon$ and observe that $\lim_{N\to\infty}(\gamma_x(N/\mu_\beta) - \gamma_x((N - R_\varepsilon)/\mu_\beta)) = 0$. Afterwards letting $\varepsilon \downarrow 0$ shows that the upper bound in (2.4) holds. The proof of the corresponding lower bound is similar, and we omit it. $\square$

References

[G07] G. Giacomin, *Random Polymer Models*, Imperial College Press (2007).

[dH09] F. den Hollander, *Random Polymers*, Springer (2009).